Target problems, Second order BSDEs, and probabilistic numerical methods for fully nonlinear PDEs

Nizar TOUZI
Ecole Polytechnique Paris
Collège de France
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Outline

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   - Second order target problems

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   - Second order BSDEs

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   - Monte Carlo Simulation of BSDEs
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The standard model in frictionless markets

- \((\Omega, \mathcal{F}, \mathbb{P}), W\) Brownian motion in \(\mathbb{R}^d\), \(\mathcal{F} = \{\mathcal{F}_t, t \geq 0\} = \mathcal{F}^W\)
- The financial market consists of a riskless asset \(S^0 \equiv 1\), and a risky asset with price process

\[
dS_t = \text{diag}[S_t] (\mu_t dt + \sigma_t dW_t)
\]

\(\mu, \sigma\) adapted, \(\sigma\) invertible + ...
- Portfolio \(Z^i_t\): amount invested in asset \(i\) time \(t\):

\[
\{Z_t, t \geq 0\} \quad \mathcal{F} - \text{adapted with values in } \mathbb{R}^d
\]

- Self-financing condition \(\implies\) dynamics of portfolio value:

\[
dY_t = Z_t \cdot \text{diag}[S_t]^{-1} dS_t
\]

Super-hedging problem of \(\mathcal{F}_T\)-measurable \(G \geq 0\)

\[
V_0 := \inf \{Y_0 : Y_T \geq G \text{ a.s. for some } Z \in \mathcal{A}\}
\]
Solution: the Black-Scholes model

- We may assume $\mu \equiv 0$: equivalent change of measure

- Then for $Y_0 > V_0$, $\mathbb{E}[Y_T] \geq \mathbb{E}[G] \implies V_0 \geq \mathbb{E}[G]$

- From the martingale representation in Brownian filtration

  \[ \hat{Y}_t := \mathbb{E}[G | \mathcal{F}_t] = \mathbb{E}[G] + \int_0^T \phi_t \cdot dW_t = \hat{Y}_0 + \int_0^T \hat{Z}_t \cdot \sigma_t dW_t \]

  Since $Y_T = G$, we deduce that $\mathbb{E}[G] \geq V_0$

  **Hence** $V_0 = \mathbb{E}[G]$ and $Y_T = G$ a.s. for some portfolio $Z \in \mathcal{A}$
Stochastic target problems

- Controlled process

\[ dX_t = \mu(t, X_t, \nu_t)dt + \sigma(t, X_t, \nu_t)dW_t \]

where the control process \( \nu \in \mathcal{U} \) takes values in \( \mathcal{U} \subset \mathbb{R}^k \)

- Given a Borel set \( \Gamma_0 \subset \mathbb{R}^d \), find

\[ \mathcal{V}_0 := \{ X_0 \in \mathbb{R}^d : X_T \in \Gamma_0 \text{ for some } \nu \in \mathcal{U} \} \]

- If \( X = (S, Y) \in \mathbb{R}^{n-1} \times \mathbb{R} \) where \( Y \) is increasing in \( Y_0 \), find

\[ V_0 := \inf \{ Y_0 : X_T = (S_T, Y_T) \in \Gamma_0 \text{ for some } \nu \in \mathcal{U} \} \]
Main ingredient for target problems

- Define the dynamic problems $\mathcal{V}_t$ and $V_t$

**Geometric Dynamic Programming** for any stopping time $\theta$ valued in $[t, T]$

$$\mathcal{V}_t = \{ X_t : X_\theta \in \mathcal{V}_\theta \text{ for some } \nu \in \mathcal{U} \}$$

if $Y$ is increasing in $Y_0$:

**Geometric Dynamic Programming** for any stopping time $\theta$ valued in $[t, T]$

$$V_t = \inf \{ Y_t : Y_\theta \geq V_\theta \text{ for some } \nu \in \mathcal{U} \}$$
Dynamic Programming Equation for $V$

- If $U = \mathbb{R}^k$. Assume that $V$ is locally bounded. Then $V(t, s)$ is a (discontinuous) viscosity solution of

$$
- \frac{\partial V}{\partial t}(t, s) - \mathcal{L}^{\nu_0(t,s)} V(t, s) + \mu^Y(t, s, V(t, s), \nu_0(t, s)) = 0
$$

where

$$
\mathcal{L}^\nu V(t, s) = \mu^S(t, s, V(t, s), \nu) \cdot DV(t, s) + \frac{1}{2} \text{Tr} \left[ \sigma^S \sigma^S \ast D^2 V(t, s) \right]
$$

and

$$
\sigma^Y(t, s, V(t, s), \nu_0(t, s)) = \sigma^S(t, s, V(t, s), \nu_0(t, s)) \cdot DV(t, s)
$$

