Vortices in rotating Bose Einstein condensates

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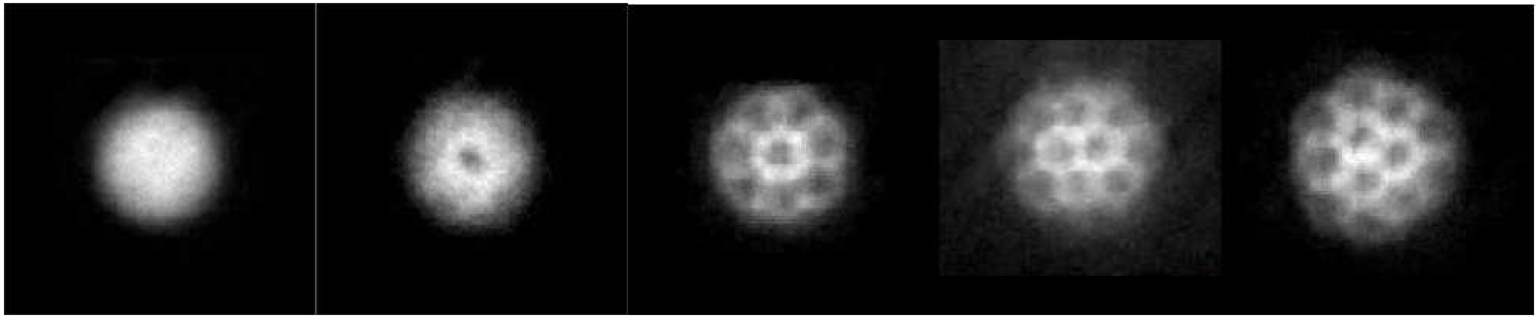
Outline

- Bose-Einstein condensates – rotation
- Model: Gross-Pitaevskii energy
  - Reduction to a 2D model
  - Reduction to the lowest Landau level (LLL)
- LLL problem
  - Numerics
  - Bargmann transform and Bargmann spaces
  - Θ functions
- Supersolid models

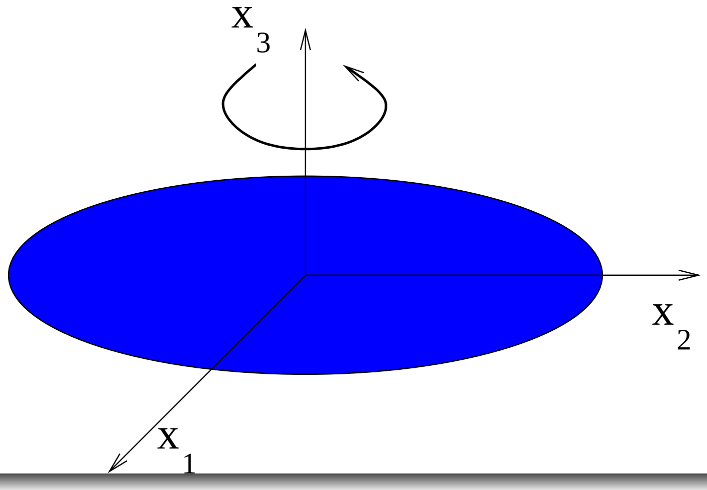
Experiment: rotation

The condensate is rotated along the x_3 axis.

- Ω small: no modification is observed
- Ω larger: vortices are nucleated



Dalibard et al., PRL **84**, 806 (2000)



Model: Gross-Pitaevskii energy

- In a BEC, atoms are described by a collective wave function ψ[a]
- The corresponding energy is

$$
\begin{aligned}
E_{3D}(\psi) &= \int_{\mathbb{R}^3} \frac{1}{2}|\nabla\psi|^2 - i\overrightarrow{\Omega}\cdot\left(x\times\overline{\nabla\psi}\right)\psi + \frac{1}{2}|x|^2|\psi|^2 + \frac{1}{2}G|\psi|^4, \\
&= \int_{\mathbb{R}^3} \frac{1}{2}|\nabla\psi - i\Omega r^{\perp}\psi|^2 + \frac{1}{2}(1-\Omega^2)|r|^2|\psi|^2 + \frac{1}{2}G|\psi|^4 \\
&\quad + \frac{1}{2}|\partial_3\psi|^2 + \frac{1}{2}x_3^2|\psi|^2,
\end{aligned}
$$

with the constraint $\int_{\mathbb{R}^3}|\psi|^2 = 1$, and $r = (x_1, x_2)$.

[a] see Lieb-Seiringer 2005, Lieb-Seiringer-Yngvason 2000.

