Examples of compact hypersurfaces in $\mathbb{R}^{2p}$, $2p \geq 6$, with no periodic orbits[1]

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1. Introduction

Let (M^{2p}, w_0) be a C^∞ symplectic, metrizable, manifold. For example $T^*\mathbb{R}^p$, with its canonical symplectic form. Let $H_0 : M \to \mathbb{R}$ be a C^∞ function, c a regular value of H_0; then $H_0^{-1}(c)$ is a codimension 1 C^∞ manifold. Given any codimension 1 C^∞ [*closed*] manifold $N_0 \subset M^{2p}$, then N_0 is a connected component of $H_0^{-1}(c)$, for some C^∞ function H_0 and c a regular value.

Let X_{H_0} be the Hamiltonian vector field associated to H_0 and w_0: $i_{X_{H_0}} w_0 = dH_0$; and $f_t^{H_0}$ the local Hamiltonian flow of X_{H_0}, that leaves invariant H_0 and N_0. If we consider the various possible choices of H_0, for a given N_0, then X_{H_0} changes to φX_{H_0}, $\varphi \in C^\infty(N_0, \mathbb{R})$, $\varphi \neq 0$ (*i.e.* the parametrisation of the orbits). Reparametrisation does not change the periodic orbits but usually does change the periods.

Theorem. *We suppose that* $2p \geq 8$ *(resp.* $2p = 6$*) and that* $f_t^{H_0}|N_0$ *has only a finite number of periodic orbits. Then there exist* $H : M \to \mathbb{R}$ *of class* C^∞ *(resp.* $C^{3-\varepsilon}$*, for any* $\varepsilon > 0$*) and a regular value* c *of* H *such that* $H^{-1}(c)$ *is* C^∞ *(resp.* $C^{3-\varepsilon}$*) diffeomorphic to* $H_0^{-1}(c)$ *and* $f_t^H|N$ *has no periodic orbits, where* N *is [a] component of* $H^{-1}(c)$ *diffeomorphic to* N_0*.*[2]

Example. Using a Darboux chart, we can glue in a symplectic manifold the following example on $T^*\mathbb{R}^p \cong \mathbb{C}^p$:

$$H_0(z) = \sum_{j=1}^{p} \alpha_j |z_j|^2 ,$$

where the α_j's are positive and independent over $\mathbb{Q}$. Then, on $H_0^{-1}(c)$, $c > 0$, $f_t^{H_0}$ has only p periodic orbits and we can apply the above theorem.

The idea of the proof is very elementary. When $p \geq 4$, we use Wilson's plugs (Ann. Math. 1966) and, when $p = 3$, Schweitzer's plugs (Ann. Math. 1974). On $T^*\mathbb{R}^p \cong \mathbb{R}^p \times \mathbb{R}^p$, with the coordinates $(x, r) = (x_1, \ldots, x_p, r_1, \ldots, r_p)$, given a vector field $X : \mathbb{R}^p \times \{0\} \to \mathbb{R}^p$, we can obtain a Hamiltonian extension whose Hamiltonian is

$$\sum_{j=1}^{p} X_j(x) r_j .$$

[1]The handwritten original paper looks like a preliminary version; references and some subsections remained empty. Typing, minor corrections (written in italics between brackets) and some comments (in footnotes) are due to F. Laudenbach.

[2]Actually the proof shows that, given any tubular neighbourhood T of N_0, N can be chosen as a C^0 section of T.

Then we modify it to create a C^∞ symplectic plug and destroy a periodic orbit that passes through a given symplectic flow box on a given energy surface. When $p = 3$, the extension $\sum_{j=1}^{p} X_j(x) r_j$ has to be modified to gain one unit of differentiability (using convolution operators).

The case $2p = 4$ is unknown to the author but the reader is referred to a recent preprint of G. Kuperberg (quoted by E. Ghys's Bourbaki seminar n^o 785). For *volume preserving flows on oriented manifolds of dimension* ≥ 4, C^∞–volume preserving plugs can be constructed to destroy a finite number of periodic orbits (the case of dimension 3 is treated by G. Kuperberg).

Similar results were obtained independently by V.L. Ginzburg[3].

(Abstract for Trieste Conf. Oct. 1994.[4])

2. Flow box theorem

Let M^n be a C^∞ manifold (metrizable and connected), $x_0 \in M^n$, and X a C^∞ vector field such that $X(x_0) \neq 0$. We recall the flow box theorem.

2.1. **Theorem.** *There exists a* C^∞ *local diffeomorphism* $\varphi : (V, x_0) \subset M^n \to (\mathbb{R}^n, 0)$ *such that* $\varphi(x_0) = 0$ *and*

$$\varphi_* X = (D\varphi . X) \circ \varphi^{-1} = e_1 = (1, 0, \ldots, 0) .$$

For a proof see [ ].

2.2. We suppose furthermore $n = 2p$, $p \in \mathbb{N}^*$, and (M^{2p}, w_0) is a C^∞ symplectic manifold (*i.e.* w_0 is a C^∞ closed 2-form of maximal rank). We suppose that $H_0 : M^n \to \mathbb{R}$ is a C^∞ function, $dH_0(x_0) \neq 0$ and X_{H_0} is the Hamiltonian vector field associated to H_0 and w_0:

$$dH_0 = i_{X_{H_0}} w_0 = w_0(X_{H_0}, -) .$$

On $\mathbb{R}^{2p} = T^*\mathbb{R}^p \cong \mathbb{R}^p \times (\mathbb{R}^p)^*$ we put the canonical symplectic form

$$w = -d\left(\sum_{j=1}^{p} r_j \, dx_j\right) = \sum_{j=1}^{p} dx_j \wedge dr_j$$

where $x = (x_1, \ldots, x_p) \in \mathbb{R}^p$, $(r_1, \ldots, r_p) \in (\mathbb{R}^p)^*$ are the canonical coordinates.

[3] Viktor L. Ginzburg, *An embedding* $S^{2n-1} \to \mathbb{R}^{2n}$, $2n - 1 \geq 7$, *whose Hamiltonian flow has no periodic trajectories*, Internat. Math. Res. Notices 1995, no. 2, 83–97.

[4] It seems that M. Herman did not attend that conference. Actually, the text of this section appeared in the proceedings of a NATO conference helded at S'Agaró (Spain), 19-30 June 1995: Hamiltonian systems with three or more degrees of freedom, C. Simo (ed.), p. 126, Kluwer Acad. Pub., 1999.