- If $U \neq \mathbb{R}^k$: similar PDE, with gradient constraint, boundary layer...
Dynamic Programming equation for $\mathcal{V}$

Set $u(t, x) := \mathbb{1}_{\mathcal{V}(t) \in \mathcal{C}}(x)$

**Theorem** Under some conditions, $u$ is a (discontinuous) viscosity solution of the geometric equation

$$-rac{\partial v}{\partial t}(t, x) + F(t, x, Dv(t, x), D^2v(t, x)) = 0$$

where

$$F(t, x, p, A) = \sup \left\{ \mu(t, x, \nu) \cdot p + \frac{1}{2} \text{Tr} \left[ \sigma \sigma^T(t, x, \nu) A \right] : \nu \in \mathcal{N}(t, x, p) \right\}$$

and

$$\mathcal{N}(t, x, p) := \left\{ \nu \in U : \sigma(t, x, \nu)^T p = 0 \right\}$$

$\implies$ Stochastic representation for a class of geometric equations
(exp: mean curvature flow)
Quantile target problems

• Controlling the probability of reaching the target:

\[ V(t, s, p) := \inf \{ y : P[(S, Y)_T \in \Gamma_0] \geq p \text{ for some } \nu \in \mathcal{U} \} \]

• Introduce an additional controlled state:

\[ dP_t = \alpha_t \cdot dW_t \]

Then

\[ V(t, s, p) := \inf \{ y : 1_{(S, Y)_T \in \Gamma_0} - P_T \geq 0 \text{ for some } (\alpha, \nu) \in \mathcal{U} \} \]

thus converting the quantile target problem into a target problem

• \[ V(T, s, p) !! <\text{Bouchard, Elie, T.}> \]
Hedging under liquidity costs (1)

<Çetin, Jarrow and Protter 2004, 2006>

- Risky asset price is defined by a supply curve:
  \[ S(S_t, \nu) : \text{price per share of } \nu \text{ risky assets} \]

\[ S(S_t, 0) = S_t \]

- \( X_t \) : holdings in cash, \( Z_t \) : holdings in risky asset (number of shares)

\[
X_{t+dt} - X_t + (Z_{t+dt} - Z_t) S(S_t, Z_{t+dt} - Z_t) = 0
\]

\[ \implies X_T = X_0 - \sum (Z_{t+dt} - Z_t) S(S_t, Z_{t+dt} - Z_t) \]

\[ = X_0 + \sum Z_t (S_t - S_{t+dt}) + \ldots \]
Hedging under liquidity costs (2)

Direct computation leads to

\[ Y_T := X_T + Z_T S_T = Y_0 + \sum Z_t (S_{t+dt} - S_t) \]

\[ - \sum (Z_{t+dt} - Z_t) [S (S_t, Z_{t+dt} - Z_t) - S_t] \]

Assume \( \nu \mapsto S(S_t, \nu) \) is smooth, then:

\[ Y_T = Y_0 + \int_0^T Z_t dS_t \]

\[ - \int_0^T \frac{\partial S}{\partial \nu} (S_t, 0) d \langle Z^c \rangle_t \]

\[ - \sum_{t\leq T} \Delta Z_t [S (S_t, \Delta Z_t) - S_t] \]

Super-hedging problem

\[ V_0 := \inf \{ y : Y_T \geq g(S_T) \text{ a.s. for some } Y \in \mathcal{A} \} \]
Second order target problems

- The controlled state is defined by

\[ dY_t = f(t, S_t, Y_t, Z_t, \Gamma_t) \, dt + Z_t \cdot dS_t \]

and the control \( Z \) satisfies the dynamics

\[ dZ_t = dA_t + \Gamma_t dS_t \]

- Given a function \( g \), find

\[ V_0 := \inf \{ y : Y_T \geq g(S_T) \text{ for some } Z \in \mathcal{A} \} \]