2D Gross-Pitaevskii energy

$$
\begin{aligned}
E_{3D}(\psi) &= \int_{\mathbb{R}^3} \frac{1}{2}|\nabla\psi - i\Omega r^\perp \psi|^2 + \frac{1}{2}(1-\Omega^2)|r|^2|\psi|^2 + \frac{1}{2}G|\psi|^4 \\
&\quad + \frac{1}{2}|\partial_3\psi|^2 + \frac{1}{2}x_3^2|\psi|^2, \\
E_{2D}(\phi) &= \int_{\mathbb{R}^2} \frac{1}{2}|\nabla\phi - i\Omega r^\perp \phi|^2 + \frac{1}{2}(1-\Omega^2)|r|^2|\phi|^2 + \frac{1}{2}\tilde{G}|\phi|^4,
\end{aligned}
$$

where $\tilde{G} = G\int_{\mathbb{R}} \xi^4, \quad \xi(x_3) = \dfrac{e^{-\frac{1}{2}{x_3}^2}}{(2\pi)^{1/4}}$.

$$
\psi(x_1,x_2,x_3) = \phi(x_1,x_2)\xi(x_3) \Rightarrow E_{3D}(\psi) = \frac{1}{2} + E_{2D}(\phi).
$$

2D Gross-Pitaevskii energy

Theorem (A. Aftalion, XB): Let

$$I_{3D}(\Omega) = \inf \left\{ E_{3D}(\phi), \quad \int_{\mathbb{R}^3} |\phi|^2 = 1 \right\},$$

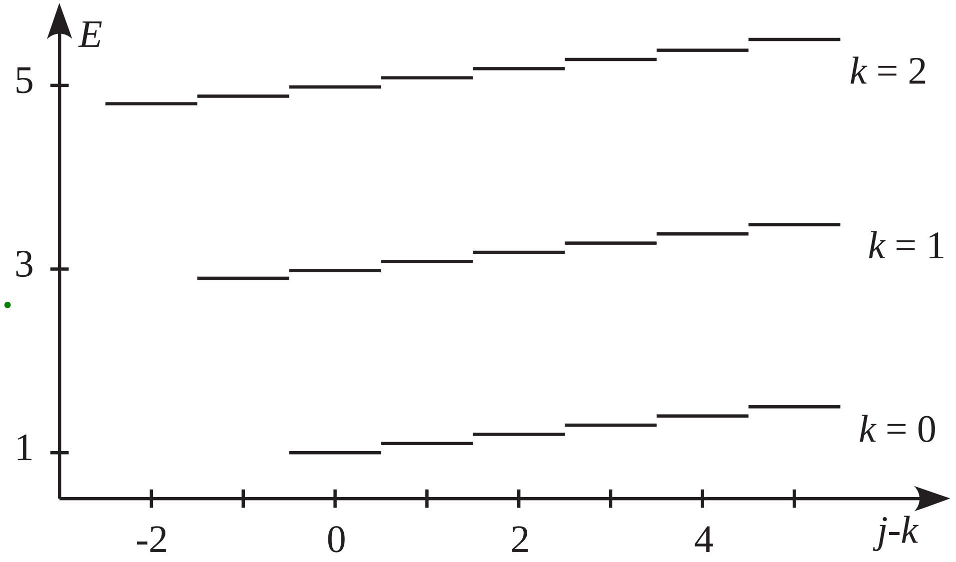
$$I_{2D}(\Omega) = \inf \left\{ E_{2D}(\phi), \quad \int_{\mathbb{R}^2} |\phi|^2 = 1 \right\},$$

Then,
$I_{3D}(\Omega) = \frac{1}{2} + I_{2D}(\Omega) + o(\sqrt{1-\Omega})$, $I_{2D}(\Omega) = 1 + O(\sqrt{1-\Omega})$,
and

$\phi^{3D}(x_1, x_2, x_3) = \eta(x_1, x_2)\xi(x_3) + w(x_1, x_2, x_3)$,
with $\|w\|_{C^{0,1/2}} + \|w\|_{L^2} \longrightarrow 0.$

Lowest Landau Level

$$H_\Omega = -\frac{1}{2}\Delta + \frac{1}{2}|r|^2 - i\Omega(x_2\partial_{x_1} - x_1\partial_{x_2})$$



Eigenvalues:
$E_{j,k} = 1 + (1-\Omega)j + (1+\Omega)k.$
$k = 0$: lowest Landau level.

Energy bounded below $\iff \Omega < 1.$

$\boxed{\text{We study } \Omega \xrightarrow[\Omega<1]{} 1.}$

LLL approximation

$$E_{2D}(\phi) = \int_{\mathbb{R}^2} \frac{1}{2}|\nabla\phi - i\Omega r^{\perp}\phi|^2 + \frac{1-\Omega^2}{2}|r|^2|\phi|^2 + \frac{1}{2}\tilde{G}|\phi|^4,$$

Lowest Landau level: first eigenvectors of $H_\Omega = -\frac{1}{2}\Delta + \frac{1}{2}|r|^2 - \Omega L$

$$LLL = \left\{ \phi(z) = f(z)e^{-|z|^2/2}, \quad f \text{ holomorphic, } \phi \in L^2(\mathbb{R}^2) \right\}.$$

$$z = x_1 + ix_2, \quad r = (x_1, x_2).$$

$$\phi \in LLL \quad \Rightarrow \quad E_{2D}(\phi) = \Omega + \int_{\mathbb{R}^2} \frac{1-\Omega^2}{2}|r|^2|\phi|^2 + \frac{1}{2}\tilde{G}|\phi|^4.$$

