Cours au Collège de France - Février 2016

Towards the ultimate precision limits in parameter estimation: An introduction to quantum metrology

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Instituto de Física - Universidade Federal do Rio de Janeiro

Dates of Lectures

Lecture 1 - February 4th
Lecture 2 - February 11th $\Bigg\}$ Thursday
Lecture 3 - February 18th

Lecture 4 - February 29th - Monday

All lectures will be at 11:00, salle 2 du Collège de France.

But du cours

These lectures will focus on recent developments in quantum metrology. The main questions to be answered are: What are the ultimate precision limits in the estimation of parameters, according to classical mechanics and quantum mechanics? Are there fundamental limits? Is quantum mechanics helpful in reaching better precision? How to cope with the deleterious effects of noise?

Our discussion is restricted to local quantum metrology: in this case, one is not interested in an optimal globally-valid estimation strategy, valid for any value of the parameter to be estimated, but one wants instead to estimate a parameter confined to some small range. The techniques to be developed are useful, for instance, for estimating parameters that undergo small changes around a known value, like sensing phase changes in gravitational detectors; or yet if one has some prior (eventually rough) knowledge about the value of the parameter.

Summary of the lectures

The lectures will be organized as follows:

LECTURE 1. Examples of metrological tasks. Quantum metrology and optical interferometers. Classical bounds on precision: Derivation of the Cramér-Rao bound and introduction of the Fisher information.

LECTURE 2. Extension of Cramér-Rao bound and Fisher information to quantum mechanics. Quantum Fisher information for pure states. The role of entanglement. Application to atomic interferometry and weak-value amplification.

LECTURE 3. Noisy quantum-enhanced metrology: General framework for evaluating the ultimate precision limit in the estimation of parameters. Quantum channels. Application to optical interferometers, force estimation, and atomic spectroscopy.

LECTURE 4. Quantum metrology and the energy-time uncertainty relation. Quantum speed limit and the geometry of quantum states. Generalization to open systems. Application to atomic decay and dephasing.

SOME REFERENCES

- C. W. Helstrom, Quantum detection and estimation theory (Academic press, 1976).
- C. M. Caves et al., On the measurement of a weak classical force coupled to a quantum-mechanical oscillator I: Issues of principle, Rev. of Mod. Phys. **52**, 341 (1980).
- C. M. Caves, Quantum-mechanical noise in an interferometer, Phys. Rev. D **23**, 1693 (1981).
- A. S. Holevo, Probabilistic and Statistical Aspects of Quantum Theory (North Holland, Amsterdam, 1982).
- S. M. Kay, Fundamentals of Statistical Signal Processing: Estimation Theory (Prentice Hall, USA, 1993).
- S. L. Braunstein, C. M. Caves, and G. J.Milburn, Generalized Uncertainty Relations: Theory, Examples, and Lorentz Invariance, Annals of Physics **247**, 135 (1996).
- V. Giovannetti, S. Lloyd, and L. Maccone, Quantum metrology, Phys. Rev. Lett. **96**, 010401 (2006).
- Rafal Demkowicz-Dobrzański, Marcin Jarzyna and Jan Kołodyński, Quantum Limits in Optical Interferometry, in Progress in Optics, vol. **60**, Ed. Emil Wolf (2015).

SOME REFERENCES

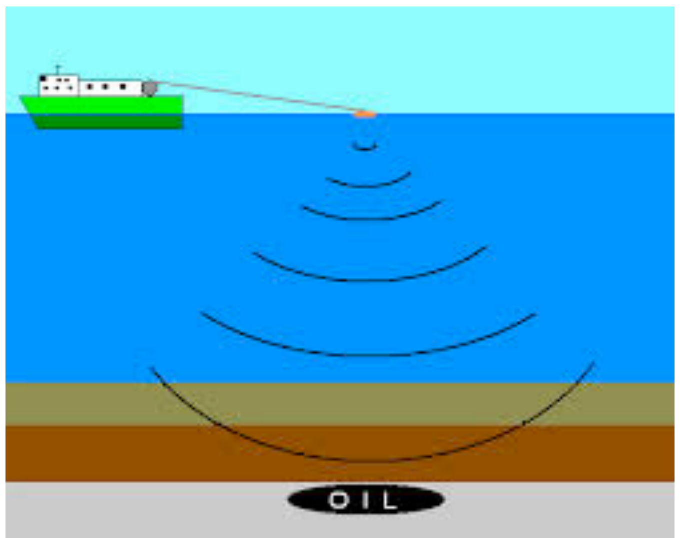
- B. M. Escher, R. L. Matos Filho, and L. Davidovich, General framework for estimating the ultimate precision limit in noisy quantum-enhanced metrology, Nature Physics **7**, 406 (2011).
- B. M. Escher, R. L. de Matos Filho, and L. Davidovich, Quantum metrology for noisy systems, Braz. J. Phys **41**, 229 (2011).
- B. M. Escher, L. Davidovich, N. Zagury, and R. L. de Matos Filho, Quantum Metrological Limits via a Variational Approach, Phys. Rev. Lett. **109**, 190404 (2012).
- C. L. Latune, B. M. Escher, R. L. de Matos Filho, and L. Davidovich, Quantum limit for the measurement of a classical force coupled to a noisy quantum-mechanical oscillator, Phys. Rev. A **88**, 042112 (2013).
- M. M. Taddei, B. M. Escher, L. Davidovich, and R. L. de Matos Filho, Quantum Speed Limit for Physical Processes, Phys. Rev. Lett. **109**, 050402 (2013).
- G. Bié Alves, B. M. Escher, R. L. de Matos Filho, N. Zagury, and L. Davidovich, Weak-value amplification as an optimal metrological protocol, Physical Review A **91**, 062107 (2015).
- M. M. Taddei, Ph. D. Thesis, Quantum speed limits for general physical processes, Ph. D. Thesis (2014), available at arxiv.org/pdf/1407.4343

Leçon 1

I.1 - General introduction: parameter estimation and classical limits on precision

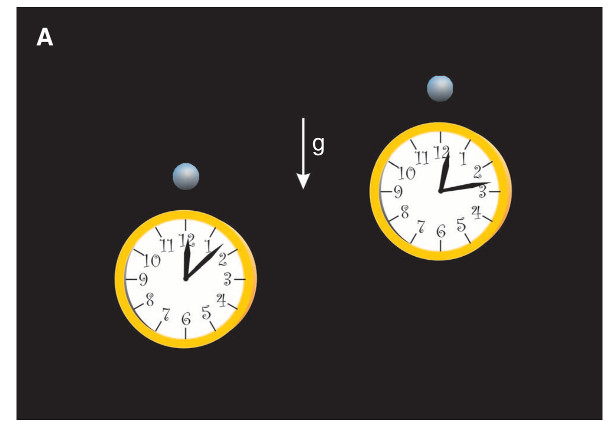
Parameter estimation

Depth of an oil well



Time duration of a process

Transition frequency



$$\Delta h = 33\,\mathrm{cm}$$

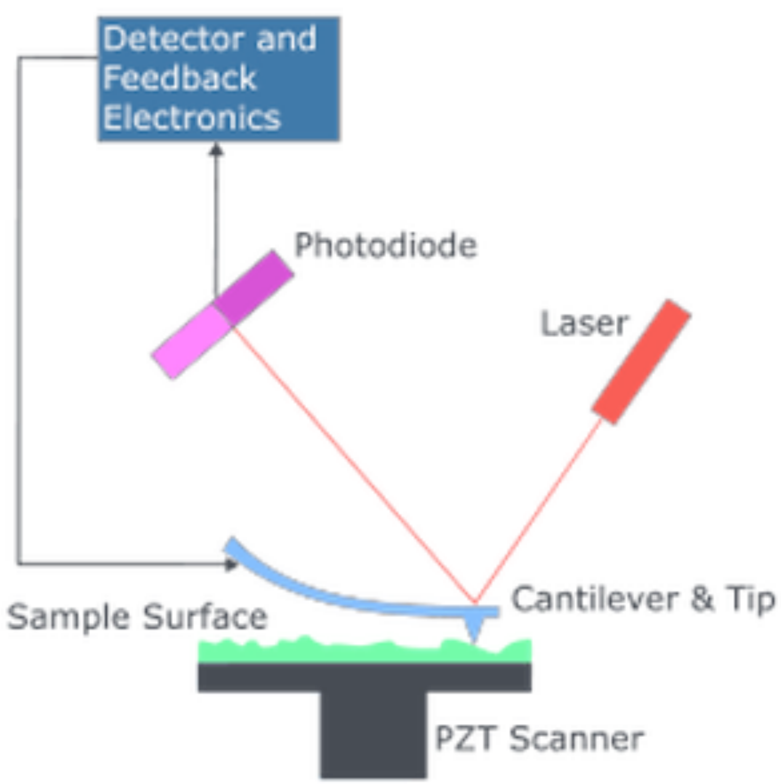
$$\frac{\Delta f}{f} = (4.1 \pm 1.6) \times 10^{-17}$$

Optical Clocks and Relativity

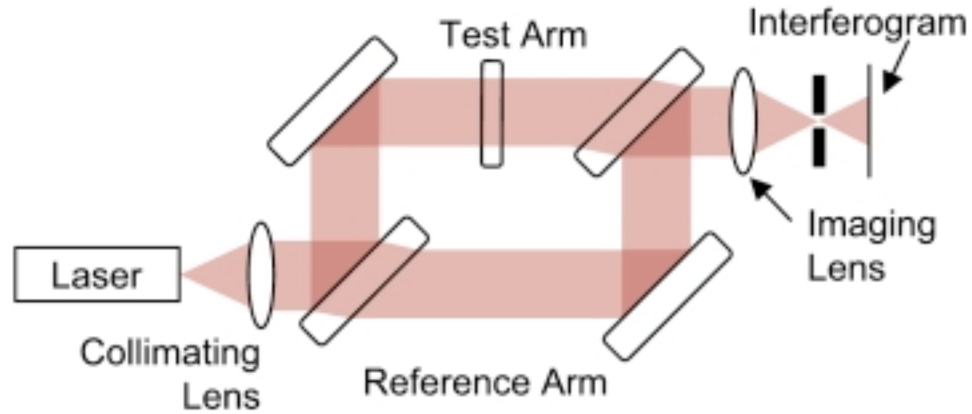
C. W. Chou,* D. B. Hume, T. Rosenband, D. J. Wineland

24 SEPTEMBER 2010 VOL 329 **SCIENCE**

Weak forces or small displacements



Phase displacements in interferometers



Laser Interferometer Gravitational Wave Observatory

nature photonics

LETTERS

PUBLISHED ONLINE: 21 JULY 2013 | DOI: 10.1038/NPHOTON.2013.177

Enhanced sensitivity of the LIGO gravitational wave detector by using squeezed states of light