**Theorem** \( V(t, s) \) is a (discontinuous) viscosity solution of

\[
- \frac{\partial V}{\partial t} - \mathcal{L}^S V(t, s) - \hat{f} \left( t, s, V(t, s), DV(t, s), D^2 V(t, s) \right) = 0
\]

where \( \hat{f}(t, s, r, p, A) := \sup_{\beta \geq 0} f(t, s, r, p, A + \beta) \) (elliptic envelope)
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Backward SDE: Definition

Find an $\mathbb{F}^W$-adapted $(Y, Z)$ satisfying:

$$Y_t = G + \int_t^T F_r(Y_r, Z_r)dr - \int_t^T Z_r \cdot dW_r$$

i.e.

$$dY_t = -F_t(Y_t, Z_t)dt + Z_t \cdot dW_t$$

and $Y_T = G$

where the generator $F : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$, and

$$\{F_t(y, z), \ t \in [0, T]\} \text{ is } \mathbb{F}^W - \text{adapted}$$

If $F$ is Lipschitz in $(y, z)$ uniformly in $(\omega, t)$, and $G \in L^2(\mathbb{P})$, then there is a unique solution satisfying

$$\mathbb{E} \sup_{t \leq T} |Y_t|^2 + \mathbb{E} \int_0^T |Z_t|^2 dt < \infty$$
Markov BSDE’s

Let $X_t$ be defined by the (forward) SDE

$$dX_t = b(t, X_t) dt + \sigma(t, X_t) dW_t$$

and

$$F_t(y, z) = f(t, X_t, y, z), \quad f : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$$

$$G = g(X_T) \in L^2(\mathbb{P}), \quad g : \mathbb{R}^d \rightarrow \mathbb{R}$$

If $f$ continuous, Lipschitz in $(x, y, z)$ uniformly in $t$, then there is a unique solution to the BSDE

$$dY_t = -f(t, X_t, Y_t, Z_t) dt + Z_t \cdot \sigma(t, X_t) dW_t, \quad Y_T = g(X_T)$$

Moreover, there exists a measurable function $V :$

$$Y_t = V(t, X_t), \quad 0 \leq t \leq T$$
BSDE’s and semilinear PDE’s

• By definition,
  \[ Y_{t+h} - Y_t = V(t + h, X_{t+h}) - V(t, X_t) \]
  \[ = - \int_t^{t+h} f(X_r, Y_r, Z_r) dr + \int_t^{t+h} Z_r \cdot \sigma(X_r) dW_r \]

• If \( V(t, x) \) is smooth, it follows from Itô’s formula that:
  \[ \int_t^{t+h} \mathcal{L} V(r, X_r) dr + \int_t^{t+h} D V(r, X_r) \cdot \sigma(X_r) dW_r \]
  \[ = - \int_t^{t+h} f(X_r, Y_r, Z_r) dr + \int_t^{t+h} Z_r \cdot \sigma(X_r) dW_r \]

where \( \mathcal{L} \) is the Dynkin operator associated to \( X \):
\[
\mathcal{L} V = V_t + b \cdot D V + \frac{1}{2} \text{Tr}[\sigma \sigma^T D^2 V]
\]
Introduction
Second order BSDEs and fully nonlinear PDEs
Probabilistic numerical methods for fully nonlinear PDEs

Stochastic representation of solutions of a semilinear PDE

- Under some conditions, the semilinear PDE

\[- \frac{\partial V}{\partial t} - \frac{1}{2} \text{Tr} \left[ \sigma \sigma^T (x) D^2 V(t, x) \right] - f(x, V(t, x), DV(t, x)) = 0 \]

\[V(T, x) = g(x)\]

has a unique solution which can be represented as \( V(t, x) = Y_{t,x} \)
where \( Y_{t,x} \) solves the BSDE

\[ Y_T = g(X_T), \quad dY_s = -f(X_s, Y_s, Z_s) ds + Z_s \cdot \sigma(X_s) dW_s \]

\[ X_t = x, \quad dX_s = \sigma(X_s) dW_s, \quad t \leq s \leq T \]

- Extension to semilinear PDEs with obstacle is available by introducing Reflected BSDEs
- For \( f \equiv 0 \), we recover the Feynman-Kac formula
Second order BSDEs: Definition

\[ \hat{f}(x, y, z, \gamma) := f(x, y, z, \gamma) + \frac{1}{2} \text{Tr}[\sigma \sigma^T(x)\gamma] \text{ non-decreasing in } \gamma \]