2.3. **Theorem.** *There exist a C^∞ local diffeomorphism $\varphi : (V, x_0) \subset M^{2p} \to (\mathbb{R}^{2p}, 0)$ such that $\varphi^* w = w_0$, and H, a C^∞ function depending only on the coordinate r_1, such that $H \circ \varphi = H_0$.*

2.4. Let $X_H = \left(\frac{\partial H}{\partial r_1}(r_1), \dots\right)$ be the Hamiltonian vector field of H for the symplectic form w of $\mathbb{R}^{2p}$. We have $\varphi_* X_{H_0} = X_H$. Let us remark on each energy surface of H (*i.e.* $r_1 = c$) that

$$\frac{\partial H}{\partial r_1}(r_1) = \frac{\partial H}{\partial r_1}(c)$$

is constant.

2.5. If we replace the Hamiltonian H_0 by $\Phi \circ H_0$, where $\Phi'(H_0(x_0)) \neq 0$ we can suppose that $\Phi \circ H(r_1) = r_1$. Hence

$$X_{\Phi \circ H} = e_1 = (1, 0, \dots, 0)\,.$$

2.6. We will prove the *volume preserving flow box theorem.* Let M^n be a C^∞ oriented manifold and Ω_0 a volume form. A C^∞ vector field X is Ω_0-volume preserving if its local flow preserves Ω_0. This is equivalent to

$$L_X \Omega_0 = i_X \circ d\Omega_0 + d \circ i_X \Omega_0 = d\left(i_X \Omega_0\right) = 0$$

or, equivalently, the $(n-1)$–form $i_X \Omega_0$ is closed.

Theorem. *There exists a local C^∞ diffeomorphism $\varphi : (M^n, x_0) \to (\mathbb{R}^n, 0)$ such that $\varphi_* X = (D\varphi.X) \circ \varphi^{-1} = e_1$ near x_0 and $\varphi^* \Omega = \Omega_0$, where $\Omega = dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$.*

Proof.[5] Let φ_1 be a local diffeomorphism given by theorem 2.1. We leave to the reader the case $n = 1$ and we suppose that $n \geq 2$. We can suppose that φ_1 is orientation preserving (by considering $S \circ \varphi_1$, where $S(x_1, x_2, \dots) = (x_1, -x_2, x_3, \dots x_n)$). Let $\left(\varphi_1^{-1}\right)^* \Omega_0 = \Omega_1$. Considering $h_a \circ \varphi$, $a > 0$, where

$$h_a(x_1, x_2, \dots, x_n) = (x_1, a x_2, \dots, x_n),$$

we can suppose that $\Omega_1(0) = dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$. By conjugation, the vector field e_1 is Ω_1–volume preserving. Hence

$$\Omega_1 = a(x_2, \dots, x_n)\, dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$$

where a is a positive C^∞ function, depending only on $(x_2, \dots, x_n)$, and $a(0) = 1$. Let $\check{\Omega}_i = i_{e_1} \Omega_i$ for $i = 0, 1$.

Lemma. *There exists a local C^∞ diffeomorphism f of $(\mathbb{R}^{(n-1)}, 0)$ such that*

$$f_1^* \check{\Omega}_1 = dx_2 \wedge \ldots \wedge dx_n = \check{\Omega}_0\,.$$

[5] It could be easily deduced from Moser's Theorem on volume forms.

Proof. Let $\check{\Omega}_s = (1-s)\check{\Omega}_0 + s\check{\Omega}_1$, $0 \le s \le 1$. We want to find a C^∞ vector field X_s, $X_s(0) = 0$ for $0 \le s \le 1$, such that the solution f_s of

$$\frac{\partial f_s}{\partial s} = X_s \circ f_s,\ f_0 = Id,$$

satisfies $f_s^* \check{\Omega}_s = \check{\Omega}_0$. Differentiating one obtains

$$f_s^* \left(L_{X_s} \check{\Omega}_s + \frac{\partial \check{\Omega}_s}{\partial s} \right) = 0\,,$$

which is equivalent to $d\left(i_{X_s}\Omega_s\right) = \check{\Omega}_1 - \check{\Omega}_0$. We can choose

$$\left(i_{X_s}\check{\Omega}_s\right)(z) = \int_0^1 t^{n-2}\, i_z\left(\check{\Omega}_1 - \check{\Omega}_0\right)(tz)\,dt,\ z \in \mathbb{R}^{n-1},\ \|z\| < \delta$$

(see). We have $\left(i_{X_s}\check{\Omega}_s\right)(0) = 0$ since $\left(\check{\Omega}_1 - \check{\Omega}_0\right)(0) = 0$. The form $i_{X_s}\check{\Omega}_s$ is only defined for $\|z\|$ small and, since $\check{\Omega}_s(0)$ is of maximal rank, we obtain the vector X_s, uniquely defined for $\|z\| < \delta_1 \le \delta$. Since $X_s(0) = 0$, the vector field X_s can be integrated up to $s = 1$ when $\|z\| < \delta_2 < \delta_1$. Since $f_s^*\check{\Omega}_s = \check{\Omega}_0$ for all $s \in [0,1]$, we have $f_1^*\check{\Omega}_1 = \check{\Omega}_0$ and this prove the lemma.

The flow box theorem now follows by conjugation of e_1 by the local diffeomorphism

$$f(x_1,\ldots,x_n) = (x_1, f_1(x_2,\ldots,x_n))\ .$$

3. Reflection principle

3.1. Let $H : \mathbb{R}^{2p} \to \mathbb{R}$ be a C^∞ function such that X_H is C^∞ tangent to e_1 along $\{x_1 = 0\}$[6]. Let $(X_1,\ldots,X_{2n})$ be the components of X_H. One defines the vector field $\widehat{X}$ by the following formulas. If $x_1 \ge 0$, let $\widehat{X}(z) = X_H(z)$, where $z = (x,r)$. When $x_1 \le 0$, one defines:

$$\begin{array}{lcl} \widehat{X}_1(z) & = & X_1(-x_1, x_2, \ldots, -r_1, \ldots, r_p), \\ \widehat{X}_j(z) & = & -X_j(-x_1, x_2, \ldots, -r_1, \ldots, r_p),\ \text{for } j \ne 1, p+1, \\ \widehat{X}_{p+1}(z) & = & X_{p+1}(-x_1, x_2, \ldots, -r_1, \ldots, r_p), \end{array}$$

the Hamiltonian, for $x_1 \le 0$, being $-H(-x_1, x_2, \ldots, x_p, -r_1, r_2, \ldots, r_p)$.
Let $A : (x_1, \ldots, r_1, \ldots, r_p) \mapsto (-x_1, x_2, \ldots, x_p, -r_1, r_2, \ldots, r_p)$. We have:

$$A_*\widehat{X}_H = -\widehat{X}_H\,. \tag{3.2}$$

3.3. We suppose that $z \in \{r_1 = 0\}$, $z = (-a, x_2, \ldots, x_p, 0, r_2, \ldots, r_p)$, with $a > 0$, and that the flow f_t of $\widehat{X}_H$ (that is supposed to exist in the flow box) is such that $f_{t_1}(z) \in \{x_1 = r_1 = 0\}$ [for some $t_1 > 0$]. Then we have:

$$f_{2t_1}(z) = (a, x_2, \ldots, x_p, 0, r_2, \ldots, r_p).$$

This follows from (3.2) (A leaves invariant the [*set of*] flow lines of $\widehat{X}_H$) and from the fact that $\{x_1 = r_1 = 0\}$ is [*pointwise*] invariant by A.