LLL approximation

Theorem (A. Aftalion, XB): Let

$$I_{2D}(\Omega) = \inf \left\{ E_{2D}(\phi), \quad \int_{\mathbb{R}^2} |\phi|^2 = 1 \right\},$$

$$I_{LLL}(\Omega) = \inf \left\{ E_{2D}(\phi), \quad \phi \in LLL, \quad \int_{\mathbb{R}^2} |\phi|^2 = 1 \right\}.$$

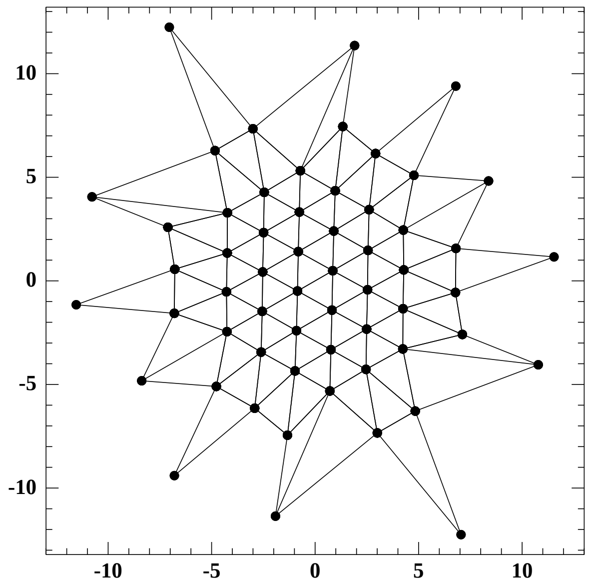
Then, $I_{2D}(\Omega) = I_{LLL}(\Omega) + o(\sqrt{1-\Omega})$.
Moreover,

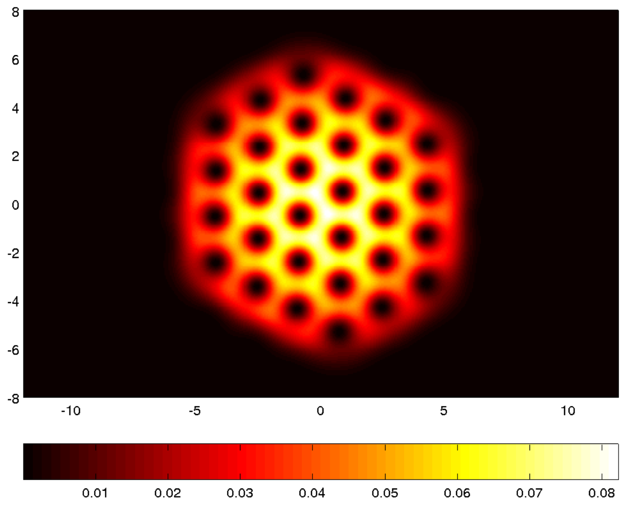
$$\left\| \phi^{2D} - \Pi_{LLL}\left(\phi^{2D}\right) \right\|_{C^{0,1/2}} + \left\| \phi^{2D} - \Pi_{LLL}\left(\phi^{2D}\right) \right\|_{L^2} \longrightarrow 0.$$

Numerics

In the LLL, the wave function is defined by its zeroes:

$$\phi(z) = A \prod_{i=1}^{n} (z - z_i) e^{-\frac{|z|^2}{2}}, \quad \int_{\mathbb{R}^2} |\phi|^2 = 1.$$





$\Omega = 0.999$, $G = 3$, $n = 58$

Scaling

$$E(\phi) = \int_{\mathbb{R}^2} \frac{1}{2}|\nabla\phi - i\Omega x^\perp \phi|^2 + \frac{1}{2}|x|^2|\phi|^2(1-\Omega^2) + \frac{1}{2}G|\phi|^4$$

Rescaling: $\phi(x) = \gamma u(\gamma x)$, with $\gamma = \sqrt{\Omega}\left(\frac{1-\Omega^2}{2}\right)^{\frac{1}{4}}$.

$$\begin{aligned} E(\phi) &= \int_{\mathbb{R}^2} \left|\nabla u - i\left(\frac{2}{1-\Omega^2}\right)^{\frac{1}{2}} x^\perp u\right|^2 \\ &\quad + \sqrt{\frac{1-\Omega^2}{2\Omega^2}}\left(\int_{\mathbb{R}^2} |x|^2|u|^2 + \frac{G\Omega^2}{2}|u|^4\right) \end{aligned}$$

Set $h = \sqrt{\frac{1-\Omega^2}{2\Omega^2}}$, small parameter.