Consider the 2nd order BSDE:

\[
\begin{align*}
 dX_t &= \sigma(X_t) dW_t \\
 dY_t &= -f(t, X_t, Y_t, Z_t, \Gamma_t) dt + Z_t \sigma(X_t) dW_t, \quad Y_T = g(X_T) \\
 dZ_t &= \alpha_t dt + \Gamma_t \sigma(X_t) dW_t
\end{align*}
\]

A solution of (2BSDE) is

a process \((Y, Z, \alpha, \Gamma)\) with values in \(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times S^n\)

Question: existence? uniqueness? in which class?  
<Cheridito, Soner, Touzi and Victoir CPAM 2007>
(i) Suppose a solution exists with $Y_t = V(t, X_t)$, then

$$Y_{t+h} - Y_t = V(t+h, X_{t+h}) - V(t, X_t)$$

$$= - \int_t^{t+h} f(X_r, Y_r, Z_r, \Gamma_r) dr + \int_t^s Z_r \cdot dW_r$$

$$= - \int_t^{t+h} f(X_r, Y_r, Z_r, \Gamma_r) dr$$

$$+ \int_t^{t+h} \left( Z_t + \int_t^r \alpha_u du + \int_t^r \Gamma_u dW_u \right) \cdot dW_r$$

($\sigma(.) = \text{Identity matrix for simplification}$)

(ii) 2× Itô’s formula to $V$, identify terms of different orders

$\implies$ Need short time asymptotics of double stochastic integrals

$$\int_0^t \int_0^r b_u dW_u \cdot dW_r , \quad t \geq 0$$
Second order BSDE : Uniqueness Assumptions

Assumption (f) \( f : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times S_d(\mathbb{R}) \rightarrow \mathbb{R} \) continuous, Lipschitz in \( y \) uniformly in \( (t, x, z, \gamma) \), and for some \( C, p > 0 \):

\[
|f(t, x, y, z)| \leq C (1 + |y| + |x|^p + |z|^p + |\gamma|^p)
\]

Assumption (Comp) If \( w \) (resp. \( u \)) : \([0, T] \times \mathbb{R}^d \rightarrow \mathbb{R} \) is a l.s.c. (resp. u.s.c.) viscosity supersolution (resp. subsolution) of \((E)\) with

\[
w(t, x) \geq -C(1 + |x|^p), \quad \text{and} \quad u(t, x) \leq C(1 + |x|^p)
\]

then \( w(T, .) \geq u(T, .) \) implies that \( w \geq u \) on \([0, T] \times \mathbb{R}^d\)
Let $A_{t,x}^m$ be the class of all processes $Z$ of the form

$$Z_s = z + \int_t^s \alpha_r \, dr + \int_t^s \Gamma_r \, dX_{r,t,x}^s, \quad s \in [t, T]$$

where $z \in \mathbb{R}^d$, $\alpha$ and $\Gamma$ are respectively $\mathbb{R}^d$ and $S_d(\mathbb{R}^d)$ progressively measurable processes with

$$\max \{ |Z_s|, \|\alpha\|_b, |\Gamma_s| \} \leq m \left( 1 + |X_{s,t,x}^t|^p \right),$$

$$|\Gamma_r - \Gamma_s| \leq m \left( 1 + |X_{r,t,x}^t|^p + |X_{s,t,x}^t|^p \right) \left( |r - s| + |X_{r,t,x}^t - X_{s,t,x}^s| \right)$$

We shall look for a solution $(Y, Z, \alpha, \Gamma)$ of (2BSDE) such that

$$Z \in A_{t,x} : = \bigcup_{m \geq 0} A_{t,x}^m$$
Theorem  Suppose that the nonlinear PDE $E$ satisfies the comparison Assumption $\text{Com}$. Then, under Assumption $(f)$, for every $g$ with polynomial growth, there is at most one solution to (2BSDE) with

$$Z \in A_{t,x}$$
2BSDE: Idea of proof of uniqueness

Define the stochastic target problems

\[ V(t, x) := \inf \left\{ y : Y_T^{t,y,Z} \geq g(X_T^{t,x}) \text{ a.s. for some } Z \in \mathcal{A}_{t,x} \right\} \]

(Seller super-replication cost in finance), and

\[ U(t, x) := \sup \left\{ y : Y_T^{t,y,Z} \leq g(X_T^{t,x}) \text{ a.s. for some } Z \in \mathcal{A}_{t,x} \right\} \]

(Buyer super-replication cost in finance)