[6] One should say that H is C^∞ tangent to r_1 along $\{x_1 = 0\}$.

3.4. We modified the mirror image construction of Wilson [ ] to obtain a symplectic vector field $\widehat{X}_H$. If we want an Ω-volume preserving diffeomorphism ($\Omega = dx_1 \wedge \ldots \wedge dx_n$) we could directly take the mirror image construction of Wilson.

4. Wilson plugs in $\mathbb{R}^p$, $p \geq 4$

4.1. We will follow H. Rosenberg. Let Z be a complete C^∞ vector field on $\mathbb{R}^{p-1}$ such that the flow g_t of Z leaves invariant a compact set $K \neq \emptyset$ and $g_t|K$ has no periodic orbits. We suppose that $K \subset \{\|\hat{x}\| < 1/2\}$, where $\hat{x} = (x_2, \ldots, x_p) \in \mathbb{R}^{p-1}$ and $\|\hat{x}\|^2 = \sum_{j=2}^{p} x_j^2$.

4.2. We will suppose that $0 \in K$.

4.3. **Examples.**

a. As $p-1 \geq 3$, we can C^∞ embed $\mathbb{T}^q$, $2 \leq q \leq p-2$, into $\mathbb{R}^{p-1}$: $\mathbb{T}^q \xrightarrow{\cong} T \subset \mathbb{R}^{p-1}$. We can put on T a minimal flow Z_1 (*e.g.* an irrational flow). Using N_T, an [*open*] tubular neighbourhood of T in $\mathbb{R}^{p-1}$, and a partition of unity, we can extend Z_1 to a vector field Z such that $Z|T = Z_1$ and $Z(x) = 0$, if $x \notin N_T$.

b. Let M^q be a C^∞ compact manifold, Z_1 a C^∞ vector field [*on* M] such that its flow is minimal. As we can embed $M^q \hookrightarrow \mathbb{R}^{2q+1}$, we perform the construction we did in **a**. There are many examples. For instance, we can suspend the examples of minimal diffeomorphisms we constructed in [ ].

c. Let f be a C^∞ (area preserving) diffeomorphism of $\mathbb{R}^2$ having a periodic homoclinic orbit. Using a horseshoe, we can find $K_0 \cong \{0,1\}^{\mathbb{Z}} \hookrightarrow \mathbb{R}^2$, $k \geq 1$, such that $f(K_0) = K_0$ and $f|K_0$ is conjugated to the shift $S\left(x_j\right)_j = \left(x_{j+1}\right)_{j\in\mathbb{Z}}$. The transformation S has an uncountable number of compact invariant minimal sets. Let K_1 be one of them (*e.g.* expansive Denjoy minimal sets). We suspend the diffeomorphism f and obtain in $\mathbb{R}^3$ a flow Z with a minimal set K, such that a Poincaré section of $Z|K$ is topologically conjugated to $S|K_1$.

4.4. Let $\lambda : [-1, 0] \to I$ be a C^∞ function, $1 \geq \lambda \geq 0$, $\lambda = 0$ near $\partial[-1, 0]$ and $\lambda(t) = 1 \Leftrightarrow t = -\frac{1}{2}$. Let $\varphi : \mathbb{R}^{p-1} \to [0, 1]$ be a C^∞ function with compact support such that $\varphi^{-1}(1) = K$. We extend λ on $[-1, +1]$ by $-\lambda(-t) = \lambda(t)$. We define the C^∞ Wilson plug on $[-1, +1] \times \mathbb{R}^{p-1}$ by:

$$X(x_1, \hat{x}) = (1 - \varphi(\hat{x})|\lambda(x_1)|)\,\frac{\partial}{\partial x_1} + \varphi(\hat{x})\lambda(x_1)Z\,. \tag{4.5}$$

When $\hat{x} \in K$, we have:

$$X(x_1, \hat{x}) = (1 - |\lambda(x_1)|)\,\frac{\partial}{\partial x_1} + \lambda(x_1)Z\,. \tag{4.6}$$

4.7. We recall $0 \in K$.

4.8. Let $A_1(x_1, \hat{x}) = (-x_1, \hat{x})$. We have the mirror image property:

$$(A_1)_* X = -X .$$

Hence, every orbit of X starting at $(-1, \hat{x})$ and that arrives to $(0, v)$ exists at $(1, \hat{x})$ (this is the mirror image property of Wilson). Since $(1 - \varphi(\hat{x})|\lambda(x_1)|) > 0$, except when $x_1 = \pm\frac{1}{2}$ and $\hat{x} \in K$, and since X is tangent to $\{\pm\frac{1}{2}\} \times K$ and $X|\{\pm\frac{1}{2}\} \times K = (0, Z|K)$, the flow f_t of X has no periodic orbit. Indeed, if p_1 denotes the projection $p_1(x) = x_1$, one has:

$$\frac{\partial\,(p_1 \circ f_t)}{\partial t}(x) = (1 - \varphi(\hat{x})|\lambda(x_1)|) \circ f_t(x) \,,$$

which is positive except if $x = \left(-\frac{1}{2}, \hat{x}\right)$, $\hat{x} \in K$: hence $t \mapsto p_1 \circ f_t(x)$ is strictly increasing [*except when it is constant*]. Every orbit of $(-1, \hat{x})$, $\hat{x} \in K$, does not exist at $x_1 = +1$; indeed, it satisfies (4.6) and, when $t \to +\infty$,

$$p_1\left(f_t(x)\right) \to -\frac{1}{2} \,.$$

[*We will refer to this as the* trapping property*. In particular, according to 4.7, the orbit of* $(-1, 0)$ *is trapped.*]

4.9. In the construction, K can be a manifold with boundary, $\dim K = p - 1$; the only thing we need is that the flow g_t of X leaves invariant K[7]. In this way, when $p - 1 = 2$ [*and* K *is an annulus*], we can suppose $g_t|K$ has only 2 periodic orbits. For instance, we can suspend the diffeomorphism $f : [1, 2] \to [1, 2]$ given by

$$f(x) = \frac{1}{2}\big(x + (x-1)^3\big) + \frac{1}{2} \,,$$

$(f(1) = 1,\ f(2) = 2)$.

