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On the values of the characters of compact Lie groups

The lecture discussed three loosely related theorems on the characters of a compact Lie group G.

1. A generalization of a theorem of Burnside

Theorem 1 – *Let χ be the character of an irreducible complex representation of G. Assume $\chi(1) > 1$. Then there exists an element x of G, of finite order, with $\chi(x) = 0$.*

When G is finite, this is a well-known result of Burnside.

2. Positive characters with mean value equal to 1

Let f be a virtual character of G having the following two properties :

1. $f(x)$ is real ≥ 0 for every $x \in G$.
2. The mean value $< f, 1 >$ of f is equal to 1.

There are many examples of such characters when G is finite (e.g. permutation characters relative to a transitive action). Not so when G is connected. More precisely :

Question – If G is connected and simply connected, is it true that every character f having properties (a) and (b) is equal to $\chi.\bar{\chi}$, where χ is an irreducible (complex) character of G ?

Theorem 2 – *The answer to the question above is* "yes" *when G is of rank 1, i.e. when $G = \mathbf{SU}_2(\mathbf{C})$.*

3. The character of the adjoint representation

Consider the adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}\ (\mathrm{Lie}\ G)$.

Theorem 3 – *One has* $\mathrm{Tr}\ \mathrm{Ad}(x) \geq -\ \mathrm{rank}(G)$ *for every $x \in G$.*

The minimal value of $\mathrm{Tr}\ \mathrm{Ad}(x)$ can be determined explicitly :

Choose a maximal torus T of G ; let N be its normalizer and let W be the quotient N/T (so that W is the Weyl group if G is connected). For any $w \in W$, let $\mathrm{Tr}_T(w)$ be the trace of w acting on Lie T. Theorem 3 can be refined as :

Theorem 3′ – *One has* $\inf_{x \in G} \text{Tr} \quad \text{Ad}(x) = \inf_{w \in W} \text{Tr}_T(w)$.

This shows in particular that inf Tr Ad(x) is *an integer*, a fact which was not *a priori* obvious. It also shows that inf Tr Ad(x) is equal to $-$ rank(G) if and only if W contains an element which acts on T by $t \mapsto t^{-1}$.

When G is connected and simple, Theorem 3′ gives :

$$\inf \text{ Tr Ad}(x) = -\text{ rank}(G) \text{ if } G \text{ is of type } A_1, B_n, C_n, D_n \ (n \text{ even}), G_2, F_4, E_7, E_8 \ ,$$

$$\inf \text{ Tr Ad}(x) = \begin{cases} -1 & \text{if } G \text{ is of type } A_n \ (n \geq 1) \\ 2-n & \text{if } G \text{ is of type } D_n \ (n \text{ odd} \geq 3) \\ -3 & \text{if } G \text{ is of type } E_6 \ . \end{cases}$$

4. Proofs

They are not published yet. Here are some hints for the interested reader :

Theorem 1 : Use the properties of the "principal A_1 subgroup" of G.

Theorem 2 : An exercise on positive-valued trigonometric polynomials.

Theorem 3′ : If $w \in W$ is such that $\text{Tr}_T(w)$ is minimum, any representative x of w in N is such that Tr Ad$(x) = \text{Tr}_T(w)$; this proves the inequality inf Tr Ad$(x) \leq$ inf $\text{Tr}_T(w)$. The opposite inequality can be checked by a case-by-case explicit computation. The classical groups are easy enough, but F_4, E_6, E_7 and E_8 are not (especially E_6, which I owe to Alain Connes). I hope there is a better proof.

Bibliography

W. Burnside – *On an arithmetical theorem connected with roots of unity and its application to group characteristics*, Proc. London Math. Soc. **1** (1903), 112-116.