• By definition: \( V(t, X_t) \leq Y_t \leq U(t, X_t) \) for every solution \((Y, Z, \alpha, \Gamma)\) of (2BSDE) with \(Z \in \mathcal{A}_{0,x}\)

• Main technical result: \( V \) is a (discontinuous) viscosity super-solution of the nonlinear PDE (E) \( \implies U \) is a (discontinuous) viscosity subsolution of (E)

• Assumption Com \( \implies V \geq U \)
Second order BSDE : Existence

• Consider the fully nonlinear PDE (with \( \mathcal{L}V = V_t + \frac{1}{2} \text{Tr}[\sigma \sigma^T D^2V] \))

\[-\mathcal{L}v(t, x) - f (t, x, v(t, x), Dv(t, x), D^2v(t, x)) = 0\]

\((E)\)

\[v(T, x) = g(x)\]

• If \((E)\) has a smooth solution, then

\[\bar{Y}_t = v(t, X_t), \quad \bar{Z}_t := Dv(t, X_t),\]

\[\bar{\alpha}_t := \mathcal{L}Dv(t, X_t), \quad \bar{\Gamma}_t := V_{xx}(t, X_t)\]

is a solution of (2BSDE), immediate application of Itô’s formula

• Existence is an open problem, is there a weak theory of existence?
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Discrete-time approximation of BSDEs

<Bally-Pagès SPA03, Zhang AAP04, Bouchard-Touzi SPA04>

Numerical solution of a semi-linear PDE by simulating the associated backward SDE by means of Monte Carlo methods

Start from Euler discretization: \( Y^n_t = g(X^n_t) \) is given, and

\[
Y^n_{t_{i+1}} - Y^n_{t_i} = -f(X^n_{t_i}, Y^n_{t_i}, Z^n_{t_i}) \Delta t_i + Z^n_{t_i} \cdot \sigma(X^n_{t_i}) \Delta W_{t_{i+1}}
\]
Discrete-time approximation of BSDEs

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Numerical solution of a semi-linear PDE by simulating the associated backward SDE by means of Monte Carlo methods.

Start from Euler discretization: $Y^n_{t_n} = g(X^n_{t_n})$ is given, and

$$
\mathbb{E}^n_{t_i} \left[ Y^n_{t_{i+1}} - Y^n_{t_i} = -f \left( X^n_{t_i}, Y^n_{t_i}, Z^n_{t_i} \right) \Delta t_i + Z^n_{t_i} \cdot \sigma \left( X^n_{t_i} \right) \Delta W_{t_{i+1}} \right]
$$

$\Rightarrow$ Discrete-time approximation: $Y^n_{t_n} = g(X^n_{t_n})$ and

$$
Y^n_{t_i} = \mathbb{E}^n_{t_i} \left[ Y^n_{t_{i+1}} \right] + f \left( X^n_{t_i}, Y^n_{t_i}, Z^n_{t_i} \right) \Delta t_i, \quad 0 \leq i \leq n - 1,
$$
Discrete-time approximation of BSDEs

Numerical solution of a semi-linear PDE by simulating the associated backward SDE by means of Monte Carlo methods.

Start from Euler discretization: $Y^n_{t_n} = g(X^n_{t_n})$ is given, and

$$
E^n_i[\Delta W_{t_{i+1}}] \rightarrow Y^n_{t_{i+1}} - Y^n_{t_i} = -f(X^n_{t_i}, Y^n_{t_i}, Z^n_{t_i}) \Delta t_i + Z^n_{t_i} \cdot \sigma(X^n_{t_i}) \Delta W_{t_{i+1}}
$$

$\Rightarrow$ Discrete-time approximation: $Y^n_{t_n} = g(X^n_{t_n})$ and

$$
Y^n_{t_i} = E^n_i[Y^n_{t_{i+1}}] + f(X^n_{t_i}, Y^n_{t_i}, Z^n_{t_i}) \Delta t_i, \quad 0 \leq i \leq n - 1
$$

$$
Z^n_{t_i} = (\Delta t_i)^{-1} E^n_i[Y^n_{t_{i+1}} \Delta W_{t_{i+1}}]
$$
Discrete-time approximation of BSDEs

Numerical solution of a semi-linear PDE by simulating the associated backward SDE by means of Monte Carlo methods