When $p - 1 \geq 3$, we can embed $\mathbb{T}^{p-3}$ in $\mathbb{R}^{p-2}$; this gives an embedding of $\mathbb{T}^{p-3} \times [1, 2]$. Suspending the diffeomorphism $R_\alpha \times f$, where $R_\alpha : \theta \mapsto \theta + \alpha$ is a minimal translation of $\mathbb{T}^{p-3}$, we obtain Z such that $K \cong \mathbb{T}^{p-3} \times [1, 2]$ is invariant by g_t, and g_t has no periodic orbits on K.

5. Symplectic plugs

5.1. [*We look at a box:*][8]

$$B_c = \{(x_1, x_2, \ldots, r_1, \ldots, r_p) \mid |x_1| \leq 1, |r_1| \leq c, \sum_{j=2}^{p} x_j^2 \leq 1, \sum_{j=2}^{p} r_j^2 \leq 1\}$$

[*We also introduce the left part* $B_c^- = B_c \cap \{x_1 \leq 0\}$.] We assume that $p \geq 4$ (*i.e.* $2p \geq 8$) and we consider $X = (X_1, \ldots, X_p)$, a Wilson vector field (after homothety and rescaling)[9] in $\{-1 \leq x_1 \leq 0, \sum_{j \geq 2} x_j^2 \leq 1\}$ such that:

[7]This means that the construction works even when K is not minimal.

[8]Here M.H. speaks of a *conformally symplectic flow box* for H_0. We postpone this rescaling argument to the end of the proof in 5.4.

[9]Probably, M. H. thinks of the "left part" of a Wilson vector field.

$X = e_1$ in a neighborhood of $\{x_1 = -1\} \cup \{x_1 = 0\} \cup \{\sum_{j\geq 2} x_j^2 = 1\}$ and $X_1 \geq 0$ [*everywhere*].
Let $\eta_j : \mathbb{R} \to [0,1]$ be C^∞ functions, $j = 1, 2$, such that:

$$\begin{cases} \eta_1(u) = 1 & \text{if} \quad u \leq \frac{1}{16} \\ \eta_1(u) = 0 & \text{if} \quad u \geq \frac{1}{4}\ ; \end{cases}$$

$$\begin{cases} \eta_2(u) = u & \text{if} \quad u \leq \frac{1}{16} \\ \eta_2(u) = 1 & \text{if} \quad u \geq \frac{1}{4} \\ \eta_2(u) > 0 & \text{if} \quad u > 0. \end{cases}$$

We denote $\hat{r} = (r_2, \ldots, r_p)$ and $\|\hat{r}\|^2 = \sum_{j\geq 2} r_j^2$. We introduce the Hamiltonian

$$G(x,r) = \frac{1}{\eta_1(\|\hat{r}\|^2) + \eta_2(\|\hat{r}\|^2)} \left(\eta_1(\|\hat{r}\|^2) \Big(\sum_{j=1}^{p} X_j(x) r_j \Big) + \eta_2(\|\hat{r}\|^2) r_1 \right) + r_1^3\,.$$

When $X(x) = e_1$ or when $\|\hat{r}\| \geq \frac{1}{2}$, we have $G(x,r) = r_1 + r_1^3$.

Let $X_G = (Y_1, \ldots, Y_{2p})$ be the symplectic gradient of G. We have:

$$\frac{\partial G}{\partial r_1}(x,r) = Y_1(x,r) = \frac{\eta_1(\|\hat{r}\|^2) X_1 + \eta_2(\|\hat{r}\|^2)}{\eta_1(\|\hat{r}\|^2) + \eta_2(\|\hat{r}\|^2)} \ + \ 3r_1^2 \ \geq \ 0\,.$$

We have

$$X_G(x,0) = \begin{cases} \dfrac{\partial G}{\partial r}(x,0) = X(x) \\[2ex] \dfrac{\partial G}{\partial x}(x,0) = 0\,. \end{cases}$$

If $\|r\| \neq 0$ or $X_1(x) \neq 0$, we have

$$(*) \qquad Y_1(x,r) > 0\,.$$

We have $dG(x,r) \neq 0$ since, if $\|r\| = 0$, for every x we have $X_j(x) \neq 0$ for some j, $1 \leq j \leq p$.
The function G is C^∞ and defined on

$$E = \{(x,r) \mid \|\hat{r}\| \leq 1,\ r_1 \in \mathbb{R},\ -1 \leq x_1 \leq 0,\ \|\hat{x}\| \leq 1\}\ .$$

Since $dG \neq 0$, $G^{-1}(0)$ is a closed codimension-one submanifold of E. Near

$$\{x_1 = -1\} \cup \{x_1 = 0\} \cup \{\|\hat{x}\| = 1\} \cup \{\|\hat{r}\| = 1\}\,,$$

$G^{-1}(0)$ is equal to $\{r_1 = 0\}$. The submanifold $G^{-1}(0)$ contains $\{r = 0\}$ and is the graph of a C^∞ function over $\{r_1 = 0\}$ when $r \neq 0$, or when $r = 0$ and $X_1(x) \neq 0$ (we can solve [*the equation* $G(x, r_1, \hat{r}) = 0$ *as*] $r_1 = \Psi_1(x, \hat{r})$).
When $r = 0$ and $X_1(x_0) = 0$ then, at $(x_0, 0)$, $G^{-1}(0)$ is locally the graph of a C^∞ function over $\{r_j = 0\}$ where [j *is so that*] $X_j(x_0) \neq 0$, since $\dfrac{\partial G}{\partial r_j}(x_0, 0) \neq 0$.

5.2. The manifold $G^{-1}(0)$ is compact since there are positive constants c_1 and c_2 so that $|G(x,r)| \geq |r_1^3| - c_1|r_1| - c_2$.

If $(x, r_1, 0) \in G^{-1}(0)$ (and $X_1(x) = 0$), then $r_1 = 0$ [10]. Hence, if $(x, \hat{r}) \to (\bar{x}, 0)$ with $X_1(\bar{x}) = 0$, then $\Psi_1(x, \hat{r}) \to 0$[11]. It follows that Ψ_1 extends continuously to $r_1 = 0$ at $(\bar{x}, 0)$ and $G^{-1}(0)$ is the graph of a continuous function. Therefore $G^{-1}(0)$ is *homeomorphic* to $B_c^- \cap \{r_1 = 0\}$.