Bargmann space

LLL-functional:

$$E^h(f) = \int_{\mathbb{C}} |z|^2 \, |u(z)|^2 + \frac{G\Omega_h^2}{2} \, |u|^4$$

$$h \longrightarrow 0 \qquad\qquad \Omega_h \longrightarrow 1$$

(Fock-)Bargmann space:

$$u(z) = e^{-\frac{|z|^2}{2h}} f(z)$$

$$f \in \mathcal{F}_h = \left\{ g \in L^2(\mathbb{C}; e^{-\frac{|z|^2}{h}} dz), \; g \text{ holomorphic} \right\}$$

$$\langle f | g \rangle = \int f(z) \overline{g(z)} e^{-\frac{|z|^2}{h}}.$$

Bargmann transform

Bargmann transform[a]

$$[B_h\varphi](z) = \frac{1}{(\pi h)^{3/4}} e^{\frac{z^2}{2h}} \int_{\mathbb{R}} e^{-\frac{(\sqrt{2}z-y)^2}{2h}} \varphi(y)\, dy.$$

$$B_h : L^2(\mathbb{R}, dy) \xrightarrow{unitary} \mathcal{F}_h \quad \subset L^2(\mathbb{C}; e^{-\frac{|z|^2}{h}} dz).$$

$$B_h^* B_h = \mathrm{Id}_{L^2(\mathbb{R};dy)} \qquad B_h B_h^* = \Pi_h\,.$$

Szegö projector :

$$(\Pi_h f)(z) = \frac{1}{\pi h} \int_{\mathbb{C}} e^{\frac{z\overline{z'} - |z'|^2}{h}} f(z') dz'.$$

[a]REF: V. Bargmann (61), G.B. Folland (89), A. Martinez (02)

Bargmann transform

$$\begin{array}{rcl}
\text{on } \mathcal{F}_h & & \text{on } L^2(\mathbb{R}, dy) \\
\Pi_h z \Pi_h = z & \leftrightarrow & \dfrac{1}{\sqrt{2}}(-h\partial_y + y) = a_h^* \\
\Pi_h \overline{z} \Pi_h = h\partial_z & \leftrightarrow & \dfrac{1}{\sqrt{2}}(h\partial_y + y) = a_h \\
N_h = z(h\partial_z) = \Pi_h[|z|^2 - h]\Pi_h & \leftrightarrow & \dfrac{1}{2}(-h^2\partial_y^2 + y^2 - 1) \\
\dfrac{1}{(\pi h)^{1/2} h^{n/2}\sqrt{n!}} z^n & \leftrightarrow & H_n^h = \dfrac{1}{h^{n/2}\sqrt{n!}}(a_h^*)^n H_0, \\
 & & H_0(y) = (\pi h)^{-1/4} e^{-y^2/h}\,. \\
\Pi_h \alpha(z) \Pi_h \ (\text{Toeplitz}) & \leftrightarrow & \alpha^{A-Wick}(y, hD_y)\,.
\end{array}$$

Bargmann space

spaces $\mathcal{F}_h^s$, $s \in \mathbb{R}$:

$$\mathcal{F}_h^s = \left\{ f \text{ holomorphic} \, , \left((1+|z|^2)^{s/2} f \in L^2(\mathbb{C}; e^{-\frac{|z|^2}{h}} dz) \right) \right\}$$
$$= \{ f \text{ holomorphic} \, , ((1+N_h)^s f \in \mathcal{F}_h) \} \, .$$

The imbedding $\mathcal{F}_h^s \subset \mathcal{F}_h$ is compact as soon as $s > 0$.

$$E^h(f) = \langle f \, | \, (N_h + h) f \rangle_{\mathcal{F}_h} + \frac{G\Omega_h^2}{2} \int_{\mathbb{C}} |u|^4 \, dz$$

$$f \in \mathcal{F}_h^1 \qquad u(z) = f(z) e^{-\frac{|z|^2}{h}}$$

Bargmann space

Consequence:

For $h > 0$ the minimization problem ($u(z) = f(z)e^{-\frac{|z|^2}{2h}}$)

$$\inf_{\|f\|_{\mathcal{F}_h}=1} E^h(f); \quad E^h(f) = \int_{\mathbb{C}} |z|^2 \, |u(z)|^2 + \frac{G\Omega_h^2}{2} \, |u|^4 \, dz \, .$$

admits a solution. A solution belongs to $\mathcal{F}_h^1$.

$$\frac{2\Omega_h}{3}\sqrt{\frac{2G}{\pi}} < \min_{\|f\|_{\mathcal{F}_h}=1} E^h(f) \leq \frac{2\Omega_h}{3}\sqrt{\frac{2Gb}{\pi}} + O(h^{1/4}) \, .$$

$$(b \sim 1.1596)$$

Lower bound

Forget the holomorphic constraint:

$$\inf\left\{\int_{\mathbb{C}}|r|^2|u|^2+\frac{G\Omega_h}{2}|u|^4,\quad u\in L^2\cap L^4,\quad \int_{\mathbb{C}}|u|^2=1\right\}.$$

Unique minimizer up to a phase:

$$u_{\min}(r)=\frac{1}{\sqrt{G\Omega_h}}\left(\lambda-|r|^2\right)_+^{1/2},\qquad \lambda=\sqrt{\frac{2G\Omega_h}{\pi}}$$

energy

$$E=\frac{2\Omega_h}{3}\sqrt{\frac{2G}{\pi}}.$$

Euler-Lagrange equation

$$zh\partial_z f + G\Omega_h^2 \Pi_h \left(e^{-\frac{|z|^2}{h}} |f|^2 \right) \Pi_h f - (\lambda - h) f = 0$$

$$\Pi_h \left[\left(|z|^2 + G\Omega_h^2 e^{-\frac{|z|^2}{h}} |f|^2 - \lambda \right) f \right] = 0$$

or $$zh\partial_z f + \frac{G\Omega_h^2}{2} \bar{f}(h\partial_z)[f^2(2^{-1}.)] - (\lambda - h) f = 0$$