The LIGO Scientific Collaboration*

High-precision interferometry: Advanced LIGO

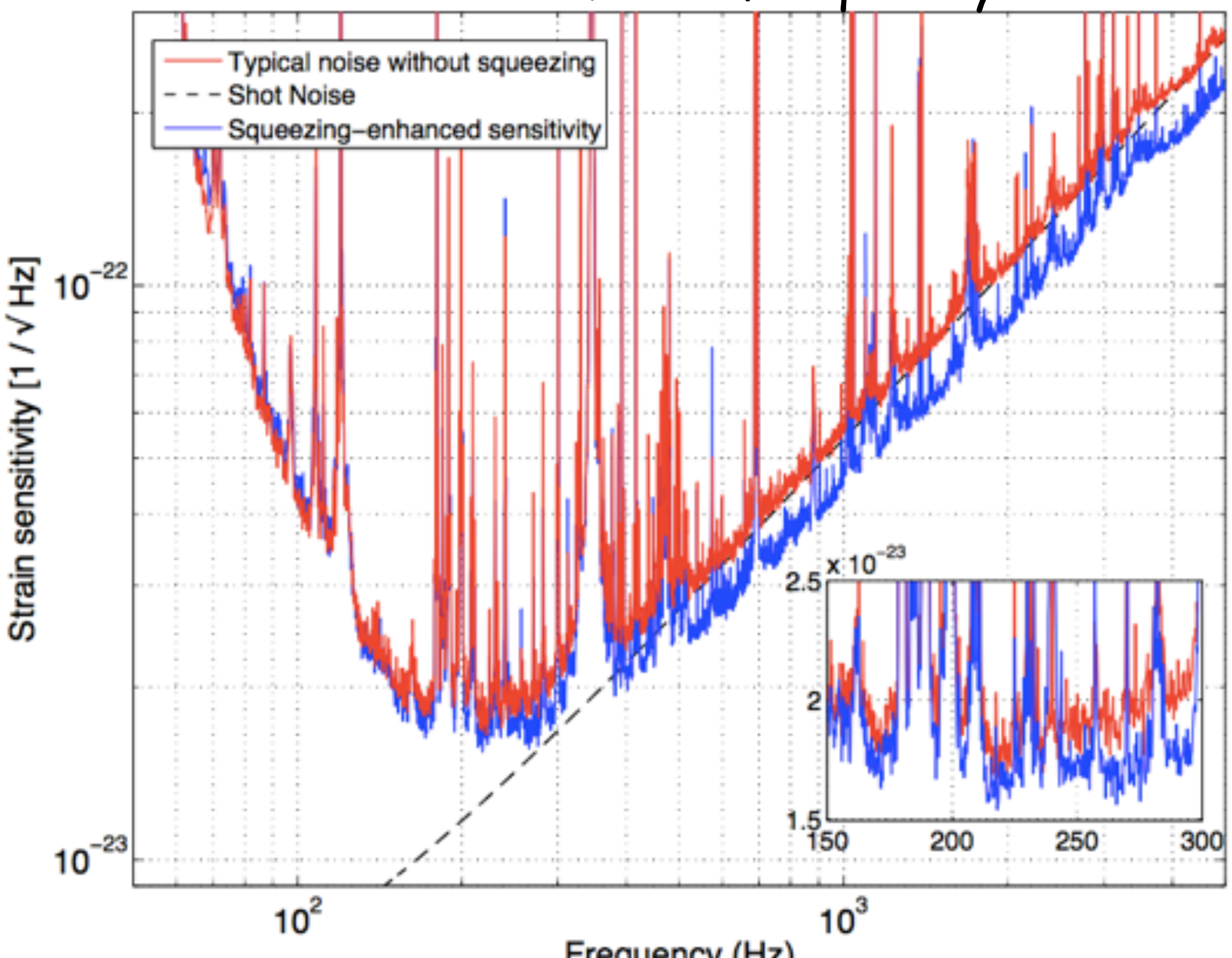
Home » Physics » General Physics » January 12, 2016

Gravitational wave rumors ripple through science world

January 12, 2016

Differential displacement sensitivity $\simeq 10^{-19}\,\mathrm{m}$

Up to 2.15dB improvement in sensitivity in the shot-noise- limited frequency band



Differential strain sensitivity $\simeq 3\times10^{-23}$

Experiments: Parameter estimation beyond classical physics in the XXI century

Phase resolution

Letter

Nature Photonics **4**, 357 - 360 (2010)
Published online: 4 April 2010 | doi:10.1038/nphoton.2010.39

Experimental quantum-enhanced estimation of a lossy phase shift

M. Kacprowicz[1], R. Demkowicz-Dobrzański[1,2], W. Wasilewski[2], K. Banaszek[1,2] & I. A. Walmsley[3]

NATURE PHOTONICS | LETTER

Entanglement-enhanced measurement of a completely unknown optical phase

G. Y. Xiang, B. L. Higgins, D. W. Berry, H. M. Wiseman & G. J. Pryde

Letters to Nature

Nature **429**, 161-164 (13 May 2004) | doi:10.1038/nature02493; Received 22 December 2003; Accepted 16 March 2004

Super-resolving phase measurements with a multiphoton entangled state

M. W. Mitchell, J. S. Lundeen & A. M. Steinberg

1. Department of Physics, University of Toronto, 60 St George Street, Toronto, Ontario M5S 1A7, Canada

New Journal of Physics
The open access journal at the forefront of physics

Beating the standard quantum limit: phase super-sensitivity of *N*-photon interferometers

Ryo Okamoto[1,2,5], Holger F Hofmann[3], Tomohisa Nagata[1], Jeremy L O'Brien[4], Keiji Sasaki[1] and Shigeki Takeuchi[1,2]

Science 4 May 2007:
Vol. 316 no. 5825 pp. 726–729
DOI: 10.1126/science.1138007

REPORT

Beating the Standard Quantum Limit with Four-Entangled Photons

Tomohisa Nagata[1], Ryo Okamoto[1,2], Jeremy L. O'Brien[3,4], Keiji Sasaki[1], Shigeki Takeuchi[1,2,*]

Experiments: Parameter estimation beyond classical physics in the XXI century

Atomic clocks

10960–10965 | PNAS | July 7, 2009 | vol. 106 | no. 27

Mesoscopic atomic entanglement for precision measurements beyond the standard quantum limit

J. Appel, P. J. Windpassinger, D. Oblak, U. B. Hoff, N. Kjærgaard, and E. S. Polzik[1]

Danish National Research Foundation Center for Quantum Optics, The Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

Letter

Nature **443**, 316-319 (21 September 2006) | doi:10.1038/nature05101; Received 5 May 2006; Accepted 18 July 2006

'Designer atoms' for quantum metrology

C. F. Roos[1,2], M. Chwalla[1], K. Kim[1], M. Riebe[1] & R. Blatt[1,2]

Science 4 June 2004:
Vol. 304 no. 5676 pp. 1476-1478
DOI: 10.1126/science.1097576

REPORT

Toward Heisenberg-Limited Spectroscopy with Multiparticle Entangled States

D. Leibfried*, M. D. Barrett†, T. Schaetz, J. Britton, J. Chiaverini, W. M. Itano, J. D. Jost, C. Langer, D. J. Wineland

Physics ABOUT BROWSE JOURNALISTS

Focus: Atomic Clock Beats the Quantum Limit

June 25, 2010 • *Phys. Rev. Focus* 25, 24

Researchers beat the quantum-mechanical fluctuations in an atomic clock by linking many atoms into an entangled quantum state and pushing the fluctuations into a realm that doesn't influence the time measurement.

New Journal of Physics
The open-access journal for physics

Entanglement-assisted atomic clock beyond the projection noise limit

Anne Louchet-Chauvet[1], Jürgen Appel, Jelmer J Renema, Daniel Oblak, Niels Kjaergaard[2] and Eugene S Polzik[3]

Implementation of Cavity Squeezing of a Collective Atomic Spin

Ian D. Leroux, Monika H. Schleier-Smith, and Vladan Vuletić

Phys. Rev. Lett. **104**, 073602 – Published 17 February 2010; Erratum Phys. Rev. Lett. **106**, 129902 (2011)

Experiments: Parameter estimation beyond classical physics in the XXI century

Magnetometers

NATURE | LETTER

Interaction-based quantum metrology showing scaling beyond the Heisenberg limit

M. Napolitano, M. Koschorreck, B. Dubost, N. Behbood, R. J. Sewell & M. W. Mitchell

NATURE | LETTER

日本語要約

Nature **510**, 376–380 (19 June 2014)

Measurement of the magnetic interaction between two bound electrons of two separate ions

Shlomi Kotler, Nitzan Akerman, Nir Navon, Yinnon Glickman & Roee Ozeri

Magnetic Sensitivity Beyond the Projection Noise Limit by Spin Squeezing

R. J. Sewell, M. Koschorreck, M. Napolitano, B. Dubost, N. Behbood, and M. W. Mitchell

Phys. Rev. Lett. **109**, 253605 – Published 19 December 2012

Quantum Noise Limited and Entanglement-Assisted Magnetometry

W. Wasilewski, K. Jensen, H. Krauter, J. J. Renema, M. V. Balabas, and E. S. Polzik

Phys. Rev. Lett. **104**, 133601 – Published 31 March 2010; Erratum Phys. Rev. Lett. **104**, 209902 (2010)

REPORTS

29 MAY 2009 VOL 324 **SCIENCE**

Magnetic Field Sensing Beyond the Standard Quantum Limit Using 10-Spin NOON States

Jonathan A. Jones,[1] Steven D. Karlen,[2] Joseph Fitzsimons,[2,3] Arzhang Ardavan,[1] Simon C. Benjamin,[2,4] G. Andrew D. Briggs,[2] John J. L. Morton[1,2*]

Increasing Sensing Resolution with Error Correction

G. Arrad, Y. Vinkler, D. Aharonov, and A. Retzker

Phys. Rev. Lett. **112**, 150801 – Published 16 April 2014

Quantum Error Correction for Metrology

E. M. Kessler, I. Lovchinsky, A. O. Sushkov, and M. D. Lukin

Phys. Rev. Lett. **112**, 150802 – Published 16 April 2014

Parameter estimation and uncertainty relations

What is the meaning of

★ Time-energy uncertainty relation?

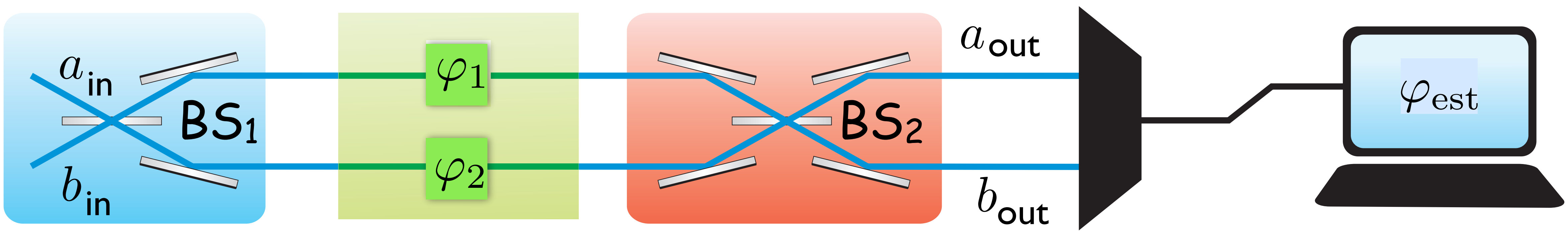
$$\Delta E \Delta T \geq \hbar / 2$$

★ Number-phase uncertainty relation?

$$\Delta N \Delta \phi \geq \hbar / 2$$

We shall see that quantum parameter estimation allows to understand these relations in terms of uncertainties in the estimation of parameters: while Heisenberg uncertainty relations are associated with Hermitian operators, the theory of parameter estimation allows one to obtain uncertainty relations for parameters, like time or phase, with no need to associate them to suitable Hermitian operators.

