

# About the method of characteristics

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Joint work with Z. Ammari about bosonic mean-field dynamics

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- The problem
- An example
- Infinite dimensional measure transportation
- Why a probabilistic trajectory picture ?

# The problem

Let  $E$  be a separable real Hilbert space.

We want to prove that a Borel probability measure  $\mu_t$  on  $E$ , which depends continuously on the time  $t \in \mathbb{R}$  and fulfills

$$\partial_t \mu + \operatorname{div}(v(t, x)\mu) = 0 \quad , \quad \mu_{t=0} = \mu_0$$

equals

$$\mu_t = \Phi(t, 0)_* \mu_0 ,$$

when  $\Phi(t, s)$  is a well-defined flow on  $E$ , associated with the ODE

$$\dot{x} = v(t, x) .$$

# The problem

## Finite dimensional case:

Assume that  $\Phi(t, s)$  is a diffeomorphism on  $E$  for all  $t, s \in \mathbb{R}$ .  
The transport equation is

$$\int_{\mathbb{R}} \int_E (\partial_t \varphi + \langle v(t, \cdot), \nabla_x \varphi \rangle_E) d\mu_t(x) dt = 0, \quad \forall \varphi \in C_0^\infty(\mathbb{R} \times E),$$

or

$$\int_0^T \int_E (\partial_t \varphi + \langle v(t, x), \nabla_x \varphi \rangle_E) d\mu_t(x) dt + \int_E \varphi(0, x) d\mu_0(x) - \int_E \varphi(T, x) d\mu_T(x) = 0, \quad \forall \varphi \in C_0^\infty([0, T] \times E).$$

# The problem

## Finite dimensional case:

The method of characteristics consists in taking

$$\varphi(x, t) = a(\Phi(T, t)x) \quad a \in C_0^\infty(E).$$

We get

$$(\partial_t \varphi + \langle v, \nabla \varphi \rangle)(\Phi(t, 0)x, t) = \frac{d}{dt} [a(\Phi(T, 0)x)] = 0$$

and

$$\int_E a(\Phi(T, 0)x) d\mu_0(x) = \int_E a(x) d\mu_T(x), \quad \forall a \in C_0^\infty(E),$$

which is

$$\mu_T = \Phi(T, 0)_* \mu_0.$$

## Infinite dimensional case:

The weak formulation

$$\int_{\mathbb{R}} \int_E (\partial_t \varphi + \langle v(t, \cdot), \nabla_x \varphi \rangle_E) d\mu_t(x) dt = 0,$$

is given for a class of test functions: the cylindrical functions  $\varphi \in \mathcal{C}_{0,cyl}^\infty(\mathbb{R} \times E)$  or some polynomial functions on  $E$ .

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is given for a class of test functions: the cylindrical functions  $\varphi \in \mathcal{C}_{0,cyl}^\infty(\mathbb{R} \times E)$  or some polynomial functions on  $E$ .

Problem: A nonlinear flow does not preserve these classes.  
We cannot take  $\varphi(x, t) = a(\Phi(T, t)x)!!!$

# An example, the Hartree flow

The Hartree equation is given by

$$i\partial_t z = -\Delta z + (V * |z|^2)z \quad , \quad z_{t=0} = z_0 .$$

with  $V(-x) = V(x)$ .

Example :  $V(x) = \pm \frac{1}{|x|}$  in dimension 3.