Theorem (A. Aftalion, XB, F. Nier):

Any minimizer has an infinite number of zeros as soon as

$$h(1 + O_G(h^{1/4})) < \frac{729}{1024} \frac{\sqrt{G}}{(2\pi b)^{3/2}} c_0^2 \approx 0.04276\sqrt{G}.$$

Approximation by polynomials.

$$
\begin{aligned}
e^h_{LLL} &= \min_{\|f\|_{\mathcal{F}_h}=1} E^h(f) \\
e^h_{LLL,K} &= \min_{P\in\mathbb{C}_K[z],\ \|P\|_{\mathcal{F}_h}=1} E^h(P) \\
& \Pi_{h,K} = 1_{[0,hK]}(N_h) \text{ orth. proj. on } \mathbb{C}_K[z].
\end{aligned}
$$

Theorem (A. Aftalion, XB, F. Nier):

- $0 < e^h_{LLL,K} - e^h_{LLL} \leq \frac{C}{h^2K}$, and $\|f - \Pi_{h,K} f\|_{\mathcal{F}_h} \leq \frac{C}{h^2K}$, when f is a minimizer.
- If P_K is a minimizer of $e^h_{LLL,K}$, then after extraction

$$\lim_{n\to\infty} \|f - P_{K_n}\|_{\mathcal{F}_h} + \|N_h(f - P_{K_n})\|_{\mathcal{F}_h} = 0\,.$$

Theta function

Definition:

$$\Theta(v,\tau) = \frac{1}{i}\sum_{n=-\infty}^{+\infty}(-1)^n e^{i\pi\tau(n+1/2)^2} e^{(2n+1)\pi i v}.$$

Proposition: If f is holomorphic and cancels on each site of the lattice $\frac{1}{\delta}\mathbb{Z}\oplus\frac{\tau}{\delta}\mathbb{Z}$, with $\delta\in\mathbb{R}^{+*}$ and $\tau\in\mathbb{C}\setminus\mathbb{R}$, and if the function $|u(z)| = |f(z)|e^{-\frac{|z|^2}{h}}$ is periodic, then

$$\delta = \sqrt{\frac{\tau_I}{\pi h}} \quad \text{and} \quad f(z) = e^{\frac{z^2}{2h}}\Theta\left(\sqrt{\frac{\tau_I}{\pi h}}z,\tau\right).$$

Abrikozov problem

Abrikosov: $\inf \left\{ \frac{\fint |u|^4}{\left(\fint |u|^2\right)^2}, \quad u(z) = f(z)e^{-|z|^2/2h}, \quad f \in \mathcal{F}_h \right\}.$

Euler-Lagrange equation: $\Pi_h\left(|u|^2 f\right) = \lambda f.$

LLL problem: $zh\partial_z f + G\Omega_h^2 \Pi_h \left(|u|^2 f\right) = (\lambda - h)f.$

Lattice $\mathbb{Z} \oplus \tau\mathbb{Z}$:

$$\gamma(\tau) := \frac{\fint |u_\tau|^4}{\left(\fint |u_\tau|^2\right)^2}, \quad u_\tau(z) = e^{\frac{z^2}{2h}} \Theta\left(\sqrt{\frac{\tau_I}{\pi h}} z, \tau\right) e^{-\frac{|z|^2}{2h}}.$$

γ has a unique minimizer[a] at $\tau = e^{\frac{2i\pi}{3}}$

Open problem: prove that the (global) minimizer is a lattice.

[a] Nonnenmacher, Voros, 1998.

Limit $h \to 0$

$$f(z) = e^{\frac{z^2}{2h}}\Theta\left(\sqrt{\frac{\tau_I}{\pi h}}z,\tau\right), \quad \gamma(\tau) = \frac{\int_{Q_\tau}|\Theta|^4}{\left(\int_{Q_\tau}|\Theta|^2\right)^2}, \quad \boxed{f_\alpha = \Pi_h\left(\alpha f\right)\ ,}$$

where $\alpha \in \mathcal{C}^{0,\beta}(\mathbb{C})$ with compact support.

$$(N_h + h - \lambda)f_\alpha + G\Omega_h^2\Pi_h\left(e^{-\frac{|z|^2}{h}}|f_\alpha|^2 f_\alpha\right) =$$

$$\Pi_h\left[\left(|z|^2 - \lambda + \gamma(\tau)G|\alpha|^2\right)\alpha\right]\Pi_h f + O_{\mathcal{F}_h}(h^{\beta/2})\,,$$

and $E^h(f_\alpha) = \int_{\mathbb{C}} |r|^2|\alpha|^2 + \gamma(\tau)\frac{G}{2}|\alpha|^4 + O\left(h^{\beta/2}\right).$

Minimum for $|\alpha| = \frac{1}{G\gamma(\tau)}\left(\sqrt{\frac{G\gamma(\tau)}{\pi}} - |r|^2\right)_+^{1/2}$ and $\tau = e^{2i\pi/3}$.