Proposition. *The manifold $G^{-1}(0)$ is diffeomorphic to $B_c^- \cap \{r_1 = 0\}$.*

Proof. Let $G_\varepsilon(x,r) = G(x,r) + \varepsilon r_1$, $\varepsilon > 0$. For every $\varepsilon > 0$, $G_\varepsilon^{-1}(0)$ is compact and is the graph of a C^∞ function over $\{r_1 = 0\}$. When $\varepsilon \to 0$ the manifolds $G_\varepsilon^{-1}(0)$ tend to $G^{-1}(0)$ in the C^∞ topology: for $0 < \varepsilon < \varepsilon_0$, $G^{-1}(0)$ is included in a tubular neighbourhood of $G_\varepsilon^{-1}(0)$ and the projection $p_\varepsilon : G^{-1}(0) \to G_\varepsilon^{-1}(0)$ (induced by the tubular neighbourhood projection) is a C^∞-diffeomorphism. (The compactness of $G_\varepsilon^{-1}(0)$, $\varepsilon \geq 0$, is absolutely essential). □

We choose c large enough in order that $G^{-1}(0) \subset B_{c/2}^-$. Let $g(r_1) = r_1 + r_1^3$; that is a diffeomorphism of $\mathbb{R}$. Let $G_1 = g^{-1} \circ G$ [12]. We have $G_1(x,r) = r_1$ near $\{x_1 = -1\} \cup \{x_1 = 0\} \cup \{\|\hat{x}\| = 1\} \cup \{\|\hat{r}\| = 1\}$. We can easily extend G_1 outside a neighbourhood of $G_1^{-1}(0) = G^{-1}(0)$ to a function G_2 with no critical point and equal to r_1 in a neighbourhood of [*the boundary of*] B_c^-; but we will not need this in what follows.

Remark. We used the h-cobordism theorem.[13]

In the above construction the main point is that $\{r = 0\}$ is a C^∞ Lagrangian submanifold L. [*When $G^{-1}(0)$ contains L, the latter is foliated by characteristic lines of $G^{-1}(0)$.*]

5.3. We consider the symplectic vector field $\widehat{X}_{G_1}$ [*notations from 3.1 for the symplectic mirror construction*[14]]. The hypersurface $G_1^{-1}(0)$ [*coincides with* $\{r_1 = 0\}$] near $\{x_1 = -1, 0, 1\}$ [*and is invariant by the flow f_t of $\widehat{X}_{G_1}$*], $f_t(z) = (f_{t,1}(z), \ldots, f_{t,2p}(z))$.

According to formula $(*)$ the function $t \mapsto f_{t,1}(z)$ is strictly increasing except when $z = (x_1, \hat{x}, r)$ satisfies $x_1 = \pm\frac{1}{2}$, $\hat{x} \in K$, $r = 0$ [15]. We conclude that f_t has no periodic orbits in B_c. If $z \in G^{-1}(0) \cap \{x_1 = -1\}$, with $\hat{x} \notin K$ or $r \neq 0$, then, for some $t_1 > 0$,

$$f_{t_1}(z) \in G^{-1}(0) \cap \{x_1 = 0\}$$

and, according to the mirror image property,

$$f_{2t_1}(z) = z + 2e_1.$$

[10] $G(x, r_1, 0) = 0$ is equivalent to $r_1\big(X_1(x) + r_1^2\big) = 0$ where $X_1(x) \geq 0$.

[11] Use the compactness of $G^{-1}(0)$.

[12] Do not be confused with G_ε for $\varepsilon = 1$.

[13] The h-cobordism theorem is involved for having an extension G_2 without critical points.

[14] One should also consider the mirror construction of the Hamiltonian as in 3.1. Let G_1 still denote the Hamiltonian of $\widehat{X}_{G_1}$ on $\{x_1 \in [-1,1]\}$; the corresponding hypersurface $G_1^{-1}(0)$ lies in B_c.

[15] Here K is the compact invariant set used in the construction of the Wilson plug which we started with.

On the contrary, [*we have the trapping property*]: if $\hat{x} \in K$, the orbit of $(-1, \hat{x}, 0)$ does not exist at [*i.e. does not reach*] $\{x_1 = 1\}$.

5.4. *End of the proof of the main theorem when* $2p \geq 8$.

Let $P_1, \dots, P_k$ be the periodic orbits of N_0 (notations from the beginning). Let $x_j \in P_j$; we suppose $\varphi_j : (V_j, x_j) \to (\mathbb{R}^{2p}, 0)$ is a symplectic flow box for X_{H_0} so that $H_0 \circ \varphi_j^{-1} = r_1$; let $W_j \subset \mathbb{R}^{2p}$ be its image. [*For* $\delta > 0$, *we introduce the rescaled Hamiltonian*[16] G_δ *defined by the following formula:*

$$G_\delta(x, r) = \delta\, G_1(\frac{x}{\delta}, \frac{r}{\delta})\,.$$

We have $G_\delta(x, r) = r_1$ *outside some compact set. When* δ *is small enough, the support of* $G_\delta - r_1$ *is contained in* W_j *for every* j. *The dynamics of the hypersurface* $G_\delta^{-1}(0)$ *has properties similar to those of* $G_1^{-1}(0)$.] In each V_j, we replace H_0 by $G_\delta \circ \varphi_j$ and $N_0 \cap V_j$ by $\varphi_j^{-1}\big(G_\delta^{-1}(0)\big)$; the hypersurface N is the result of this change. According to the trapping property of $G_\delta^{-1}(0)$, the orbits P_j are destroyed. Since $G_\delta^{-1}(0)$ carries no periodic characteristic line, and due to its mirror image property, no periodic characteristic line have been created on N. □

6. The case $2p = 6$

6.1. We will use P. Schweitzer's examples [ ]. We consider $S = \mathbb{T}^2 \setminus \Delta$, where Δ is diffeomorphic to the open 2-disk, $\Delta \cong \{z \in \mathbb{C} \mid |z| < 1\}$. Let Z be a vector field on $\mathbb{T}^2$, suspension of some diffeomorphism h of the circle with rotation number $\rho(h) = \alpha \in \mathbb{R}_{/\mathbb{Z}} \setminus \mathbb{Q}_{/\mathbb{Z}}$, that is a Denjoy counterexample. By [ ], h and Z can be chosen of class $C^{2-\epsilon}$ for every $\epsilon > 0$. We will suppose $\overline{\Delta} \subset \mathbb{T}^2 \setminus K$, where K is the exceptional minimal set of the flow of Z.

On $[-1, +1] \times S$ we consider the vector field $X = (X_1, X_2, X_3)$ given by formula (4.5)[17]. Let us notice that the first component X_1 of X is of class C^∞. On $T^*([-1, +1] \times S)$, equipped with coordinates $(x_1, x_2, x_3, r_1, r_2, r_3)$, we introduce the same Hamiltonian G as before (see 5.1) and the hypersurface $\Sigma = G^{-1}(0)$ in

$$B_c = \{(x, r) \mid x \in [-1, +1] \times S, |r_1| \leq c, r_2^2 + r_3^2 \leq 1\}.$$

The dynamics of Σ has no periodic orbits and meets the mirror and trapping properties: the orbit of $(-1, \hat{x}, 0, 0, 0)$ is trapped when $\hat{x} \in K$.

