Non existence of Lagrangian graphs *

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We will consider C^1 exact symplectic diffeomorphisms of $T^*(\mathbb{T}^n)$ of the following form

$$F_\varphi(\theta, r) = (\theta + Lr, r + d\varphi(\theta + Lr)),$$

where L is a symmetric positive definite matrix of order n, where $\varphi \in C^2(\mathbb{T}^n, \mathbb{R})$ and

$$d\varphi = (\frac{\partial\varphi}{\partial\theta_1}, \ldots, \frac{\partial\varphi}{\partial\theta_n}).$$

If the graph of $\psi \in C^1(\mathbb{T}^n, \mathbb{R}^n)$ is a Lagrangian invariant torus of F_φ, then

$$\psi \circ f - \psi = d\varphi \circ f$$

with $f = \mathrm{Id} + L\psi$. This is equivalent to

$$\frac{1}{2}\left(f + f^{-1}\right) = \mathrm{Id} + \frac{1}{2} L\, d\varphi. \tag{1}$$

The graph of $\psi = (\psi_1, \ldots, \psi_n)$ being Lagrangian is equivalent to the fact that the 1-form $\sum \psi_i\, d\theta_i$ of $\mathbb{T}^n$ is closed. Let $L^{\frac{1}{2}}$ be the positive square root of L. We differentiate (1)

$$\frac{1}{2}\left(Df(\theta) + (Df)^{-1}(f^{-1}(\theta)\right) = I + \frac{1}{2} L\, E(\theta)$$

where $E(\theta)$ is the derivate matrix of $d\varphi$ (i.e. $Dd\varphi$). We write $G = L^{-\frac{1}{2}}\, Df\, L^{\frac{1}{2}}$. It is symmetric positive definite and we obtain

$$\frac{1}{2}\left(G + G^{-1} \circ f^{-1}\right) = I + \frac{1}{2} L^{\frac{1}{2}}\, E\, L^{\frac{1}{2}}. \tag{2}$$

Thus we have the following necessary condition

$$\text{the matrix } B_1(\theta) = I + \frac{1}{2} L^{\frac{1}{2}}\, E(\theta)\, L^{\frac{1}{2}} \text{ is positive definite.} \tag{3}$$

If φ is non constant then there exists $t_\varphi > 0$ such that if $t \in \mathbb{R}$, $|t| \geq t_\varphi$, (3) is violated for $F_{t\varphi}$.

The condition (3) is not optimal. We have

$$\begin{aligned} \frac{1}{2n}\mathrm{tr}(G) + \frac{1}{2n}\mathrm{tr}(G^{-1} \circ f^{-1}) &= 1 + \frac{1}{2n}\mathrm{tr}(L^{\frac{1}{2}}\, E\, L^{\frac{1}{2}}) \\ &= 1 + \frac{1}{2} e \end{aligned} \tag{4}$$

*Typing and minor corrections are due to P. Le Calvez where

$$e = \frac{1}{n}\mathrm{tr}(L^{\frac{1}{2}}\, E\, L^{\frac{1}{2}}) = \frac{1}{n}\mathrm{tr}(L\, E).$$

Let

$$M = \max\left(\max_{\theta} \frac{1}{n}\mathrm{tr}(G(\theta)),\ \max_{\theta} \frac{1}{n}\mathrm{tr}(G^{-1}(\theta))\right).$$

We have

$$m = \min\left(\min_{\theta} \frac{1}{n}\mathrm{tr}(G(\theta)),\ \min_{\theta} \frac{1}{n}\mathrm{tr}(G^{-1}(\theta))\right) \geq \frac{1}{M} \tag{5}$$

since, by Cauchy-Schwarz inequality, we have

$$\frac{1}{n}\mathrm{tr}(G(\theta))\,\frac{1}{n}\mathrm{tr}(G^{-1}(\theta)) \geq 1.$$