Extension: quartic trap

Change the potential:

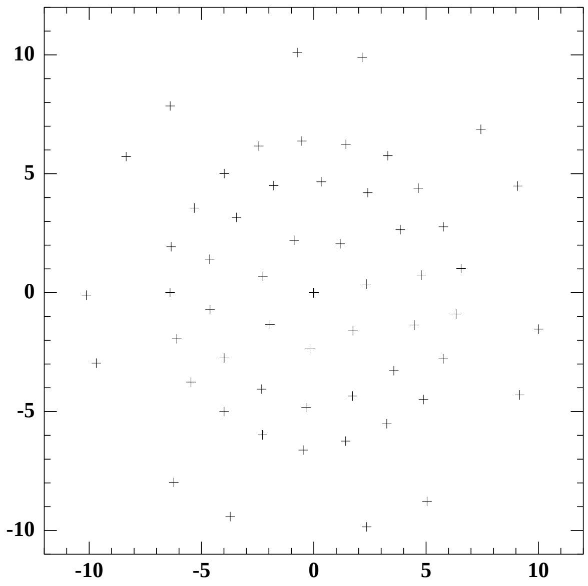
$$E_{2D}(\phi) = \int_{\mathbb{R}^2} \frac{1}{2}|\nabla\phi - i\Omega r^{\perp}\phi|^2 + \frac{1}{2}(1-\Omega^2)|r|^2|\phi|^2 + \frac{k}{2}|r|^4|\phi|^2 + \frac{1}{2}G|\phi|^4,$$

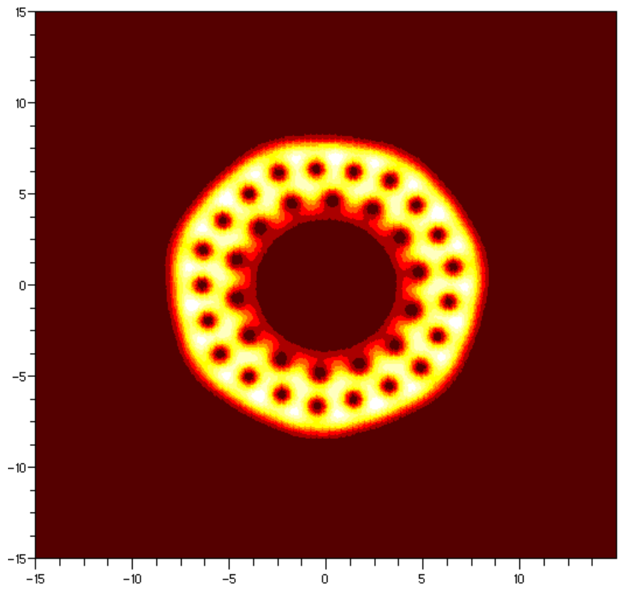
$\boxed{\Omega > 1 \text{ is allowed.}}$

If $k \propto (\Omega - 1)^{3/2}$, the LLL approximation is valid, and reduces to

$$E^h(u) = \int_{\mathbb{C}} (k_0|z|^4 - |z|^2)|u|^2 + \frac{G}{2}|u|^4, \quad u(z) = f(z)e^{-\frac{|z|^2}{2h}}.$$

Extension: quartic trap





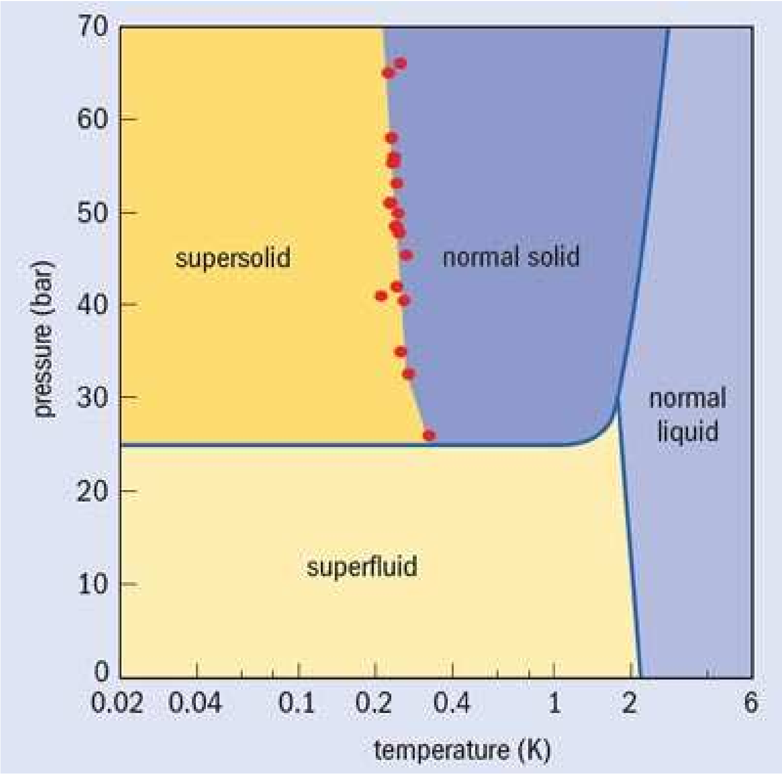
Numerical simulations (N. Rougerie) for
$G = 3, k_0 = 1, \Omega = 1 + 2.15 \times 10^{-3}$.