In the formula of G, $X_1(x) r_1$ is of class C^∞ and we do not change it. The terms $X_j(x) r_j$, $j = 2, 3$, are only $C^{2-\varepsilon}$. The only properties we need for $H_j(x, r_j) = X_j(x) r_j$, when $x \in [-1, 0] \times S$ and $j = 2, 3$, are the following:

[16] The rescaling of the Hamiltonian is not explicitly mentioned in the original text; some more complicated argument is used.

[17] We refer to the canonical coordinates (x_2, x_3) on $\mathbb{T}^2$: $(X_2, X_3) = \varphi(\hat{x})\lambda(x_1) Z$.

$$(6.2)\qquad \begin{cases} H_j(x,0)=0 \\ \\ \dfrac{\partial H_j}{\partial r_j}(x,0)=X_j(x) \\ \\ H_j(x,r_j)=0 \quad \text{near } \{x=-1\}\cup\{x=0\} \end{cases}$$

For $x \in [0,+1]\times S$ we take $H_j(x_1,x_2,x_3,r) = -H_j(-x_1,x_2,x_3,r)$. According to the next proposition, there exists a function of class $C^{3-\varepsilon}$ having the first two properties. [The third one is a consequence of them since $X_j(x)=0$ in the considered neighbourhood.]

6.3. **Proposition.** *Let $X:\mathbb{T}^n\to\mathbb{R}$, $n\geq 1$, be a C^k function, $k\in\mathbb{R}_+$. Then there exists a function*

$$(x,r)\in\mathbb{T}^n\times\mathbb{R}\mapsto H(x,r)\in\mathbb{R}$$

of class C^{k+1} such that $H(x,0)=0$ and $\dfrac{\partial H}{\partial r}(x,0)=X(x)$.

Proof. We also denote $X:\mathbb{R}^n\to\mathbb{R}$ the periodic function associated with X. Let $\varphi:\mathbb{R}^n\to\mathbb{R}$ be a C^∞ function with compact support and whose integral equals 1. We define

$$H(x,r)=r\int_{\mathbb{R}^n}\varphi(y)X(x-ry)dy\,.$$

This periodic function has the required properties (see Theorem 1.3.3 in Lars Hörmander, *The Analysis of Linear Partial Differential Operators, I*, second edition, Springer-Verlag, 1990)[18].□

6.4. *End of the proof of the main theorem when $2p=6$.*[19]

We consider a C^∞ embedding

$$f_0: S=\mathbb{T}^2\setminus\Delta\to\{(y_1,y_2,y_3)\in\mathbb{R}^3 \mid -1<y_1<+1,\ y_2^2+y_3^2<\frac{1}{2}\}$$

transversal to $\dfrac{\partial}{\partial y_1}$ (its projection onto $y_1=0$ is an immersion). We enlarge it as an embedding $f:[-\delta,+\delta]\times S\to\mathbb{R}^3$ by the formula

$$f(x_1,\hat{x})=f_0(\hat{x})+x_1\frac{\partial}{\partial y_1}\,.$$

It is an embedding if δ is small enough and we have

$$f_*\left(\frac{\partial}{\partial x_1}\right)=\frac{\partial}{\partial y_1}\,.$$

By using a Riemannian metric on $\mathbb{R}^3$ (which allows us to identify tangent and cotangent spaces), f extends as a symplectic embedding[20]

$$F:T^*([-\delta,+\delta]\times S)\to T^*(\mathbb{R}^3)\,.$$

[18]M.H. wrote a complete proof; we omit it since J.-C. Yoccoz found this available reference.

[19]This subsection was partly missing in the original text.

[20]This embedding is known as “Chekanov’s trick”.

Let $(y_1, y_2, y_3, \rho_1, \rho_2, \rho_3)$ be the canonical coordinates on $T^*(\mathbb{R}^3)$.

Lemma. *We have:* $F^*\rho_1 = r_1$.

Proof. Let us only prove that the zero sets of both terms are the same. On the righthand side we have the covectors vanishing on $\dfrac{\partial}{\partial x_1}$, that is, up to the Riemannian isomorphism, the set of tangent vectors orthogonal to $\dfrac{\partial}{\partial x_1}$ with respect to the metric which is induced on the image of f. Since $F_*\left(\dfrac{\partial}{\partial x_1}\right) = \dfrac{\partial}{\partial y_1}$, this set of vectors is the orthogonal space to $\dfrac{\partial}{\partial y_1}$. □

As a consequence, the hypersurface Σ, suitably rescaled as in 5.4 so that it takes place in a small tube around the zero section in $T^*([-\delta, +\delta] \times S)$, has an image by F that we can extend by $\{\rho_1 = 0\}$ outside the image of F. Let $\widehat{\Sigma}$ denote this hypersurface in $T^*([-1, +1] \times \mathbb{R}^2)$. Arguing as in 5.2, we are done after the following lemma.

Lemma.
1) *The dynamics of* $\widehat{\Sigma}$ *meets the mirror image property.*
2) *If* π *denotes the projection* $(y_1, \hat{y}) \mapsto \hat{y}$*, then the orbit of* $(-1, \hat{y})$ *is trapped when* $\hat{y} \in \pi \circ f_0(K)$.

Proof. Look at a characteristic line λ from $(-1, \hat{y}, 0, \hat{r})$. If it enters the image of F, either it is trapped or it gets out with the same $(\hat{y}, \hat{r})$ coordinates. Indeed Σ meets the mirror image property and F maps $\dfrac{\partial}{\partial x_1}$ segments to $\dfrac{\partial}{\partial y_1}$ segments. If we continue to traverse λ, we may again enter the image of F since $(\hat{y}, \hat{r}) + \mathbb{R}\dfrac{\partial}{\partial y_1}$ may cross the image of F along two intervals (when $\pi \circ f_0$ is the standard immersion of the punctured 2-torus into the plane). But we get out still with the same $(\hat{y}, \hat{r})$ coordinates. □

7. Volume preserving plugs

[*The main result of this section is stated in proposition 7.3.*]

7.1. We can embed $\mathbb{T}^{n-2}$ in $\mathbb{R}^{n-1}$. As its normal bundle is trivial we can embed $[-1, +1]^2 \times \mathbb{T}^{n-2}$ into $\mathbb{R}^n$.