Let

$$e_- = -\min_{\theta} e(\theta),$$
$$e_+ = \max_{\theta} e(\theta).$$

By (4) and (5) we have

$$\frac{1}{M} \leq 1 - \frac{1}{2}e_-.$$

If $M = \frac{1}{n}\mathrm{tr}(G(\theta_0))$ then by (4)

$$\frac{1}{2}\left(M + \frac{1}{n}\mathrm{tr}(G^{-1}(f^{-1}(\theta_0)))\right) = 1 + \frac{1}{2}e(\theta_0) \leq 1 + \frac{1}{2}e_+$$

and therefore, since $\frac{1}{n}\mathrm{tr}(G^{-1}(f^{-1}(\theta_0))) \geq \frac{1}{M}$,

$$\frac{1}{2}\left(M + \frac{1}{M}\right) \leq 1 + \frac{1}{2}e_+. \tag{6}$$

If $M = \frac{1}{n}\mathrm{tr}(G^{-1}(f^{-1}(\theta_0)))$ for some θ_0 the same gives also (6). This condition implies

$$\frac{1}{1 - \frac{1}{2}e_-} \leq M \leq 1 + \frac{1}{2}e_+ + \left(e_+ + \frac{1}{4}e_+^2\right)^{\frac{1}{2}}. \tag{7}$$

Consequences. We suppose that the function φ is not constant. Then if $t > t_\varphi$, the function $F_{t\varphi}$ has no invariant Lagrangian torus that is a graph of a C^1 function of $C^1(\mathbb{T}^n, \mathbb{R}^n)$. If $e_+ \to 0$ and $e_- > 2\sqrt{e_+}$ then F_φ has no invariant Lagrangian torus that is a graph of a C^1 function in $C^1(\mathbb{T}^n, \mathbb{R}^n)$.

Remark. The condition (7) is optimal for $n = 1$, see [He]. If $e_- = e_+ = 1$ (for φ) we obtain $t_\varphi = \frac{4}{3}$ in the case $n = 1$ (see [M]).

Theorem : *We take $L = I$. There exists a sequence of C^∞ functions $(\varphi)_{i\in\mathbb{N}}$ converging to* 0 *in the $C^{n+2-\varepsilon}$ topology (for every $\varepsilon > 0$) such that F_{φ_i} leaves invariant no Lagrangian torus that is the graph of a C^1 function.*

Proof. It is enough to construct the sequence (φ_i) that violates (7). We have $e(\theta) = \frac{1}{n}\triangle\varphi$ with $\triangle = \sum_i \frac{\partial^2}{\partial\theta_i^2}$ the Laplacian. Let $\eta_1 \geq 0$ be a C^∞ function of $\mathbb{R}$ with support contained in $[-\frac{1}{4}, \frac{1}{4}]$ and such that

$$\begin{cases} \eta_1(x) = 1 & \text{if } -\frac{1}{8} \leq x \leq \frac{1}{8}, \\ \eta_1(-x) = \eta_1(x). \end{cases}$$

Let define $\eta(x) = \eta_1(\|x\|)$ for $x \in \mathbb{R}^n$. Let $\delta > 0$ be small and $x_0 = \left(\frac{1}{2}, 0, \dots, 0\right)$. We define the function of $\mathbb{R}^n$:

$$\begin{cases} \overline{e}_\delta(x) = \delta\eta(x) & \text{if } \|x\| \leq \frac{1}{4}, \\ \overline{e}_\delta(x) = -4\delta^{\frac{1}{2}}\eta\left((x - x_0)(4\delta^{\frac{1}{2}})^{-\frac{1}{n}}\right) & \text{if } \|x - x_0\| \leq \frac{1}{4}\left(4\delta^{\frac{1}{2}}\right)^{\frac{1}{n}}, \\ \overline{e}_\delta(x) = 0 & \text{otherwise.} \end{cases}$$

We extend these functions $\mathbb{Z}^n$ periodically to functions $e_\delta \in C^\infty(\mathbb{T}^n)$ such that $\int_{\mathbb{T}^n} e_\delta(\theta)\, d\theta = 0$. If $\delta \to 0$, the family (e_δ) is bounded in the C^n topology and from interpolation, $e_\delta \to 0$ in the $C^{n-\varepsilon}$ topology for any $\varepsilon > 0$.