Same ansatz: Θ function and

$$|\alpha(z)|^2 = \frac{1}{G}\left(\lambda + |z|^2 - k_0|z|^4\right)_+.$$

Supersolid models

Crystalline Helium exhibits superfluid properties: "supersolid"



Experiments by Kim and Chan, *Nature* **427**, 2004.

Penrose-Onsager (1956), Andreev-Lifshitz (1969), Leggett (1970), Prokof'ev and Svistunov (2005), Balibar et al (2007), ...

Supersolid models

Gross-Pitaevskii with nonlocal interaction (Josserand, Pomeau, Rica, PRL **98**, 2007):

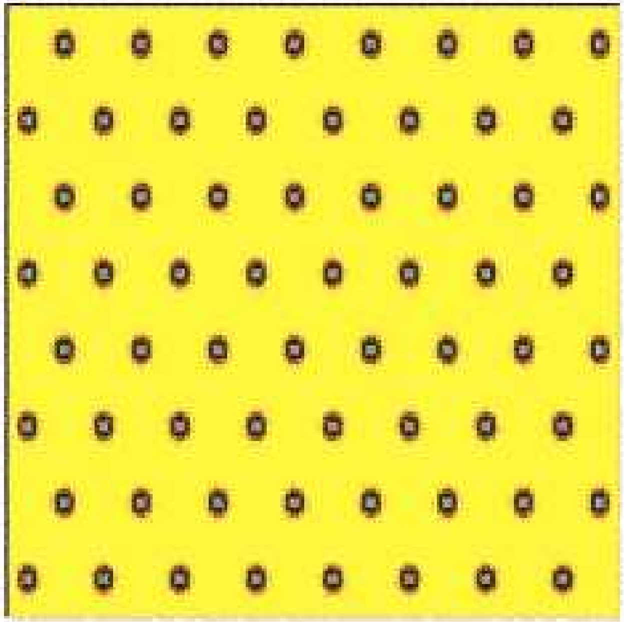
$$E(\phi) = \int_{\mathcal{D}} \frac{1}{2}|\nabla\phi|^2 + \frac{G}{2}\left(V * |\phi|^2\right)|\phi|^2.$$

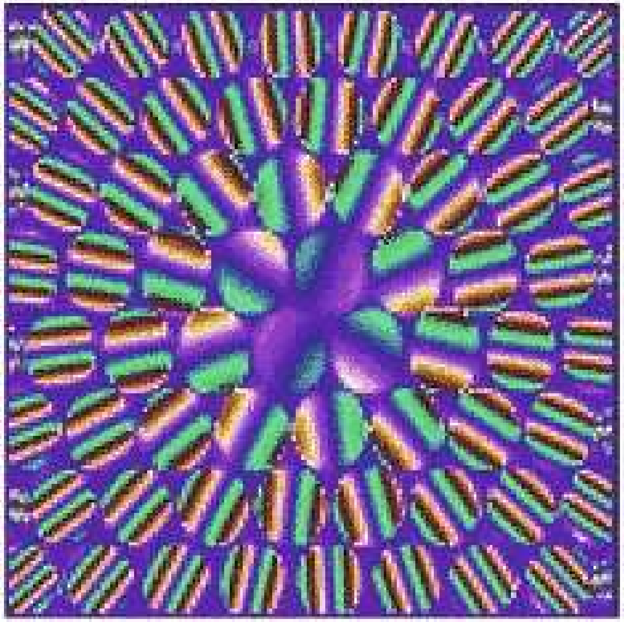
$$V(x) = \mathbf{1}_{B_1}(x) = \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

$$\inf\left\{E(\phi), \quad \phi \in H^1_{\mathrm{per}}(\mathcal{D}), \quad \int_{\mathcal{D}} |\phi|^2 = |\mathcal{D}|\right\}.$$

Supersolid models

Numerical simulation of G-P at 2D





Supersolid models

Theorem (A. Aftalion, XB, R. L. Jerrard): Define

$$n(\mathcal{D}) := \max\{k : \exists x_1, \ldots, x_k \in \mathcal{D} \text{ such that } |x_i - x_j| > 1 \ \ \forall i \neq j\}.$$

Then $$\inf\left\{\int_{\mathcal{D}} (V * \rho)\rho,\ \rho \geq 0,\ \int_{\mathcal{D}} \rho = |\mathcal{D}|\right\} = |\mathcal{D}|^2/n(\mathcal{D}),$$
and ρ is a minimizer iff there exist $n(\mathcal{D})$ pairwise disjoint closed sets $A_1, \ldots, A_{n(\mathcal{D})} \subset \bar{\mathcal{D}}$, such that

$$\mathrm{dist}\,(A_i, A_j) \geq 1 \text{ if } i \neq j, \quad \text{and} \quad \forall i, \int_{A_i} \rho\, dx = \frac{|\mathcal{D}|}{n(\mathcal{D})}$$