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The equation can be written  $i\partial_t z = \partial_{\bar{z}} \mathcal{E}(z)$  with

$$\begin{aligned} \mathcal{E}(z) &= \int_{\mathbb{R}^d} |\nabla z(x)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^{2d}} V(x-y) |z(x)|^2 |z(y)|^2 dx dy \\ &= \langle z, -\Delta z \rangle + \frac{1}{2} \langle z^{\otimes 2}, V(x-y) z^{\otimes 2} \rangle . \end{aligned}$$

# An example, the Hartree flow

Set  $\tilde{z}_t = e^{it\Delta} z_t$  and the equation becomes

$$i\partial_t \tilde{z}_t = \partial_{\bar{z}} h(\tilde{z}, t)$$

$$h(z, t) = \langle z^{\otimes 2}, V_t z^{\otimes 2} \rangle$$

$$V_t = e^{it(\Delta_x + \Delta_y)} (V(x - y) \times) e^{-it(\Delta_x + \Delta_y)}$$

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When  $V = \pm \frac{1}{|x|}$ ,  $d = 3$ , Hardy's inequality leads to

$$|\langle z_1, V_t z_2 \otimes z_3 \rangle|_{L^2} \leq C \min_{\sigma \in \mathcal{S}_3} (|z_{\sigma(1)}|_{H^1} |z_{\sigma(2)}|_{L^2} |z_{\sigma(3)}|_{L^2})$$

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More generally it works if  $V$  satisfies  $V(-x) = V(x)$  and  $V(1 - \Delta)^{-1/2} \in \mathcal{L}(L^2(\mathbb{R}^d))$ .

## Theorem

Assume  $V(x) = V(-x)$  and  $V(1 - \Delta)^{-1/2} \in \mathcal{L}(L^2(\mathbb{R}^d))$ . The equations

$$i\partial_t z = \partial_{\bar{z}} \mathcal{E}(z) \quad \text{et} \quad i\partial_t \tilde{z} = \partial_{\bar{z}} h(\tilde{z}, t)$$

define flows  $\Phi(t)$  and  $\tilde{\Phi}(t, s)$  on  $H^1(\mathbb{R}^d; \mathbb{C})$ .

The norm  $\|\cdot\|_{L^2}$  and the energy  $\mathcal{E}$  are invariant under  $\Phi$ . The norm  $\|\cdot\|_{L^2(\mathbb{R}^d)}$  is invariant under  $\tilde{\Phi}$  and the velocity field  $v(z, t) = \frac{1}{i} \partial_{\bar{z}} h(z, t)$ , satisfies

$$\|v(z, t)\|_{H^1} \leq C \|z\|_{H^1}^2 \|z\|_{L^2}.$$

# An example, the Hartree flow

The mean field analysis for bosons interacting via a pair potential  $V(x - y)$ , leads to Borel probability measures on  $H^1 = H^1(\mathbb{R}^d; \mathbb{C})$  which verifies

$$\int_{\mathbb{R}} \int_{H^1} (\partial_t \varphi + i(\partial_z h \cdot \partial_{\bar{z}} \varphi - \partial_z \varphi \cdot \partial_{\bar{z}} h)) d\mu_t(z) dt = 0$$

for all  $\varphi \in \mathcal{C}_{0,cyl}^{\infty}(\mathbb{R} \times H^1; \mathbb{R})$ .

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for all  $\varphi \in \mathcal{C}_{0,cyl}^{\infty}(\mathbb{R} \times H^1; \mathbb{R})$ .

For a cylindrical function  $f \in \mathcal{C}_{0,cyl}^{\infty}(H^1; \mathbb{R})$  on  $H^1$ , we define  $\nabla_{\bar{z}} f$  by

$$\forall u \in H^1(\mathbb{R}^d), \quad \langle u, \nabla_{\bar{z}} f \rangle_{H^1} = \langle u, \partial_{\bar{z}} f \rangle.$$

Its gradient for the real structure on  $H^1 = H^1(\mathbb{R}^d; \mathbb{C})$  with the scalar product  $\langle u_1, u_2 \rangle_{H_{\mathbb{R}}^1} = \operatorname{Re} \langle u_1, u_2 \rangle_{H^1}$  is given by

$$\nabla = 2\nabla_{\bar{z}}$$

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for all  $\varphi \in \mathcal{C}_{0,cyl}^{\infty}(\mathbb{R} \times H^1; \mathbb{R})$ .