An example: optical interferometry

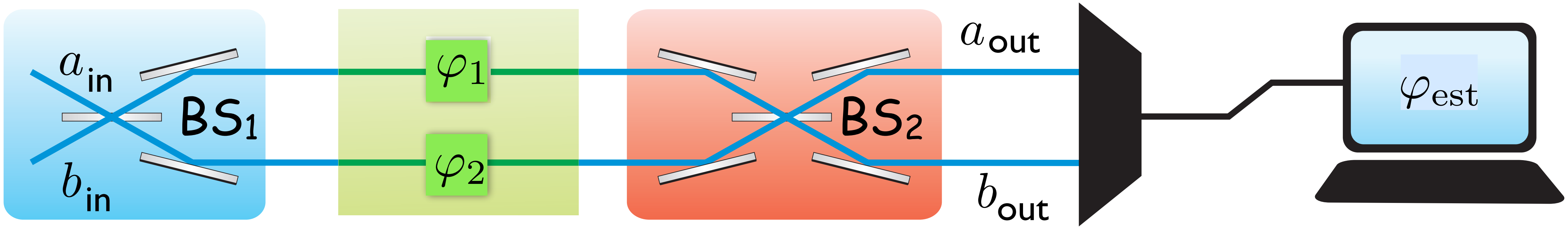


Mach-Zender interferometer: a beam with complex amplitude **a_{in}** is split on a balanced beam splitter BS_1 and the two resulting beams acquire phases φ_1 and φ_2, interfering on the second beam splitter BS_2. The photon numbers $n_{a_{out}}$ and $n_{b_{out}}$ are measured at the output ports. One could also have two incident beams, with complex amplitudes **a_{in}** and **b_{in}**.

The outgoing fields are related to the incoming ones through the transformation (note that **a_{out}**=**a_{in}**, **b_{out}**=**b_{in}** when $\varphi_1 = \varphi_2 = 0$, since $[BS_1]X[BS_2]=1$) :

$$\begin{pmatrix} a_{out} \\ b_{out} \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}}_{BS_2} \begin{pmatrix} e^{i\varphi_1} & 0 \\ 0 & e^{i\varphi_2} \end{pmatrix} \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}}_{BS_1} \begin{pmatrix} a_{in} \\ b_{in} \end{pmatrix}$$

Optical interferometry (2)



Multiplying the matrices, and replacing the complex amplitudes by the corresponding photon annihilation operators, one gets:

$$\begin{pmatrix} \hat{a}_{out} \\ \hat{b}_{out} \end{pmatrix} = e^{i(\varphi_1+\varphi_2)/2} \begin{pmatrix} \cos(\varphi/2) & -\sin(\varphi/2) \\ \sin(\varphi/2) & \cos(\varphi/2) \end{pmatrix} \begin{pmatrix} \hat{a}_{in} \\ \hat{b}_{in} \end{pmatrix}, \qquad \varphi = \varphi_1 - \varphi_2,$$

where the operator â annihilates photons in mode a: $\hat{a}|N\rangle = \sqrt{N}|N-1\rangle$ and $|N\rangle$ is the Fock state with N photons, with $\hat{a}^\dagger\hat{a}|N\rangle = N|N\rangle$, where $\hat{a}^\dagger\hat{a}$ is the number operator. The overall phase above can be neglected.

We use now the Jordan-Schwinger transformation, which allows to analyze the Mach-Zender interferometer in terms of the algebra of angular momentum operators.

Optical interferometry and Jordan-Schwinger transformation

PHYSICAL REVIEW A VOLUME 33, NUMBER 6 JUNE 1986

SU(2) and SU(1,1) interferometers

Bernard Yurke, Samuel L. McCall, and John R. Klauder
AT&T Bell Laboratories, Murray Hill, New Jersey 07974
(Received 30 October 1985)

Quantum limits in optical interferometry

R. Demkowicz-Dobrzański, M. Jarzyna, and J. Kołodyński
Faculty of Physics, University of Warsaw, ul. Hoża 69, PL-00-681 Warszawa, Poland

Progress in Optics **60** (2015)

This has the advantage of providing a unified formalism, which can also be applied to problems in atomic spectroscopy and magnetometry.

Let $\boxed{\hat{J}_x = \frac{1}{2}(\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}), \quad \hat{J}_y = \frac{i}{2}(\hat{b}^\dagger \hat{a} - \hat{a}^\dagger \hat{b}), \quad \hat{J}_z = \frac{1}{2}(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})}$

Then $[\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk}\hat{J}_k$ and $\hat{J}^2 = \frac{\hat{N}}{2}\left(\frac{\hat{N}}{2} + 1\right), \quad \hat{N} = \hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}$

so these operators obey the angular momentum algebra.

Transformations of operators $\hat{a}$ and $\hat{b}$ can be considered as rotations in spin space: $\hat{a}' = \hat{U}\hat{a}\hat{U}^\dagger$, $\hat{b}' = \hat{U}\hat{b}\hat{U}^\dagger$, with $\hat{U} = \exp(-i\theta\hat{J}\cdot\hat{n})$, where the unit vector $\hat{n}$ is along the axis of rotation, and with the correspondence:

BS$_1 \rightarrow \hat{U} = \exp(-i\pi\hat{J}_x/2)$

BS$_2 \rightarrow \hat{U} = \exp(i\pi\hat{J}_x/2)$

Phase delay $\rightarrow \hat{U} = \exp(-i\phi\hat{J}_z)$

Angular momentum operators for optical interferometry

Corresponding transformation for the operators $\hat{J}_i$ (Heisenberg picture!):

$$\begin{pmatrix} \hat{J}_x^{out} \\ \hat{J}_y^{out} \\ \hat{J}_z^{out} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{J}_x^{in} \\ \hat{J}_y^{in} \\ \hat{J}_z^{in} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{pmatrix} \begin{pmatrix} \hat{J}_x^{in} \\ \hat{J}_y^{in} \\ \hat{J}_z^{in} \end{pmatrix}$$

Therefore, Mach-Zender transformation amounts to a rotation around y axis of the angular momentum operators.

The state transforms as $|\psi\rangle_{\rm out} = e^{-i\hat{J}_x\pi/2} e^{i\hat{J}_z\varphi} e^{-i\hat{J}_x\pi/2} |\psi\rangle_{\rm in}$

Based on this formalism, we derive now an expression for the uncertainty in the estimation of the phase, for measurements of the difference in photon numbers at the two outputs of the interferometer.

Precision of phase estimation

From $\hat{J}_z = \frac{1}{2}(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})$, it is clear that $\hat{n}_a - \hat{n}_b = 2\hat{J}_z$.

On the other hand, the average of $\hat{J}_z$ in the output state is equal to the average of $\hat{J}_z^{\mathrm{out}}$, given by the previous matrix expression, in the input state.

Therefore, $\boxed{\langle \hat{J}_z \rangle_{\mathrm{out}} = \cos\varphi \langle \hat{J}_z \rangle_{\mathrm{in}} - \sin\varphi \langle \hat{J}_x \rangle_{\mathrm{in}}}$ while the variance is

$$\boxed{\Delta^2 \hat{J}_z \Big|_{\mathrm{out}} = \cos^2\varphi\, \Delta^2 \hat{J}_z \Big|_{\mathrm{in}} + \sin^2\varphi\, \Delta^2 \hat{J}_x \Big|_{\mathrm{in}} - 2\sin\varphi\cos\varphi\ \mathrm{cov}(\hat{J}_x, \hat{J}_z)\Big|_{\mathrm{in}}}$$

where the covariance cov is defined as

$$\boxed{\mathrm{cov}(\hat{J}_x, \hat{J}_z) = \tfrac{1}{2}\langle \hat{J}_x \hat{J}_z + \hat{J}_z \hat{J}_x \rangle - \langle \hat{J}_x \rangle \langle \hat{J}_z \rangle}$$

The precision of estimation can now be quantified by the error propagation formula:

$$\boxed{\Delta\varphi = \frac{\Delta \hat{J}_z \Big|_{\mathrm{out}}}{\left| \frac{d\langle \hat{J}_z \rangle_{\mathrm{out}}}{d\varphi} \right|}}$$

where $\Delta\varphi = \sqrt{\Delta^2\varphi}$ is a standard deviation (same for $\Delta\hat{J}_z$).

Optical interferometry with Fock states

Consider first that a Fock state $|N\rangle$ is injected in port a, so that $\boxed{|\psi\rangle_{\rm in} = |N\rangle_a |0\rangle_b}$. Since

$$\hat{J}_x = \frac{1}{2}(\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}), \quad \hat{J}_y = \frac{i}{2}(\hat{b}^\dagger \hat{a} - \hat{a}^\dagger \hat{b}), \quad \hat{J}_z = \frac{1}{2}(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})$$

this initial state is an eingestate of $\hat{J}_z$ and $\hat{J}^2$: $\boxed{\hat{J}_z |N,0\rangle = (N/2)|N,0\rangle,}$ $\boxed{\hat{J}^2 |N,0\rangle = \frac{N}{2}\left(\frac{N}{2}+1\right)|N,0\rangle,}$ so we may write $\boxed{|N,0\rangle \rightarrow |j,j\rangle\,.}$ Also,

$$\boxed{\langle \hat{J}_z \rangle_{\rm in} = N/2,\; \langle \hat{J}_x \rangle_{\rm in} = 0,\; \Delta^2 \hat{J}_z\Big|_{\rm in} = 0\,,\; \Delta^2 \hat{J}_x\Big|_{\rm in} = N/4,}$$

and $\boxed{{\rm cov}(\hat{J}_x, \hat{J}_z)_{\rm in} = 0\,.}$

From $\langle \hat{J}_z \rangle_{\rm out} = \cos\varphi \langle \hat{J}_z \rangle_{\rm in} - \sin\varphi \cancel{\langle \hat{J}_x} \rangle_{\rm in}$ and