Start from Euler discretization: \( Y^n_{t_i} = g(X^n_{t_i}) \) is given, and

\[
\mathbb{E}^n_i \left[ \Delta W_{t_{i+1}} \right] \rightarrow Y^n_{t_{i+1}} - Y^n_{t_i} = -f(X^n_{t_i}, Y^n_{t_i}, Z^n_{t_i}) \Delta t_i + Z^n_{t_i} \cdot \sigma(X^n_{t_i}) \Delta W_{t_{i+1}}
\]

\( \Rightarrow \) Discrete-time approximation: \( Y^n_{t_n} = g(X^n_{t_n}) \) and

\[
Y^n_{t_i} = \mathbb{E}^n_i \left[ Y^n_{t_{i+1}} \right] + f(X^n_{t_i}, Y^n_{t_i}, Z^n_{t_i}) \Delta t_i, \quad 0 \leq i \leq n - 1
\]

\[
Z^n_{t_i} = (\Delta t_i)^{-1} \mathbb{E}^n_i \left[ Y^n_{t_{i+1}} \Delta W_{t_{i+1}} \right]
\]

\( \Rightarrow \) Similar to numerical computation of American options
Discrete-time approximation of BSDEs, continued

\[ \pi : 0 = t_0 < t_1 < \ldots < t_n = T, \quad |\pi| = \max_{1 \leq i \leq n} |t_{i+1} - t_i| \]

**Theorem**  Assume \( f \) and \( g \) are Lipschitz. Then:

\[
\limsup_{n \to \infty} n^{1/2} \left\{ \sup_{0 \leq t \leq 1} \| Y^n_t - Y_t \|_{L^2} + \| Z^n - Z \|_{H^2} \right\} < \infty
\]

**Theorem** <Gobet-Labart 06> Under additional regularity conditions:

\[
\limsup_{n \to \infty} n \| Y^n_0 - Y_0 \|_{L^2} < \infty
\]

Weak error
Main observation: in our context all conditional expectations are regressions, i.e.

\[
\mathbb{E} \left[ Y_{t_{i+1}} \mid \mathcal{F}_{t_i} \right] = \mathbb{E} \left[ Y_{t_{i+1}} \mid X_{t_i} \right]
\]

\[
\mathbb{E} \left[ Y_{t_{i+1}} \Delta W_{t_{i+1}} \mid \mathcal{F}_{t_i} \right] = \mathbb{E} \left[ Y_{t_{i+1}} \Delta W_{t_{i+1}} \mid X_{t_i} \right]
\]

Classical methods from statistics:
- Kernel regression <Carrière>
- Projection on subspaces of $L^2(\mathbb{P})$ <Longstaff-Schwarz, Gobet-Lemor-Warin AAP05>

from numerical probabilistic methods
- quantization... <Bally-Pagès SPA03>

Integration by parts <Lions-Reigner 00, Bouchard-Touzi SPA04>
Simulation of Backward SDE’s

1. Simulate trajectories of the forward process $X$ (well understood)

2. Backward algorithm:

\[
\begin{align*}
\hat{Y}_{t_n}^n &= g(X_{t_n}^n) \\
\hat{Y}_{t_{i-1}}^n &= \hat{E}_{t_{i-1}}^n [\hat{Y}_{t_i}^n] + f(X_{t_{i-1}}^n, \hat{Y}_{t_{i-1}}^n, \hat{Z}_{t_{i-1}}^n) \Delta t_i, \quad 1 \leq i \leq n, \\
\hat{Z}_{t_{i-1}}^n &= \frac{1}{\Delta t_i} \hat{E}_{t_{i-1}}^n [\hat{Y}_{t_i}^n \Delta W_{t_i}]
\end{align*}
\]

(truncation of $\hat{Y}^n$ and $\hat{Z}^n$ needed in order to control the $\mathbb{L}^p$ error)
Simulation of BSDEs : bound on the rate of convergence

Error estimate for the Malliavin-based algorithm, $|\pi| = n^{-1}$

**Theorem**  For $p > 1$:

$$\limsup_{n \to \infty} \max_{0 \leq i \leq n} n^{-1-\frac{d}{4p}} N^{1/2p} \| \hat{Y}^n_{t_i} - Y^n_{t_i} \|_{L^p} < \infty$$