7.2. On $[-1, +1] \times [-1, +1] \times \mathbb{T}^{n-2}$ we put the coordinates (x, r, θ). Let $u(x)$ be an even C^∞ function, $x \in [-1, +1]$, $0 \leq u(x) \leq 1$, equal to 1 near $\{-1, 0, +1\}$ and such that $u^{-1}(0) = \left\{-\frac{1}{2}, \frac{1}{2}\right\}$. Let $\alpha(r)$ be an even C^∞ function, $r \in \mathbb{R}$, equal to 1 for $|r| \leq \frac{1}{4}$ and to 1 when $|r| \geq \frac{1}{2}$. If $b > 0$ is large enough

$$r \in [-1, +1] \xrightarrow{\;g\;} \left[\frac{\alpha(r) r}{b} + r^3\right] \in [-1, +1]$$

is a C^∞ diffeomorphism. We consider

$$H(x,r) = g^{-1} \circ \left[\frac{u(x)\alpha(r)r}{b} + r^3 \right].$$

[*We have* $H(x,r) = r$ *near the boundary of* $[-1,+1]^2$.]

Let $X_H = \left(\frac{\partial H}{\partial r}, -\frac{\partial H}{\partial x} \right)$. We have

$$X_H(-x,r) = \left(\frac{\partial H}{\partial r}(x,r), \frac{\partial H}{\partial x}(x,r) \right)$$

and $X_H(x,r) = (1,0)$ when (x,r) is close to $\partial([-1,+1]^2)$. We also have

$$\frac{\partial H}{\partial r}(x,r) = \left(\frac{u(x)}{b}\big(\alpha(r) + \alpha'(r)r\big) + 3r^2 \right).(g^{-1})'(z)$$

with $z = \frac{u(x)\alpha(r)r}{b} + r^3$. We suppose that b is large enough in order that $3r^2 - \frac{1}{b}|\alpha'(r)r + \alpha(r)| \geq \varepsilon > 0$ when $|r| \geq \frac{1}{4}$. Hence $\frac{\partial H}{\partial r}(x,r) \geq 0$ and $\frac{\partial H}{\partial r}(x,r) = 0$ if and only if $(x,r) \in \{\pm\frac{1}{2}\} \times \{0\}$. Finally

$$X_H(x,r) = 0 \Longleftrightarrow (x,r) \in \{\pm\frac{1}{2}\} \times \{0\}.$$

We consider C^∞ functions:

$-1 \leq \lambda_1(x) \leq 1,\ -\lambda_1(-x) = \lambda_1(x),\ \lambda_1^{-1}(1) = \{\frac{1}{2}\}$,
$-0 \leq \lambda_2(r) \leq 1,\ \lambda_2(0) = 1,\ \lambda_2(-r) = \lambda_2(r)$,
$\lambda_1(x) = 0$ when x is close to $\{-1,0,1\}$,
$\lambda_2(r) = 0$ when r is close to $\{-1,1\}$.

When $n \geq 4$, we define a *volume preserving plug* by

$$X(x,r,\theta) = \left(\frac{\partial H}{\partial r}(x,r), -\frac{\partial H}{\partial x}(x,r), \lambda_1(x)\lambda_2(r)(\alpha_1,\ldots,\alpha_{n-2}) \right)$$

where $(\alpha_1,\ldots,\alpha_{n-2})$ are rationally independent. We have $X(x,r,\theta) = (1,0,\ldots,0)$ when (x,r,θ) is close to $\partial(I^2 \times \mathbb{T}^{n-2})$. Let $(X_1,\ldots,X_n)$ be the coordinates of X. We have the *mirror image property*:

$$X(-x,r,\theta) = (X_1(z), -X_2(z), \ldots, -X_n(z))$$

where $z = (x,r,\theta)$. Moreover $X_1(z) = 0$ if and only if $z \in K_\pm = \left\{ (\pm\frac{1}{2}, 0) \right\} \times \mathbb{T}^{n-2}$ and X is tangent to $\left\{ (\pm\frac{1}{2}, 0) \right\} \times \mathbb{T}^{n-2}$. On $\left\{ (\pm\frac{1}{2}, 0) \right\} \times \mathbb{T}^{n-2}$, $X = (0,0,\alpha_1,\ldots,\alpha_{n-2})$. Hence X has no periodic orbit. We also have the fact that every orbit starting from $(-1,0,\theta)$ spirals to K_-.

For $c > 0$, consider the volume form $\widehat{\Omega}_c = c\, dx \wedge dr \wedge d\theta_1 \wedge \ldots \wedge d\theta_{n-2}$. We have $\mathrm{div}_{\widehat{\Omega}_c} X = 0$. In order to construct a volume preserving plug it is enough to prove the following lemma.

Lemma. *Let Ω be the Euclidian volume form of $\mathbb{R}^{n-1}$. There exists a C^∞ embedding $\varphi : (r,\theta) \in [-1,1] \times \mathbb{T}^{n-2} \mapsto \varphi(r,\theta) \in \mathbb{R}^{n-1}$ such that*

$$\varphi^*\Omega = c\, dr \wedge d\theta_1 \wedge \ldots \wedge d\theta_{n-2}\,.$$

Proof.[21] [*Let $\varphi_0 : [-1,1] \times \mathbb{T}^{n-2} \to \mathbb{R}^{n-1}$ be a smooth embedding. Let B be a ball in $\mathbb{R}^{n-1}$ enclosing the image of φ_0 whose Euclidian volume is larger than the volume of $[-1,1] \times \mathbb{T}^{n-2}$ with respect to $c\, dr \wedge d\theta_1 \wedge \ldots \wedge d\theta_{n-2}$. It is easy to construct a volume forme Ω' on B such that:*

1) Ω' and Ω coincide near the boundary of B;
2) $\varphi_0^*\Omega' = c\, dr \wedge d\theta_1 \wedge \ldots \wedge d\theta_{n-2}$;
3) $\mathrm{vol}_{\Omega'}(B) = \mathrm{vol}_\Omega(B)$.

By Moser's theorem on volume forms, there is an isotopy $\{\psi_t\}_{t\in[0,1]}$ on $\mathbb{R}^{n-1}$ with compact support in the interior of B such that $\psi_0 = Id$ and $\psi_1^*\Omega = \Omega'$. Then $\varphi = \psi_1 \circ \varphi_0$ meets the wanted property.] □

7.3. **Proposition.** *Let (M^n,Ω), $n \geq 4$, be a C^∞ manifold endowed with a volume form. Let X be an Ω-volume preserving non-singular vector field. We assume that the periodic orbits of X are included in a finite number of codimension-one compact submanifolds (possibly with boundaries and not mutually disjoint).* [Then there exists an Ω-volume preserving non-singular vector field X' without periodic orbit.]