Let φ_δ be the unique function in $C^\infty(\mathbb{T}^n, \mathbb{R})$ such that

$$\int_{\mathbb{T}^n} \varphi_\delta\, d\theta = 0, \;\; \frac{1}{n}\triangle\varphi_\delta = e_\delta.$$

By Schauder estimates one knows that for any $\varepsilon > 0$ given, $\varphi_\delta \to 0$ in the $C^{n+2-\varepsilon}$ topology, when $\delta \to 0$. For a proof, see [Ho]. The proof shows that the functions φ_δ are bounded in the space of functions that are of class $C^{n+2-\varepsilon}$ and the partial derivatives of order $\leq n-1$ are smooth in the Zygmund sense. When $\delta \to 0$, e_δ does not satisfy (7) and the theorem follows.□

Important remark. The functions e_δ constructed above are such that all the partial derivatives up to order $2n$ are bounded in $L^1(\mathbb{T}^n, d\theta)$: e_δ is bounded in $W^{2n,1}(\mathbb{T}^n)$. This implies that the functions φ_δ are bounded in $W^{2n+2*,1}(\mathbb{T}^n)$ (i.e. the set of functions such that all the partial derivatives up to order $2n+1$ are Zygmund smooth in the L^1 sense, see [Ho]). By interpolation this implies that for any $\varepsilon > 0$, $\varphi_\delta \to 0$ in the Banach space $W^{2n+2-\varepsilon,1}(\mathbb{T}^n)$.

For $F_\varphi(\theta, r) = (\theta + Lr, r + d\varphi(\theta + Lr))$ we want to indicate some other a priori inequalities. Let

$$\lambda_1 = \max_\theta \lambda_1(B_1(\theta)),$$
$$\lambda_n = \min_\theta \lambda_n(B_1(\theta)),$$

where $\lambda_1(B_1(\theta)) \geq \cdots \geq \lambda_n(B_1(\theta))$ are the eigenvalues of $B_1(\theta)$, and

$$1 \leq m = \max\left(\max_\theta \lambda_1(G(\theta)),\, \max_\theta \lambda_1(G^{-1}(\theta))\right).$$

We have

$$\min\left(\min_\theta \lambda_n(G(\theta)),\, \min_\theta \lambda_n(G^{-1}(\theta))\right) \geq \frac{1}{m}.$$

By (2) we have

$$\frac{1}{m} \leq \frac{1}{2}\left(\lambda_n(G(\theta)) + \lambda_n(G^{-1}(f^{-1}(\theta)))\right) \leq \lambda_n(B_1(\theta))$$

hence

$$\frac{1}{m} \leq \lambda_n.$$

By the same argument as in the proof of (6) we have

$$\frac{1}{2}\left(m + \frac{1}{m}\right) \leq \lambda_1$$

or

$$m \leq \lambda_1 + \left(\lambda_1^2 - 1\right)^{\frac{1}{2}},$$

hence we have the a priori estimate

$$\lambda_n^{-1} \leq \lambda_1 + (\lambda_1^2 - 1)^{\frac{1}{2}}. \tag{8}$$

Consequence. Let φ be a non constant function, then we find $t_0 > 0$ such that, if $t_0 < t < t_0 = \varepsilon$, for some $\varepsilon > 0$, then $F_{t\varphi}$ has no invariant torus that is the graph of a C^1 function, but the symmetric matrix $B_{t_0} = I + \dfrac{t_0}{2} L^{-\frac{1}{2}}\, E(\theta)\, L^{-\frac{1}{2}}$ is positive definite for every θ (let $t_1 > 0$ be such that $\det\left(B_{t_1}(\theta_0)\right) = 0$ for some θ_0 and $\det(B_t) \neq 0$ if $0 \leq t < t_1$, as $t \to t_1$, λ_1 is bounded but $\lambda_n^{-1} \to \infty$, this violates (8)).