For the time step $\frac{1}{n}$, and limit case $p = 1$:

rate of convergence of $\frac{1}{\sqrt{n}}$ if and only if

$$n^{-1-\frac{d}{4}} N^{1/2} = n^{1/2}, \text{ i.e. } N = n^{3+\frac{d}{2}}$$
By analogy with BSDE, we introduce the following discretization for 2BSDEs:

\[
Y^n_{tn} = g(X^n_{tn}),
\]
\[
Y^n_{ti-1} = \mathbb{E}^n_{i-1} \left[ Y^n_{ti} \right] + f \left( X^n_{ti-1}, Y^n_{ti-1}, Z^n_{ti-1}, \Gamma^n_{ti-1} \right) \Delta t_i, \quad 1 \leq i \leq n,
\]
\[
Z^n_{ti-1} = \mathbb{E}^n_{i-1} \left[ Y^n_{ti} \frac{\Delta W_{ti}}{\Delta t_i} \right]
\]
\[
\Gamma^n_{ti-1} = \mathbb{E}^n_{i-1} \left[ Y^n_{ti} \frac{|\Delta W_{ti}|^2 - \Delta t_i}{|\Delta t_i|^2} \right]
\]
**Intuition From Greeks Calculation**

- First, use the approximation $f''(x) \sim_{h=0} \mathbb{E}[f''(x + W_h)]$
- Then, integration by parts shows that

$$f''(x) \sim \int f''(x + y) \frac{e^{-y^2/(2h)}}{\sqrt{2\pi}} dy$$

$$= \int f'(x + y) \frac{y e^{-y^2/2}}{h \sqrt{2\pi}} dy = \mathbb{E} \left[ f'(x + W_h) \frac{W_h}{h} \right]$$

$$= \int f(x + y) \frac{y^2 - h e^{-y^2/2}}{h^2 \sqrt{2\pi}} dy = \mathbb{E} \left[ f(x + W_h) \left( \frac{W_h^2 - h}{h^2} \right) \right]$$

- Connection with Finite Differences: $W_h \sim \sqrt{h} \left( \frac{1}{2} \delta_1 + \frac{1}{2} \delta_{-1} \right)$

$$\mathbb{E} \left[ \psi(x + W_h) \frac{W_h}{h} \right] \sim \frac{\psi(x + \sqrt{h}) - \psi(x - \sqrt{h})}{2h} \quad \text{Centered FD !}$$
The Convergence Result

<Fahim and Touzi 2007>

**Theorem** Suppose in addition that $f$ is Lipschitz and $\|f_\gamma\|_{L^\infty} \leq \sigma$. Then

$$Y_0^n(t, x) \longrightarrow v(t, x) \quad \text{uniformly on compacts}$$

where $v$ is the unique viscosity solution of the nonlinear PDE.

- Proof: stability, consistency, monotonicity <Barles-Souganidis AA91>
- Bounds on the approximation error are available <Krylov, Barles-Jacobsen, Cafarelli-Souganidis>
- This convergence result is weaker than that of (first order) Backward SDEs...
Comments on the 2BSDE algorithm

- In BSDEs, the drift coefficient $\mu$ of the forward SDE can be changed arbitrarily by Girsanov theorem (importance sampling...)

- In 2BSDEs, both $\mu$ and $\sigma$ can be changed (numerical results however recommend prudence...)

- The heat equation $v_t + v_{xx} = 0$ corresponds to a BSDE with zero driver. Splitting the Laplacian in two pieces, it can also be viewed as a 2BSDE with driver $f(\gamma) = \frac{1}{2} \gamma$

  → numerical experiments show that the 2BSDE algorithm perform better than the pure finite differences scheme
Portfolio optimization (X. Warin)

With \( U(x) = -e^{-\eta x} \), want to solve:

\[
V(t, x) := \sup_{\theta} \mathbb{E} \left[ U \left( x + \int_t^T \theta_u \sigma(\lambda du + dW_u) \right) \right]
\]

- An explicit solution is available

- \( V \) is the characterized by the fully nonlinear PDE

\[
-V_t + \frac{1}{2} \lambda^2 \frac{(V_x)^2}{V_{xx}} = 0 \quad \text{and} \quad V(T, .) = U
\]
Fig.: Relative Error (Regression), dimension 1
Introduction
Second order BSDEs and fully nonlinear PDEs
Probabilistic numerical methods for fully nonlinear PDEs
Monte Carlo Simulation of BSDEs
The fully nonlinear case
Numerical example

Fig.: Relative Error (Regression), dimension 2
## Varying the drift of the FSDE

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<th>Drift FSDE</th>
<th>Relative error (Regression)</th>
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### Varying the volatility of the FSDE

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<th>Relative error (Quantization)</th>
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