The above equation is the weak version of

$$\partial_t \mu + \operatorname{div}(v_t \mu) = 0 \quad , \quad v_t = \frac{1}{i} \partial_{\bar{z}} h(z, t)$$

with cylindrical test functions on  $\mathbb{R} \times H_{\mathbb{R}}^1$ ,  $H_{\mathbb{R}}^1 = H^1(\mathbb{R}^d; \mathbb{C})$  being the real Hilbert space with the scalar product  $\langle u, v \rangle_{H_{\mathbb{R}}^1}$  and  $|z|_{H_{\mathbb{R}}^1} = |z|_{H^1}$ .



# An example, the Hartree flow

Remark : With good assumptions (on the initial mean field data), one verifies that the measure  $\mu_t$  is continuous w.r.t the Wasserstein distance \*

$$W_2(\mu_1, \mu_2) = \left( \inf \left\{ \int_{H_{\mathbb{R}}^1 \times H_{\mathbb{R}}^1} |z_2 - z_1|_{H^1}^2 d\mu(z_1, z_2), \quad \Pi_{j,*}\mu = \mu_j \right\} \right)^{1/2}.$$

Also, for all  $t \in \mathbb{R}$

$$\int_{H_{\mathbb{R}}^1} |z|_{H^1}^4 |z|_{L^2}^2 d\mu_t(z) \leq C$$

and

$$\int_0^T \int_{H_{\mathbb{R}}^1} |v(t, z)|_{H^1}^2 d\mu_t(z) \leq C_T.$$

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\* This strong continuity property is requires intermediate steps 

# Solving a transport equation in $\text{Prob}_2(E)$

$E$  real separable Hilbert space.

$\text{Prob}_2(E)$  is the space of Borel probability measures on  $E$ ,  $\mu$ , such that

$$\int_E |x|^2 d\mu(x) < +\infty$$

The Wasserstein distance  $W_2$  on  $\text{Prob}_2(E)$  is given by

$$W_2^2(\mu_1, \mu_2) = \inf \left\{ \int_{E^2} |x_2 - x_1|^2 d\mu(x_1, x_2), \quad (\Pi_j)_* \mu = \mu_j \right\} .$$

# Solving a transport equation in $\text{Prob}_2(E)$

For  $T > 0$ , set  $\Gamma_T = \mathcal{C}^0([-T, T]; E)$  endowed with the norm  $|\gamma|_\infty = \max_{t \in [-T, T]} |\gamma(t)|$

or the distance

$d(\gamma, \gamma') = \max_{t \in [-T, T]} (\sum_{n \in \mathbb{N}^*} |\langle \gamma(t) - \gamma'(t), e_n \rangle|^2 2^{-n})^{1/2}$  with  $(e_n)_{n \in \mathbb{N}^*}$  ONB of  $E$

For a Borel probability measure  $\eta$  on  $E \times \Gamma_T$  define the evaluation map at time  $t \in [-T, T]$  by

$$\int_E \varphi d\mu_t^\eta = \int_{\Gamma_T} \varphi(\gamma(t)) d\eta(x, \gamma), \quad \forall \varphi \in \mathcal{C}_{0, \text{cyl}}^\infty(E).$$

# Solving a transport equation in $\text{Prob}_2(E)$

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Actually,  $\mu_t^\eta = (e_t)_* \eta$  with

$$e_t : (x, \gamma) \in E \times \Gamma_T \rightarrow \gamma(t) \in E.$$

# Solving a transport equation in $\text{Prob}_2(E)$

The following result is the infinite dimensional version of a result by Ambrosio-Gigli-Savaré.