Supersolid models

Theorem (A. Aftalion, XB, R. L. Jerrard):

As $G \to \infty$, any subsequence of minimizers of

$$\inf \left\{ \int_{\mathcal{D}} \frac{1}{2} |\nabla \phi|^2 + \frac{G}{2} \left(V * |\phi|^2 \right) |\phi|^2, \quad \int_{\mathcal{D}} |\phi|^2 = |\mathcal{D}| \right\}$$

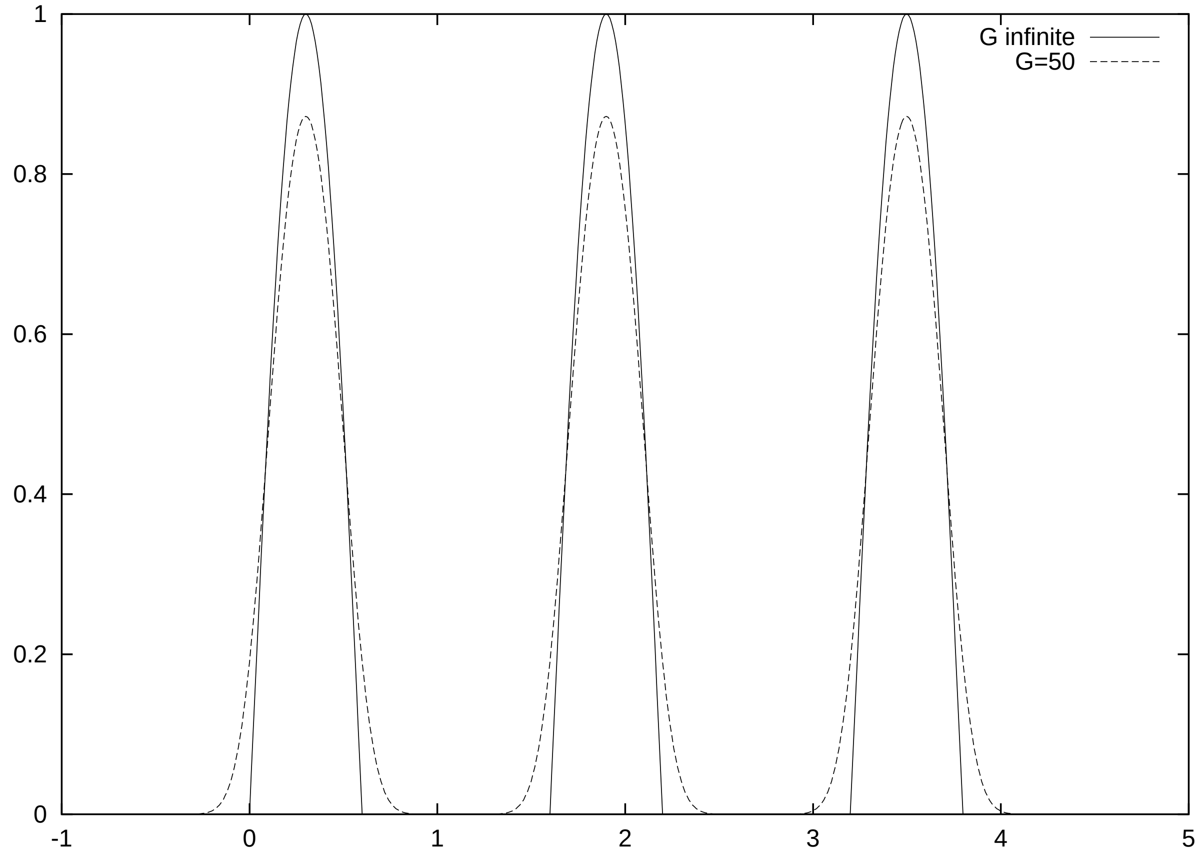
is relatively compact in $H^1(\mathcal{D})$. Any limit point is a minimizer of the Dirichlet energy in the set of minimizers of the interaction.

In dimension 1, we thus have, up to translation ($n = n(\mathcal{D})$)

$$\phi_0(x) = \begin{cases} \sqrt{\frac{2L}{h_0 n}} \sin\left(\frac{\pi}{h_0}(x - x_i)\right) & \text{if } x \in (x_i, x_i + h_0),\ 0 \le i \le n-1, \\ 0 & \text{if not} \end{cases}$$

$$n = \lceil |\mathcal{D}| \rceil, \qquad h_0 = (|\mathcal{D}| - (n-1))/n, \qquad x_i = i(1 + h_0).$$

Supersolid models



Perspectives

- Quartic trap : large rotation $\Omega \to \infty$.
- Abrikosov problem
- Anisotropic trap: $H_\Omega = -\frac{1}{2}\Delta + \frac{1}{2}x_1^2 + \frac{1+\nu^2}{2}x_2^2 - \Omega L$
- "very fast" rotation: $N\sqrt{1-\Omega} = O(1)$ correlation effects.
- Prove crystallization for the supersolid model.
- Rotation for the supersolid model.