Other inequalities.

Let $v \in \mathbb{R}^n$, $\|v\| = 1$. Let

$$\begin{aligned}
\lambda_1(v) &= \max_\theta \langle B_1(\theta)v, v\rangle \leq \lambda_1, \\
\lambda_n(v) &= \min_\theta \langle B_1(\theta)v, v\rangle \geq \lambda_n, \\
m_v &= \max\left(\max_\theta \langle G(\theta)v, v\rangle,\, \max_\theta \langle G^{-1}(\theta)v, v\rangle\right),
\end{aligned}$$

hence

$$\min\left(\min_\theta \langle G(\theta)v, v\rangle,\, \min_\theta \langle G^{-1}(\theta)v, v\rangle\right) \geq m_v^{-1}$$

because we have

$$\langle G(\theta)v, v\rangle\, \langle G^{-1}(\theta)v, v\rangle \geq 1,$$

(this is a consequence of Cauchy-Schwarz inequality :

$$|\langle G(\theta)u, v\rangle|^2 \leq \langle G(\theta)u, u\rangle\, \langle G(\theta)v, v\rangle,$$

true for every u and v in $\mathbb{R}^n$ and choosing $u = G^{-1}(\theta)v$).

We have by (2)

$$\frac{1}{2}\left(\langle G(\theta)v, v\rangle + \langle G^{-1} \circ f^{-1}(\theta)v, v\rangle\right) = \langle B_1(\theta)v, v\rangle$$

and it follows that

$$m_v^{-1} \leq \lambda_n(v)$$
$$\frac{1}{2}\left(m_v + m_v^{-1}\right) \leq \lambda_1(v)$$

which implies

$$\lambda_n(v)^{-1} \leq \lambda_1(v) + (\lambda_1(v)^2 - 1)^{\frac{1}{2}}.$$

We can write the above inequalities in the following more condensed form. For $v \in \mathbb{R}^n$, we consider the functions

$$B_+(v) = \max_\theta \langle B_1(\theta)v, v\rangle,$$
$$B_-(v) = \min_\theta \langle B_1(\theta)v, v\rangle,$$
$$M(v) = \max\left(\max_\theta \langle G(\theta)v, v\rangle, \max_\theta \langle G^{-1}(\theta)v, v\rangle\right).$$

We suppose that $B_1(\theta) > 0$ for every θ, hence the functions B_+, B_- and M are strictly positive on the unite sphere $S_{n-1} = \{v \in \mathbb{R}^n \; \langle v, v\rangle = 1\}$. The functions are homogenous of degree 2 (i.e. $B_+(kv) = k^2 B_+(v)$, $k \in \mathbb{R}$, etc) and $v \mapsto (B_+(v))^{\frac{1}{2}}$ and $v \mapsto (M(v))^{\frac{1}{2}}$ are norms on $\mathbb{R}^n$.

The inequalities obtained give on S_{n-1}

$$\frac{1}{2}\left(M + \frac{1}{M}\right) \leq B_+ \tag{9}$$

and

$$\frac{1}{M} \leq B_- \Leftrightarrow B_-^{-1} \leq M. \tag{10}$$

The inequalities (9) and (10) imply $M \geq 1$ and $B_n \geq 1$ on S_{n-1}. We obtain on S_{n-1}

$$M \leq B_+ + (B_+^2 - I)^{\frac{1}{2}}$$

hence the a priori inequality on S_{n-1}

$$B_-^{-1} \leq B_+ + (B_+^2 - I)^{\frac{1}{2}}.$$

Remark. If $P \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ is an orthogonal projection then we have

$$P\,G(\theta)\,{}^tP + P\,G^{-1} \circ f^{-1}(\theta)\,{}^tP = P\,B_1(\theta)\,{}^tP$$

for every $\theta \in \mathbb{T}^n$. On ImP we have similar inequalities as above but the ones associated to traces.

Bibliographie

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