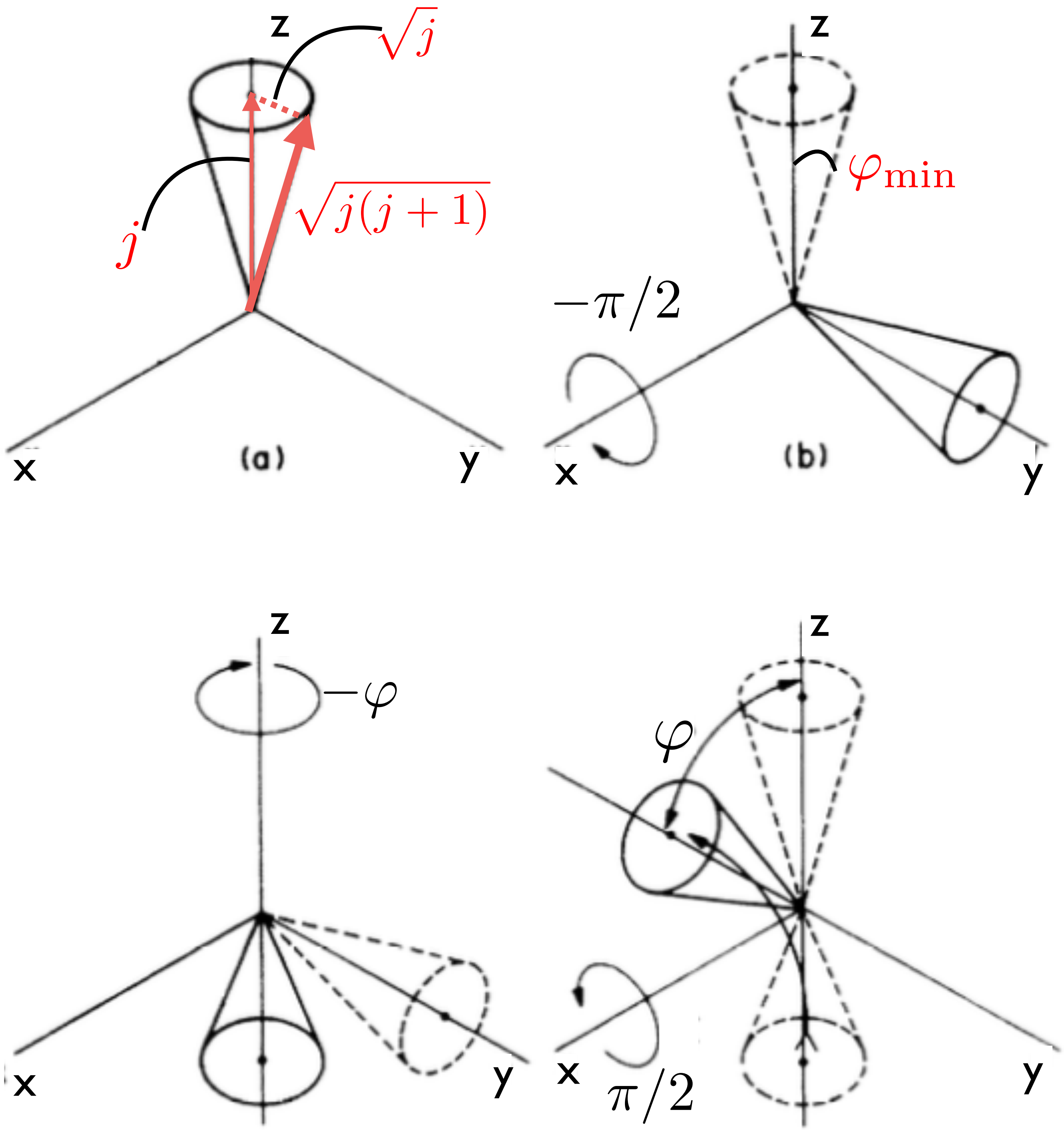
$$\Delta^2 \hat{J}_z\Big|_{\rm out} = \cos^2\varphi\; \cancel{\Delta^2 \hat{J}_z}\Big|_{\rm in} + \sin^2\varphi\; \Delta^2 \hat{J}_x\Big|_{\rm in} - 2\sin\varphi\cos\varphi\;\; {\rm cov}(\cancel{\hat{J}_x}, \hat{J}_z)\Big|_{\rm in}$$

one gets

$$\Delta\varphi = \frac{\Delta \hat{J}_z\Big|_{\rm out}}{\left|\frac{d\langle \hat{J}_z \rangle_{\rm out}}{d\varphi}\right|} = \frac{\sqrt{N}|\sin\varphi|/2}{N|\sin\varphi|/2} = \frac{1}{\sqrt{N}},$$

which is the standard (or shot-noise limit) for optical interferometry.

Geometrical interpretation



- Length of side of the cone: $\sqrt{j(j+1)}$, with j=N/2
- Distance from apex to center of base: eigenvalue of $\hat{J}_z$ —> j=N/2
- Radius of the base of the cone:

$$\sqrt{j(j+1) - j^2} = \sqrt{j}$$

(a) Initial state

(b) Action of first beam splitter

(c) Phase delay

(d) Action of second beam splitter

Minimum detectable φ is of the order of

$$\varphi_{\min} \approx \frac{\sqrt{j}}{j} = \frac{1}{\sqrt{j}} \approx \frac{1}{\sqrt{N}}$$

Optical interferometry with coherent states

Consider now that a coherent state $|\alpha\rangle$ is injected in port **a**, so that

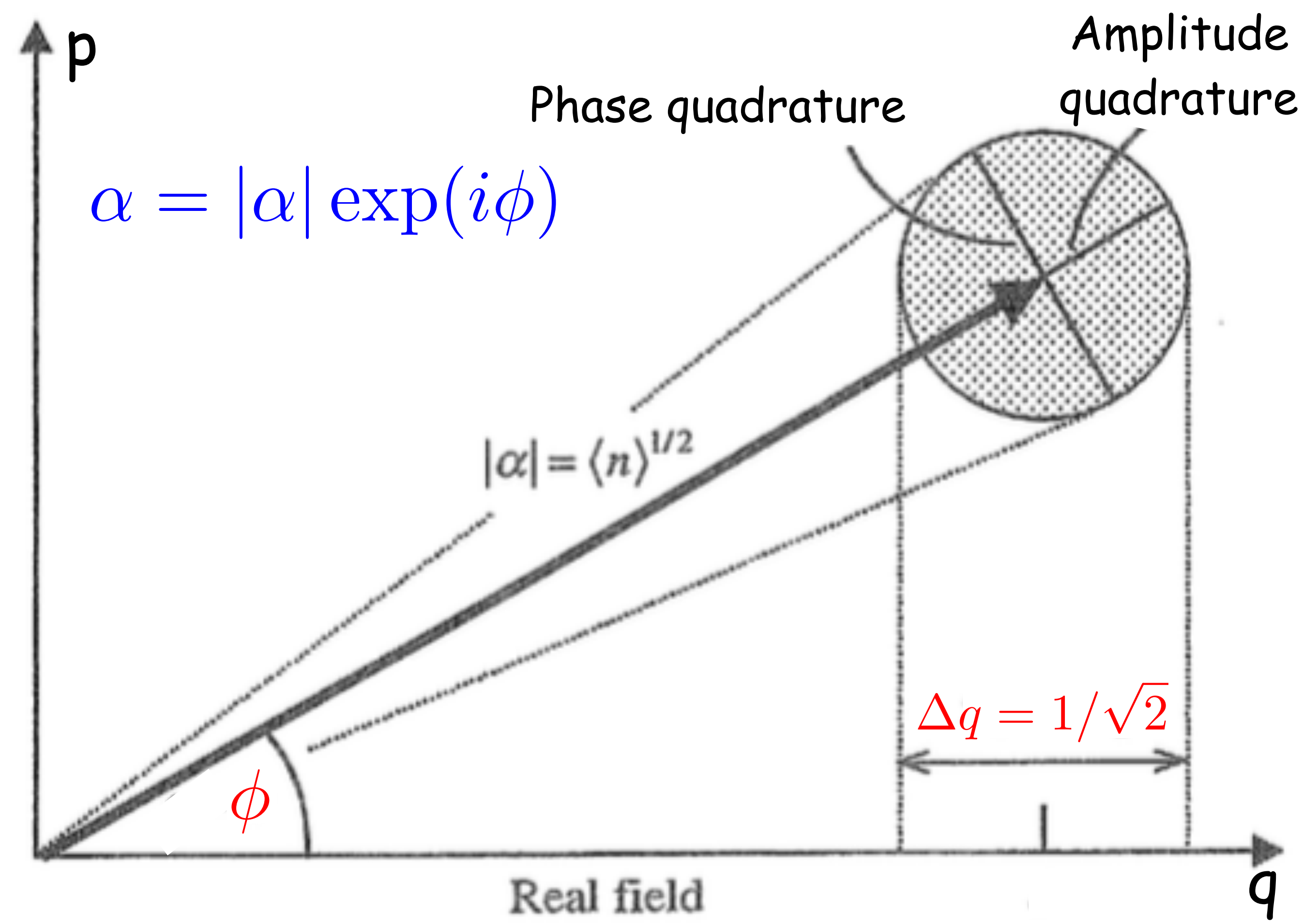
$$\boxed{|\psi\rangle_{\rm in} = |\alpha\rangle_a |0\rangle_b}$$

Just to fix the notation (and also as a reminder…), a coherent state is an eigenstate of the operator â, $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, and the average number of photons in the state is $\langle\alpha|\hat{N}|\alpha\rangle = \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = |\alpha|^2$.

Defining the quadrature operators as

$$\hat{q}_\theta = \frac{1}{\sqrt{2}}\left(\hat{a}e^{-i\theta} + \hat{a}^\dagger e^{i\theta}\right), \; \hat{p}_\theta = \hat{q}_{\theta+\pi/2} = \frac{-i}{\sqrt{2}}\left(\hat{a}e^{-i\theta} - \hat{a}^\dagger e^{i\theta}\right), \text{with } [\hat{q}_\theta, \hat{p}_\theta] = i,$$

it follows that the corresponding standard deviations in the state $|\alpha\rangle$ are $\Delta p_\theta = \Delta q_\theta = 1/\sqrt{2}$, and the coherent state is a minimum uncertainty state: $\Delta p_\theta \Delta q_\theta = 1/2$



Optical interferometry with coherent states (2)