Proof. We work in a flow box $[-1,1]\times B_\varepsilon^{n-1} = B$, where $B_\varepsilon^{n-1} = \{\hat{x} = (x_1, ..., x_{n-1}) \mid \|\hat{x}\| \leq \varepsilon\}$. We consider a periodic orbit P contained in the compact hypersurface V^{n-1}. Then, in the flow box, V intersects the manifold $\{-1\} \times B_\varepsilon^{n-1}$ transversally since $P \cap B = [-1,1] \times \{0\}$. As the problem is local, we may suppose that $V_1 = V \cap (\{-1\} \times B_\varepsilon^{n-1})$ is the graph of a local C^∞ function ψ over $\{j = 0\}$ in $\mathbb{R}^{n-1}$ [*for some j*]. We denote $\hat{z} = (x_2, ..., \hat{x}_j, ...x_n)$. As the transformation

$$(x_j, \hat{z}) \in\ \mathbb{R}^{n-1} \mapsto \big(x_j + \psi(\hat{z}), \hat{z}\big)$$

is volume preserving, we may suppose that $V_1 \subset \{(x_j, \hat{z}) \in\ \mathbb{R}^{n-1} \mid x_j = 0\}$.

[*Let us consider a plug $\varphi : [-1,1] \times \mathbb{T}^{n-2} \to B$ as in the previous lemma.*] We may also suppose that locally, after some volume preserving diffeomorphism of $\mathbb{R}^{n-1}$ [*moving φ*],

$$\varphi^{-1}(V_1) \subset \{(r,\theta) \in [-1,1] \times \mathbb{T}^{n-2} \mid r = 0\}\,.$$

Using this plug, all the periodic orbits passing through V_1 can be destroyed. If P is contained in other local hypersurfaces of periodic orbits, $V_2, ..., V_q$, we treat each V_j separately. □

[21]This proof was missing.

Appendix

References were missing in the original manuscript. We add below the copy of an email from Viktor Ginzburg to F. Laudenbach (19 Nov. 2006) with a list of further references.

Dear Francois,

Here is a brief report concerning the Seifert conjecture (mainly, Hamiltonian) and, in particular, what happened since 1994, roughly in chronological order.

*** My original counterexample (smooth, $2n \geq 8$) is published in:
V. Ginzburg, An embedding $S^{2n-1} \to \mathbb{R}^{2n}$, $2n-1 > 7$, whose Hamiltonian flow has no periodic trajectories, IMRN, 1995, no. 2, 83-98.

And Michel has a short note stating his result:
M.-R. Herman, Examples of compact hypersurfaces in $\mathbb{R}^{2p}$, $2p \geq 6$, with no periodic orbits, in *Hamiltonian systems with three or more degrees of freedom*, C. Simo (Editor), NATO Adv. Sci. Inst. Ser. C, Math. Phys. Sci., vol. 533, Kluwer Acad. Publ., Dordrecht, 1999.

*** The 1995 counterexample is followed by a smooth counterexample in dimension 6:
V. Ginzburg, A smooth counterexample to the Hamiltonian Seifert conjecture in R^6, math.DG/9703106. IMRN, 1997, no. 13, 642-650.
Both are based on the horocycle flow and the only thing that was missing in the 1995 paper is an embedding of the horocycle flow into $\mathbb{R}^5$. (In $\mathbb{R}^5$, it does not follow from Gromov's general results.) Such an embedding is constructed in the 1997 paper. (The proof there has a minor and fixable gap. A complete proof is given on pp. 118-121 in "Introduction to the h-principle" by Eliashberg and Mishachev, AMS, 2002.)

*** In 1997, you published your paper in Ann. Fac. Sci. Toulouse Math.[22].

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*** Meanwhile, in 1996, G. Kuperberg constructs a C^1-smooth (perhaps even a bit smoother) volume-preserving counterexample for S^3, based on Schweitzer's construction.
G. Kuperberg, A volume-preserving counterexample to the Seifert conjecture, *Comment. Math. Helv.*, **71** (1996), 70–97.

*** And also a paper:
G. Kuperberg, K. Kuperberg, Generalized counterexamples to the Seifert conjecture, *Ann. Math.*, **144** (1996), 239–268.
appears. (I don't quite remember their results.) I am not aware of any results on the general and volume-preserving Seifert conjecture proved after that.

[22] F. Laudenbach, Trois constructions en topologie symplectique, Ann. Fac. Sc. Toulouse, Vol. VI n^o 4 (1997), 697 - 709.

*** K. Kuperberg has two surveys where she summed up what had been established by the end of the 90s:
K. Kuperberg, Counterexamples to the Seifert conjecture, in *Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998).* Doc. Math. (1998) Extra Vol. II, 831–840.
K. Kuperberg, Aperiodic dynamical systems, *Notices Amer. Math. Soc.*, **46** (1999), 1035–1040.

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*** Back to the Hamiltonian case. Nothing much happened for some four years either. I wrote two surveys (sort of complementary to K. Kuperberg's in terms of emphasis). They cover the results prior to 1998:
V. Ginzburg, Hamiltonian dynamical systems without periodic orbits, math.DG/9811014. In Northern California Symplectic Geometry Seminar; Ed.: Y. Eliashberg et al.; Amer. Math. Soc. Transl., (2) 196 (1999), 35-48.
V. Ginzburg, The Hamiltonian Seifert conjecture: examples and open problems, math.DG/0004020. In Proceedings of the Third European Congress of Mathematics, Barcelona, 2000; Birkhauser, Progress in Mathematics, 202 (2001), vol. II, pp. 547-555.

*** Then, in 2002, Ely Kerman constructs a smooth Hamiltonian counterexample in all dimensions $2n \geq 6$ with dynamics and plugs different from mine. His construction is simpler than mine and in some sense may be closer to what Michel had in mind.
E. Kerman, New smooth counterexamples to the Hamiltonian Seifert conjecture, *J. Symplectic Geometry*, **1** (2002), 253–267.

*** Finally in 2002 and 2003, Basak Gurel and I construct a Hamiltonian counterexample in $\mathbb{R}^4$ with Hamiltonian being C^2-smooth or even a bit smoother. (Hence, the vector field is C^1 as in Kuperberg's example. The construction is also based in Schweitzer's idea.)
V. Ginzburg, B. Gurel, On the construction of a C^2-counterexample to the Hamiltonian Seifert conjecture in R^4, math.DG/0109153, Electron. Res. Announc. Amer. Math. Soc, 8 (2002), 11-19.
V. Ginzburg, B. Gurel, A C^2-smooth counterexample to the Hamiltonian Seifert conjecture in R^2, math.DG/0110047, Ann. of Math. 158 (2003), 953-976.

*** We also showed (as a more or less side thing) that aperiodic levels can accumulate to a critical level or critical points and some other (expected) results along those lines:
V. Ginzburg, B. Gurel, Relative Hofer-Zehnder capacity and periodic orbits in twisted cotangent bundles, math.DG/0301073, Duke Mathematical J. 123 (2004), 1-47.

*** Since then no real progress has been made as far as I know. The smooth Hamiltonian Seifert conjecture in R^4 is still wide open. It's not known whether the set of aperiodic energy values for a Hamiltonian on $\mathbb{R}^{2n}$ can be dense.

Best regards,
Viktor