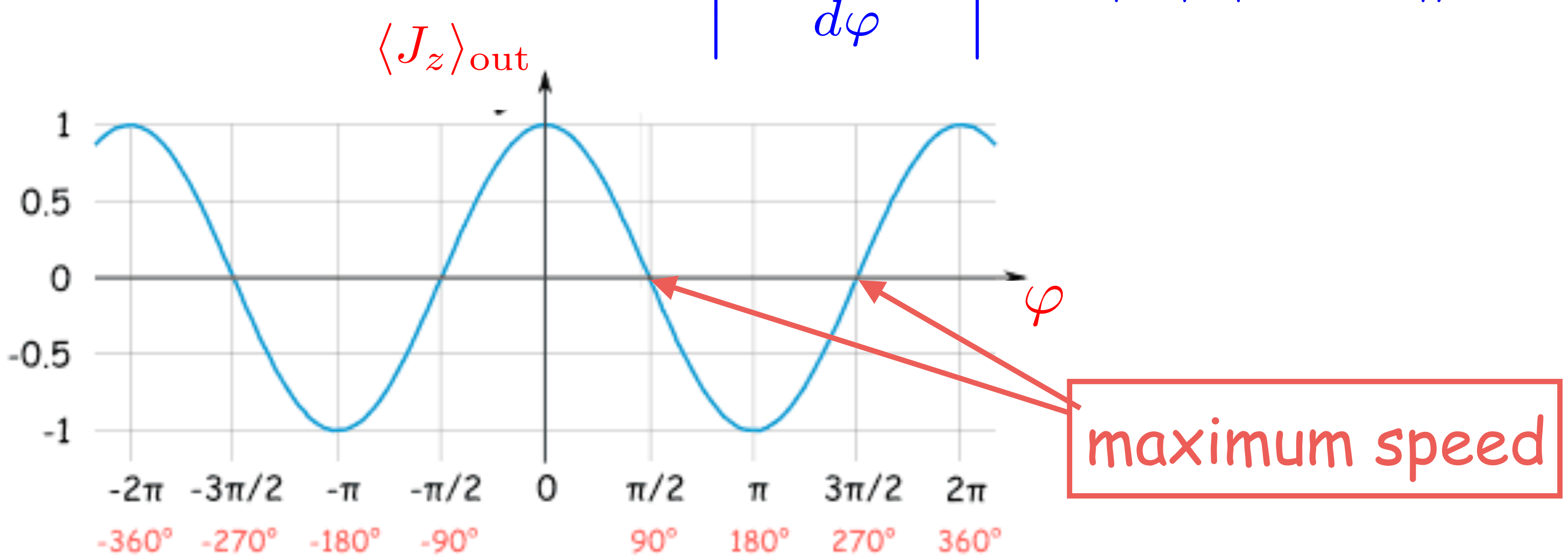
For the initial state $|\psi\rangle_{\rm in} = |\alpha\rangle_a |0\rangle_b$, one has

$$\boxed{\langle \hat{J}_z \rangle_{\rm in} = \tfrac{1}{2}|\alpha|^2,\ \ \langle \hat{J}_x \rangle_{\rm in} = 0,\ \Delta^2 \hat{J}_z\Big|_{\rm in} = \Delta^2 \hat{J}_x\Big|_{\rm in} = |\alpha|^2/4, \text{ and } {\rm cov}(\hat{J}_x, \hat{J}_z)_{\rm in} = 0\,.}$$

From $\langle \hat{J}_z \rangle_{\rm out} = \cos\varphi \langle \hat{J}_z \rangle_{\rm in} - \sin\varphi \cancel{\langle \hat{J}_x \rangle}_{\rm in}$ and

$$\Delta^2 \hat{J}_z\Big|_{\rm out} = \cos^2\varphi\ \Delta^2 \hat{J}_z\Big|_{\rm in} + \sin^2\varphi\ \Delta^2 \hat{J}_x\Big|_{\rm in} - 2\sin\varphi\cos\varphi\ \ {\rm cov}(\cancel{\hat{J}_x,} \hat{J}_z)\Big|_{\rm in}$$

one gets $$\Delta\varphi = \frac{\Delta \hat{J}_z\Big|_{\rm out}}{\left|\frac{d\langle \hat{J}_z \rangle_{\rm out}}{d\varphi}\right|} = \frac{|\alpha|/2}{|\alpha|^2|\sin\varphi|/2} = \frac{1}{|\alpha\sin\varphi|} = \frac{1}{\sqrt{\langle N \rangle}|\sin\varphi|}$$



Bound depends on incoming state and on the operating point!

The precision now depends on the operating point. The optimal operating points are at $\varphi = \pi/2$ or $\varphi = 3\pi/2$.

These two points correspond to the maximum speed of variation of $\langle \hat{J}_z \rangle_{\rm out}$ with φ, implying higher sensitivity of $\langle \hat{J}_z \rangle_{\rm out}$ to changes in this parameter.

Interferometry with coherent + squeezed states

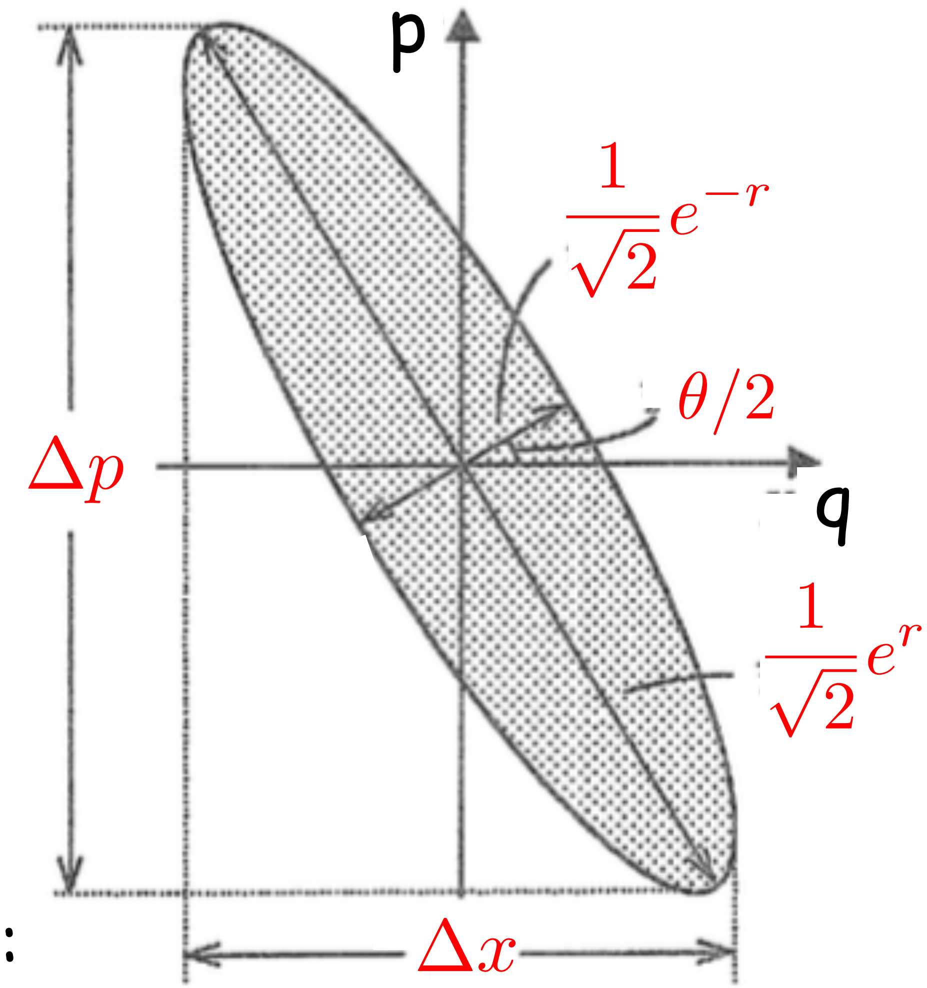
Important question: Can we do better, going beyond the shot-noise bound? This can actually be achieved, by using special quantum features of the incoming state.

Reminder on squeezed states

A squeezed state is a minimum-uncertainty state, obtained from a coherent state by a scaling transformation, which consists in squeezing a quadrature and stretching the orthogonal one. More formally, it is obtained from a coherent state through the transformation

$$\boxed{|\alpha, \xi\rangle = \hat{S}(\xi)|\alpha\rangle, \ \hat{S}(\xi) = \exp\left[\left(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2}\right)/2\right]}$$

where $\xi = r\exp(i\theta)$ is an arbitrary complex number.



For $\xi = r$ real ($\theta = 0$), the uncertainties in q and p are:

$$\Delta q = e^{-r}/\sqrt{2}, \ \Delta p = e^{r}/\sqrt{2}$$

For metrology, the squeezed vacuum states are more relevant: $\boxed{|\xi\rangle = \hat{S}(\xi)|0\rangle.}$

The average number of photons in state $|\xi\rangle$ is $\langle \hat{N}\rangle = \sinh^2 r$: a squeezed vacuum state has an average number of photons different from zero.

Interferometry with coherent + squeezed states (2)

Assume now that a coherent state is injected into one of the ports of a Mach-Zender interferometer, and a vacuum squeezed state into the other port. The initial state is then $\boxed{|\alpha\rangle \otimes |\xi\rangle}$. This scheme was proposed by Caves in 1981, and is implemented in gravitational-wave interferometers (LIGO, GEO600).

Assuming for simplicity that $\xi = r$ is real (this fixes a direction in phase space), one has:

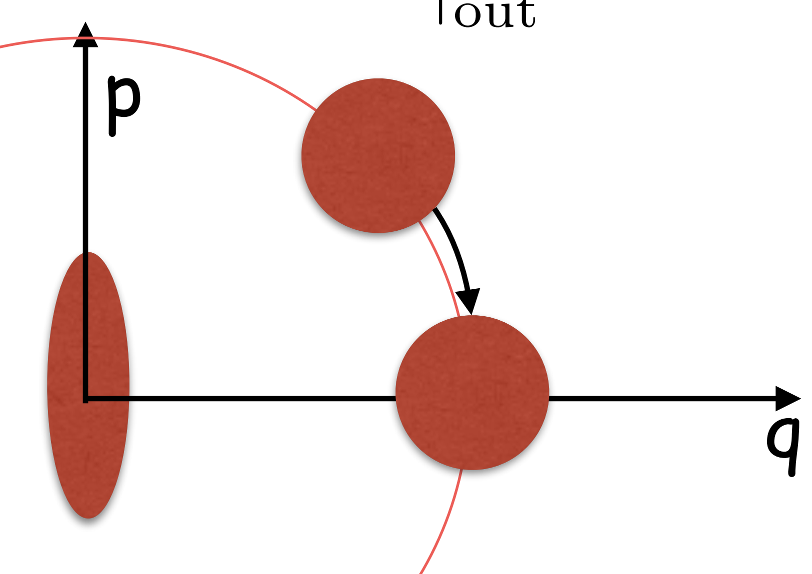
$$\langle \hat{N} \rangle = |\alpha|^2 + \sinh^2 r, \ \langle \hat{J}_z \rangle_{\text{in}} = (|\alpha|^2 - \sinh^2 r)/2, \ \langle \vec{J}_x \rangle_{\text{in}} = 0, \ \text{col}(\hat{J}_x, \hat{J}_z)\Big|_{\text{in}} = 0,$$

$$\Delta^2 \vec{J}_z\Big|_{\text{in}} = \left[|\alpha|^2 + (1/2)\sinh^2 2r\right]/4, \ \Delta^2 \vec{J}_x\Big|_{\text{in}} = \left[|\alpha|^2 \cosh 2r - \text{Re}(\alpha^2)\sinh 2r + \sinh^2 r\right]/4 .$$

This term reduces variance

Replacing these into the previous expressions for $\langle \hat{J}_z \rangle_{\text{out}}$ and $\Delta^2 \hat{J}_z\Big|_{\text{out}}$, and choosing α real, so as to minimize $\Delta^2 \hat{J}_x\Big|_{\text{in}}$ (this means that the coherent state is along the direction of highest compression):

$$\Delta\varphi = \frac{\sqrt{\cot^2\varphi(|\alpha|^2 + \frac{1}{2}\sinh^2 r) + |\alpha|^2 e^{-2r} + \sinh^2 r}}{\left||\alpha|^2 - \sinh^2 r\right|}$$



Interferometry with coherent + squeezed states (3)

We now try to minimize the expression:

$$\Delta\varphi = \frac{\sqrt{\cot^2\varphi(|\alpha|^2 + \frac{1}{2}\sinh^2 r) + |\alpha|^2 e^{-2r} + \sinh^2 r}}{||\alpha|^2 - \sinh^2 r|}$$

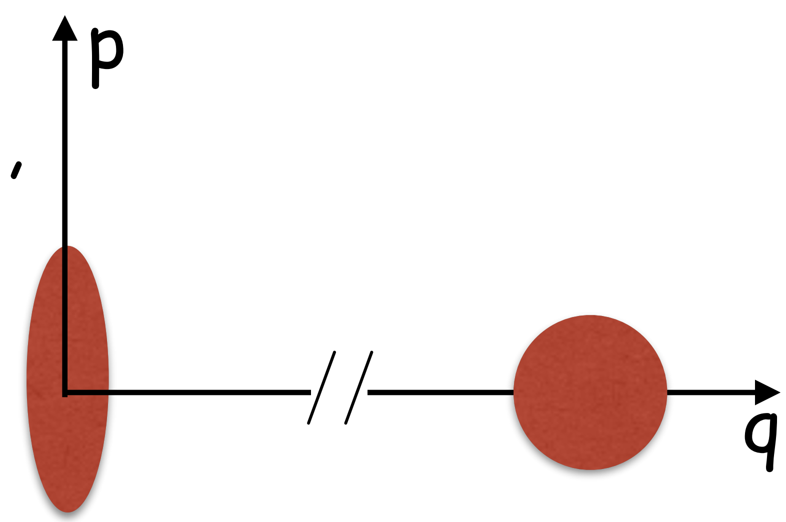
Optimal operation points: $\cot\varphi = 0 \Rightarrow \varphi = \pi/2,\, 3\pi/2.$

Then:

$$\Delta\varphi = \frac{\sqrt{|\alpha|^2 e^{-2r} + \sinh^2 r}}{||\alpha|^2 - \sinh^2 r|}$$

Consider $\langle\hat{N}\rangle \gg 1$, with the squeezed vacuum carrying approximately $\sqrt{\langle\hat{N}\rangle}/2$ photons. Then the majority of photons belong to the coherent state, and $\sinh^2 r \approx (1/4)e^{2r} \approx \sqrt{\langle\hat{N}\rangle}/2$, so that

$$\lim_{\langle N\rangle\to\infty} \Delta\varphi \approx \frac{\sqrt{\langle N\rangle/(2\sqrt{N}) + \sqrt{\langle N\rangle}/2}}{\langle N\rangle - \sqrt{\langle N\rangle}/2} \approx \frac{1}{\langle N\rangle^{3/4}},$$



implying that this scheme leads to precision better than shot noise, for the same amount of resources — in this case, the average photon number $\langle N\rangle$.

General N-photon two-mode input state

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SU(2) and SU(1,1) interferometers

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Using the angular momentum representation, a general N-photon two-mode input state can be written as

$$|\psi\rangle_{\rm in} = \sum_{m=-j}^{j} c_m |j,m\rangle,$$

where $\boxed{j=N/2}$ and $|j,m\rangle$ corresponds, in the mode-occupation notation, to the state $|j+m\rangle|j-m\rangle$. For m=j, one recovers the state $|N\rangle|0\rangle$ considered before. Assuming that N is even, consider the state

$$\boxed{|\psi\rangle_{\rm in} = \frac{1}{\sqrt{2}}(|j,0\rangle + |j,1\rangle) = \frac{1}{\sqrt{2}}\left(\left|\frac{N}{2}\right\rangle\left|\frac{N}{2}\right\rangle + \left|\frac{N}{2}+1\right\rangle\left|\frac{N}{2}-1\right\rangle\right)}.$$

Then

$$\boxed{\begin{aligned}&\langle\hat{J}_z\rangle_{\rm in} = \frac{1}{2},\ \langle\hat{J}_x\rangle_{\rm in} = \frac{1}{2}\sqrt{j(j+1)},\ \Delta^2\hat{J}_z\Big|_{\rm in} = \frac{1}{2},\\ &\Delta^2\hat{J}_x\Big|_{\rm in} = \frac{1}{2}j(j+1) - \frac{1}{4},\ {\rm cov}\left(\hat{J}_x,\hat{J}_z\right)\Big|_{\rm in} = 0.\end{aligned}}$$

General N-photon two-mode input state (2)

Replacing these into the previous expressions for $\langle \hat{J}_z \rangle_{\rm out}$ and $\Delta^2 \hat{J}_z \Big|_{\rm out}$:

$$\langle \hat{J}_z \rangle_{\rm out} = \cos\varphi \langle \hat{J}_z \rangle_{\rm in} - \sin\varphi \langle \hat{J}_x \rangle_{\rm in}$$

$$\Delta^2 \hat{J}_z \Big|_{\rm out} = \cos^2\varphi \, \Delta^2 \hat{J}_z \Big|_{\rm in} + \sin^2\varphi \, \Delta^2 \hat{J}_x \Big|_{\rm in}$$

one gets

$$\Delta\varphi = \frac{\Delta \hat{J}_z \Big|_{\rm out}}{\left| \frac{d\langle \hat{J}_z \rangle_{\rm out}}{d\varphi} \right|} = \frac{\sqrt{\cos^2\varphi + \sin^2\varphi [j(j+1) - 1]}}{|\sin\varphi + \cos\varphi \sqrt{j(j+1)}|}$$

which, for large j, has its optimal operating point at $\varphi = 0$, yielding

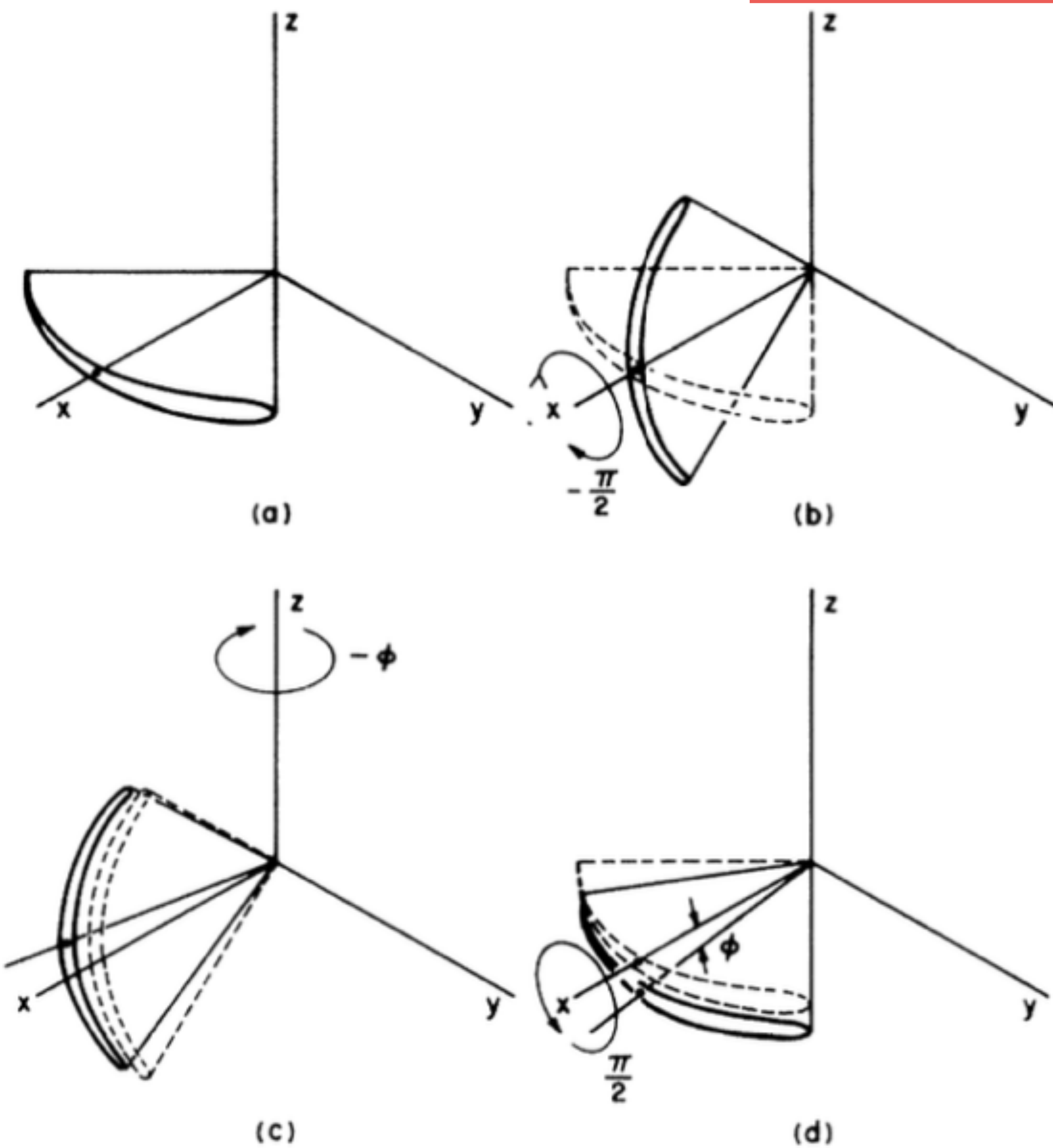
$$\Delta\varphi = \frac{1}{\sqrt{j(j+1)}} \approx \frac{2}{N}, \quad (N \to \infty)$$

For this choice of φ, the contribution from $\langle \hat{J}_x \rangle_{\rm in} = \frac{1}{2}\sqrt{j(j+1)}$ is maximized in the denominator, while that of $\Delta^2 \hat{J}_x \Big|_{\rm in} = \frac{1}{2}[j(j+1) - \frac{1}{2}]$ is minimized in the numerator.

This result is better than the one resulting from the coherent+squeezed state input, but the incoming state is much harder to prepare.

General N-photon two-mode input state: Geometrical interpretation

$$\boxed{\langle \hat{J}_z \rangle_{\text{in}} = \frac{1}{2},\ \langle \hat{J}_x \rangle_{\text{in}} = \frac{1}{2}\sqrt{j(j+1)},\ \Delta^2 \hat{J}_z \Big|_{\text{in}} = \frac{1}{2}}$$



State $|\psi\rangle_{\text{in}} = \frac{1}{\sqrt{2}}(|j,0\rangle + |j,1\rangle)$ is close to the x-y plane, as shown in (a), since $\langle \hat{J}_z \rangle_{\text{in}} = 1/2$, and it is represented by a flattened cone, with width $\Delta \hat{J}_z = 1/\sqrt{2}$.

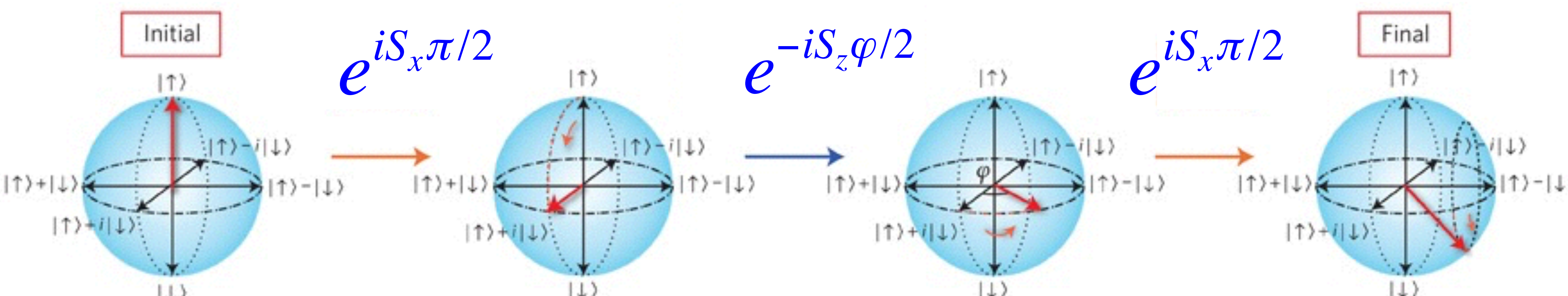
The side of the cone has length $\langle \hat{J}_x \rangle_{\text{in}} = \frac{1}{2}\sqrt{j(j+1)} \approx j/2$ so that the two flattened cones in (d) become distinguishable when

$$\varphi \approx \frac{1}{j}$$

which corresponds to the Heisenberg limit.

Other interferometric tasks

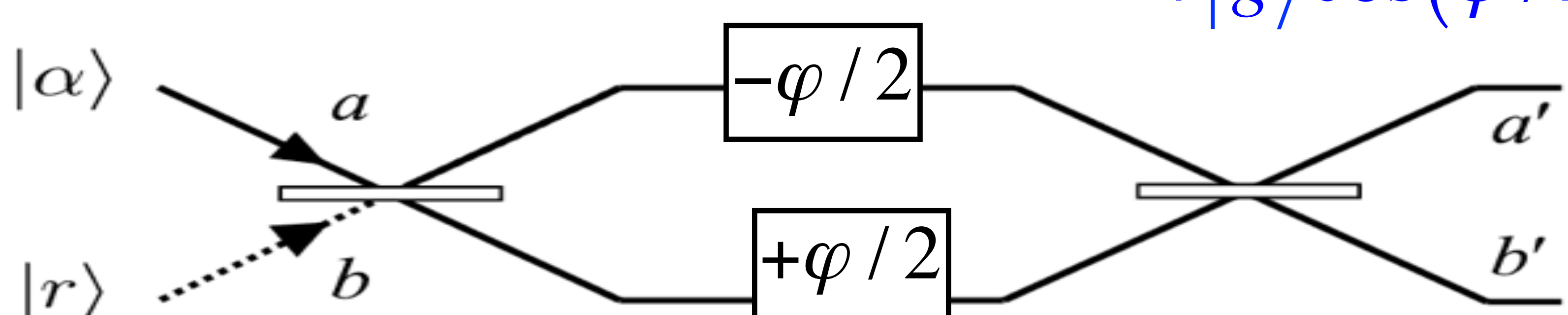
The same formalism can be applied to Ramsey interferometry: each atom in an atomic beam is subjected to a $\pi/2$ pulse, which transforms the initial state into the an equal-weighted superposition of the states $|e\rangle$ and $|g\rangle$. After evolving freely for a time t, the atom is subjected to another $\pi/2$ pulse, so that, if the pulses are resonant with the atomic transition, the atomic population is inverted. On the other hand, if the pulse is not resonant with the atom, the state acquires a phase shift $\varphi = \Delta\omega t$. Final state can be written as: $|\psi\rangle_{\text{out}} = e^{i\hat{J}_x\pi/2}e^{-i\hat{J}_z\varphi}e^{i\hat{J}_x\pi/2}|\psi\rangle_{\text{in}}$.



$$|e\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|e\rangle \rightarrow \left(|e\rangle + i|g\rangle\right)/\sqrt{2} \rightarrow |e\rangle \rightarrow \left(|e\rangle + ie^{i\varphi}|g\rangle\right)/\sqrt{2} \rightarrow -|e\rangle\sin(\varphi/2) + |g\rangle\cos(\varphi/2)$$

$$\boxed{\langle J_z\rangle = 2(P_g - P_e) = \cos\phi}$$



Other interferometric tasks

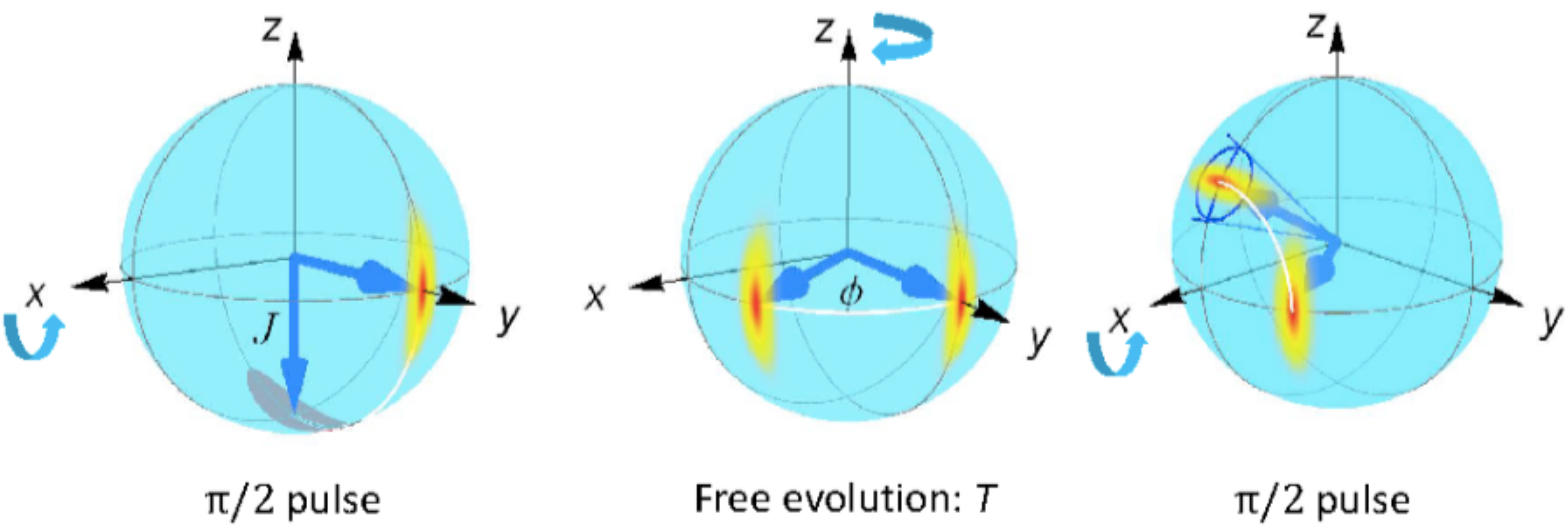
If one sends N independent atoms, we should expect the uncertainty in the phase to scale like $1/\sqrt{N}$, and this lower bound is related to the fluctuation in the number of detected atoms, called **projection noise** (the analog to the photonic shot noise). Of course, it does not make sense in this case to talk about states with indefinite number of atoms, like the coherent or squeezed states introduced before. It is possible, however, to produce special atomic states, with entangled atoms, that lead to Heisenberg scaling. This will be discussed in Lecture 2.

Other interferometric tasks

It is also possible to prepare squeezed atomic states, which lead to a 1/N scaling. Starting with atoms in a ground state, squeezed atomic states are obtained through the transformation

$$|\psi_\xi\rangle = \exp[(-\xi/2)(J_+^2 - J_-^2)]|g\rangle^{\otimes N},\ \xi\ \mathrm{real}$$

which is analogous to the corresponding transformation for electromagnetic fields. The successive transformations, applied on the collective angular momentum, are essentially the same as before — the squeezing reduces the final variance of J_z, thus increasing the precision in the estimation of the phase.



J. Ma, X. Wang, C. P. Sun, and F. Nori, arXiv:1011.2978 [quant-ph].

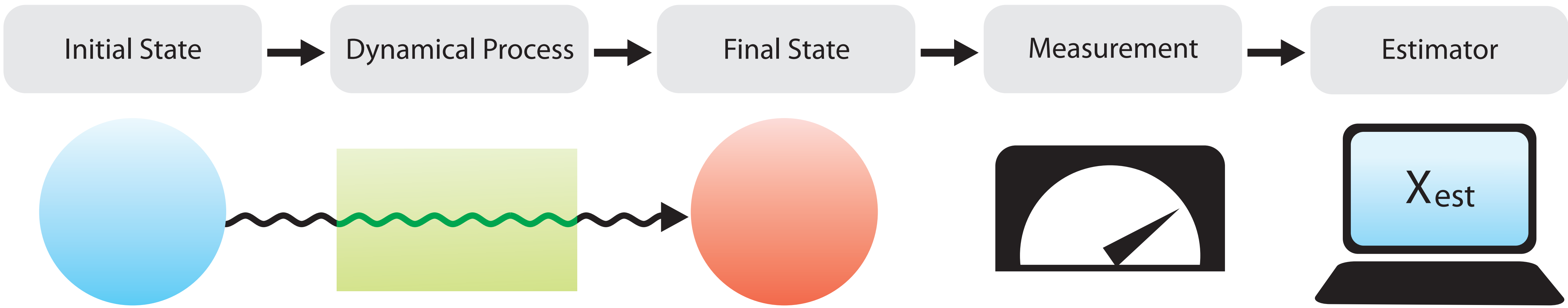
General estimation theory

We have shown that it is possible to win over the shot noise in optical interferometry, by using states with specific quantum features, like states with well-defined number of photons or squeezed states. In these examples, the estimation was obtained through measurement of the difference of photon numbers in the outgoing arms of the interferometer. It is not clear whether these are the best possible measurements, or whether better bounds can be obtained by using other incoming states.

One may ask whether it is possible to find general bounds and strategies for reaching them, which could be applied to many different systems, and could eventually help us to identify which are the best states and the best measurements for achieving the best possible precision.

This is the aim of this series of lectures: to develop, and apply to examples, a general estimation theory, capable not only to consider unitary evolutions of closed systems, like the one described here for the optical interferometer, but also open (noisy) systems.

Parameter estimation in classical and quantum physics



1. Prepare probe in suitable initial state
2. Send probe through process to be investigated
3. Choose suitable measurement
4. Associate each experimental result j with estimation

$$\delta X \equiv \sqrt{\left\langle \left[X_{\rm est}(j) - X\right]^2 \right\rangle_j \Big|_{X=X_{\rm true}}} \rightarrow \text{ Merit quantifier}$$

$$\left\langle X_{\rm est} \right\rangle = X_{\rm true}\,,\ \ d\left\langle X_{\rm est}\right\rangle / dX \Big|_{X=X_{\rm true}} = 1 \rightarrow \text{ Unbiased estimator}$$

Then $\delta X^2 = \Delta^2 X = \left\langle \left[X_{\rm est} - \langle X_{\rm est}\rangle\right]^2 \right\rangle \rightarrow$ variance of X_{est} (average is taken over all experimental results)

Estimator depends only on the experimental data.

Classical parameter estimation

H. Cramér

C. R. Rao

R. A. Fisher

Cramér-Rao bound for unbiased estimators:

$$\Delta X \geq 1/\sqrt{N\,F(X)\big|_{X=X_{\text{true}}}}\,, \quad F(X) \equiv \sum_j P_j(X)\left(\frac{d\,\ln[P_j(X)]}{dX}\right)^2$$

Fisher information

$N \to$ Number of repetitions of the experiment

$P_j(X) \to$ probability of getting an experimental result j

or yet, for continuous measurements: $F(X) \equiv \int d\xi\, p(\xi|X)\left[\frac{\partial \ln p(\xi|X)}{\partial X}\right]^2$

where ξ are the measurement results

(Average over all experimental results)

Derivation of Cramér-Rao bound (1)

Let us consider first the probability distribution $p(\xi|X)$ corresponding to a single output, where X is a single continuous parameter. The estimator is assumed unbiased, so that $\langle X_{\rm est}(\xi)\rangle = X$. Differentiating the trivial identity

$$\int d\xi\, p(\xi|X)[X_{\rm est}(\xi) - X] = 0$$

with respect to X, one gets

$$\int d\xi\, \left\{\frac{\partial p(\xi|X)}{\partial X}[X_{\rm est}(\xi) - X] + p(\xi|X)\frac{\partial}{\partial X}[X_{\rm est}(\xi) - X]\right\} = 0$$

or yet

$$\int d\xi\, p(\xi|X)\overbrace{\frac{\partial \ln p(\xi|X)}{\partial X}}^{A}\overbrace{[X_{\rm est}(\xi) - X]}^{B} = 1.$$

Square and use Cauchy-Schwarz inequality $\langle A^2\rangle\langle B^2\rangle \geq |\langle AB\rangle|^2$ —>

$$\left\langle\left[\frac{\partial \ln p(\xi|X)}{\partial X}\right]^2\right\rangle \Delta^2 X_{\rm est} \geq 1 \Rightarrow \Delta^2 X_{\rm est}\Big|_{X=X_{\rm true}} \geq \frac{1}{\left\langle\left[\frac{\partial \ln p(\xi|X)}{\partial X}\right]^2\right\rangle_{\rm X=X_{true}}}$$

$$\Delta^2 X_{\rm est} = \langle (X_{\rm est} - \langle X_{\rm est}\rangle)^2\rangle = \langle (X_{\rm est} - X)^2\rangle$$

Derivation of Cramér-Rao bound (2)

$$\Delta^2 X_{\mathrm{est}}\Big|_{X=X_{\mathrm{true}}} \geq \frac{1}{\left\langle \left[\frac{\partial \ln p(\xi|X)}{\partial X}\right]^2 \right\rangle_{\mathrm{X=X_{true}}}}$$

The denominator of the last expression is the Fisher information, which can be written in different forms:

$$F(X) \equiv \left\langle \left[\frac{\partial \ln p(\xi|X)}{\partial X}\right]^2 \right\rangle = \int d\xi \, p(\xi|X) \left[\frac{\partial \ln p(\xi|X)}{\partial X}\right]^2$$

$$= \int d\xi \frac{1}{p(\xi|X)} \left[\frac{\partial p(\xi|X)}{\partial \xi}\right]^2 = \boxed{-\left\langle \frac{\partial^2}{\partial \xi^2} \ln p(\xi|X) \right\rangle}$$

Derivation of Cramér-Rao bound (3)

If one has now several identical and independent measurements, so that the probability distribution is $p(\vec{\xi}|X) = p(\xi_1|X)\cdots p(\xi_N|X)$, where $p(\xi_i|X)$ is the probability distribution for result ξ_i , if the value of the parameter is X. Let F(X) be the Fisher information corresponding to one of the measurements.

It is immediate then from

$$F^{(N)}(X) = -\left\langle \frac{\partial^2}{\partial\xi^2} \ln p(\vec{\xi}|X) \right\rangle$$

that

$$F^{(N)}(X) = NF(X)$$

This is the additivity property of the Fisher information. It follows from this the Cramér-Rao bound for unbiased estimators:

$$\boxed{\Delta X \geq \frac{1}{\sqrt{NF(X)|_{X=X_{\text{true}}}}} \quad \text{where} \quad \Delta X \equiv \sqrt{\Delta^2 X_{\text{est}}}}$$

Is it possible to saturate the Cramér-Rao inequality?

Remember that the Cramér-Rao inequality was obtained by applying the Cauchy-Schwarz inequality $|\langle AB\rangle|^2 \leq \langle A^2\rangle\langle B^2\rangle$ to the expression

$$\int d\xi\, p(\xi|X) \overbrace{\frac{\partial \ln p(\xi|X)}{\partial X}}^{A} \overbrace{[X_{\text{est}}(\xi) - X]}^{B}$$

so that $\left|\int d\xi\, p(\xi|X) \frac{\partial \ln p(\xi|X)}{\partial X}[X_{\text{est}}(\xi) - X]\right|^2 \leq \left\langle \left[\frac{\partial \ln p(\xi|X)}{\partial X}\right]^2 \right\rangle \Delta^2 X_{\text{est}}$

Equality in this equation is reached if and only if $A(\xi, X) = c(X)B(\xi, X)$, where c(X) does not depend on ξ, but may depend on X:

$$\frac{\partial \ln p(\xi|X)}{\partial X} = c(X)[X_{\text{est}}(\xi) - X]$$

An unbiased estimator that attains the bound for all X can be found if and only if this condition is satisfied, for some functions $X_{\text{est}}(\xi)$ and c(X).

Is it possible to saturate the Cramér-Rao inequality?

Differentiating both sides of

$$\frac{\partial \ln p(\xi|X)}{\partial X} = c(X)[X_{\text{est}}(\xi) - X] \qquad [1]$$

with respect to X, taking the average over the measurement outputs, and using that $\langle X_{\text{est}}(\xi)\rangle = X$, one gets

$$c(X) = -\left\langle \frac{\partial^2 \ln p(\xi|X)}{\partial X^2} \right\rangle = F(X)$$

where F(X) is the Fisher information. Notice that a Gaussian $p(\xi|X)$, with width $\Delta X = 1/c,\ c > 0$ constant, satisfies condition [1]. This implies that Gaussians saturate the Cramér-Rao inequality, with $F = 1/\Delta X$.

Is it possible to saturate the Cramér-Rao inequality? (2)

What if $p(\xi|X)$ does not satisfy [1]? This question was answered by Fisher in a very general way, by introducing the maximum likelihood estimator.

The **maximum likelihood estimator** is the value of X that maximizes the probability distribution $p(\xi|X)$, or equivalently $\ln[p(\xi|X)]$:

$$X_{\mathrm{ML}}(\xi) = \mathrm{Arg\,max}\,[p(\xi|X)] = \mathrm{Arg\,max}\,\{\ln[p(\xi|X)]\}$$

In estimation theory, one assumes that the model-dependent mathematical expression for $p(\xi|X)$ is known as a function of the possible values of the parameter X as well as of the possible experimental results ξ. Therefore, it is possible, in principle, to calculate numerically the value of X leading to the maximum of $\ln[p(\xi|X)]$ for a data set ξ.

The Maximum Likelihood Estimator and Bayes' law

Bayes' law allows a simple interpretation of this procedure. We use that the joint probability $p(\xi, X)$ can be expressed in two different ways in terms of conditional probabilities and a priori probabilities $p(\xi)$ and $p(X)$:

$$p(\xi, X) = p(\xi|X)p(X) = p(X|\xi)p(\xi) \Rightarrow p(X|\xi) = \frac{p(\xi|X)p(X)}{p(\xi)}$$

$$= \frac{p(\xi|X)}{\int p(\xi|X)p(X)dX}p(X) = \frac{p(\xi|X)}{\int p(\xi|X)p(X)dX}$$

where $p(\xi|X)$ is the conditional probability that ξ is obtained, if the value of the parameter is X, and in the last step we have set $p(X) = 1$, which corresponds to the assumption that one does not know anything a priori about X.

Therefore, the value of X that leads to the maximum of $p(\xi|X)$ also maximizes $p(X|\xi)$: in this sense, it is the most probable value of X, considering the obtained experimental results.

More on the Maximum Likelihood Estimator

From the maximization condition, $X_{\rm ML}$ is the solution of $\left.\dfrac{\partial \ln p(\xi|X)}{\partial X}\right|_{X=X_{\rm ML}} = 0$

For N identical and independent measurements, with
$p(\vec{\xi}|X) = p(\xi_1|X)\cdots p(\xi_N|X)$, one can also write $\left.\dfrac{\partial \ln p(\vec{\xi}|X)}{\partial X}\right|_{X=X_{\rm ML}} = 0$,

so that, expanding $\ln p(\vec{\xi}|X)$ around its maximum:

$$\ln p(\vec{\xi}|X) = \ln\left[p(\vec{\xi}|X_{\rm ML})\right] + \frac{1}{2}\left(X - X_{\rm ML}\right)^2 \left(\frac{\partial^2 \ln[p(\vec{\xi}|X)]}{\partial X^2}\right)_{X=X_{\rm ML}} + \ldots$$

But
$$\left(\frac{\partial^2 \ln[p(\vec{\xi}|X)]}{\partial X^2}\right)_{X=X_{\rm ML}} = \sum_{i=1}^{N}\left(\frac{\partial^2 \ln[p(\xi_i|X)]}{\partial X^2}\right)_{X=X_{\rm ML}}$$

In the limit of large N the sum becomes

$$\lim_{N\to\infty}\sum_{i=1}^{N}\left(\frac{\partial^2 \ln[p(\xi_i|X)]}{\partial X^2}\right)_{X=X_{\rm ML}} = N\int d\xi\, p(\xi|X)\left(\frac{\partial^2 \ln[p(\xi|X)]}{\partial X^2}\right)_{X=X_{\rm ML}}$$
$$= -NF(X_{\rm ML})$$

The Maximum Likelihood Estimator - Conclusion

Therefore,

$$\lim_{N\to\infty} p(\vec{\xi}\,|X) = p(\xi|X_{\mathrm{ML}}) \exp\left[-NF(X_{\mathrm{ML}})(X - X_{\mathrm{ML}})^2/2\right]$$

Using Bayes' law, we get for $p(X|\vec{\xi}\,)$, after normalization:

$$\lim_{N\to\infty} p(X|\vec{\xi}\,) = \sqrt{\frac{NF(X_{\mathrm{ML}})}{2\pi}}\, e^{-NF(X_{\mathrm{ML}})(X-X_{\mathrm{ML}})^2/2}$$

This is a Gaussian, with $\langle X\rangle = X_{\mathrm{ML}}$ and variance

$$\Delta^2 X = \langle (X - X_{\mathrm{ML}})^2\rangle = \frac{1}{NF(X_{\mathrm{ML}})}$$

This implies that the Maximum Likelihood Estimator is an ideal estimator in the limit of large N, since it saturates the Cramér-Rao bound.

Sommaire de la deuxième leçon - 11 Février

In the next lecture, we discuss the quantum theory of parameter estimation, which leads to the ultimate precision bounds allowed by quantum mechanics, and may also lead to the best measurement procedure for each incoming state of the probe. For some problems, it may lead to analytical expressions or good approximations for the best states. We emphasize the role of entanglement in increasing the precision of estimation, and apply the quantum theory of metrology to optical interferometry, atomic spectroscopy, and weak-value amplification.