## Proposition

If  $\mu_t : [-T, T] \rightarrow \text{Prob}_2(E)$  is a  $W_2$ -continuous solution to the equation

$$\partial_t \mu + \text{div}(v(t, x))\mu = 0$$

on  $(-T, T)$ , for a Borel velocity field  $v(t, x) = v_t(x)$  on  $E$  such that  $|v_t|_{L^2(E, \mu_t)} \in L^1([-T, T])$ ; then there exists a Borel probability measure  $\eta$  on  $E \times \Gamma_T$  which satisfies

- ▶  $\eta$  is carried by the set of pairs  $(x, \gamma)$  such that  $\gamma \in AC^2([-T, T]; E)$  solves  $\dot{\gamma}(t) = v_t(\gamma(t))$  for almost all  $t \in (-T, T)$  and  $\gamma(0) = x$ .
- ▶  $\mu_t = \mu_t^\eta$  for all  $t \in [-T, T]$ .

## Corollary

*If for all  $x \in E$ , the Cauchy problem*

$$\dot{\gamma}(t) = v_t(\gamma(t)), \quad \gamma(0) = x$$

*is well posed and defines a flow  $\tilde{\Phi}(t, s)$  on  $E$ , then*

$$\mu_t = \tilde{\Phi}(t, 0)_* \mu_0.$$

# Solving a transport equation in $\text{Prob}_2(E)$

## Corollary

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$$\mu_t = \tilde{\Phi}(t, 0)_* \mu_0.$$

Proof:

$$\begin{aligned} \int_E \varphi d\mu_t &= \int_{E \times \Gamma_T} \varphi(\gamma(t)) d\eta(x, \gamma) = \int_{E \times \Gamma_T} [\varphi \circ \tilde{\Phi}(t, s)](\gamma(s)) d\eta(x, \gamma) \\ &= \int_E \varphi \circ \tilde{\Phi}(t, s) d\mu_s. \end{aligned}$$

# Why a probabilistic trajectory picture ?

Notations:

$(e_n)_{n \in \mathbb{N}^*}$  Hilbert basis of  $E$ .



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$\pi^d$ ,  $\pi^{d,T}$ ,  $\hat{\pi}^d = \pi^{d,T} \circ \pi^d$  are defined by

$$E \ni x \mapsto \pi^d(x) = (\langle e_n, x \rangle, n \leq d) \in \mathbb{R}^d,$$

$$\mathbb{R}^d \ni (y_1, \dots, y_d) \mapsto \pi^{d,T}(y) = \sum_{n=1}^d y_n e_n \in E,$$

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$\mu_t^d = \pi_*^d \mu_t$      $\hat{\mu}_t^d = \hat{\pi}_*^d \mu_t = \mu_t^d \otimes \delta_0$  when  $E = F_d \oplus F_d^\perp$ ,  $F_d \sim \mathbb{R}^d$ .  
 $\{\mu_{t,y}, y \in \mathbb{R}^d\}$  is the disintegration of  $\mu_t$  w.r.t  $\mu_t^d$ .

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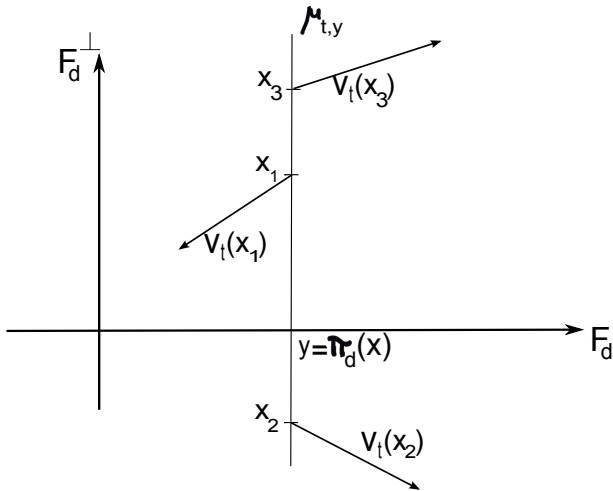
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 $\{\mu_{t,y}, y \in \mathbb{R}^d\}$  is the disintegration of  $\mu_t$  w.r.t  $\mu_t^d$ .

$$v_t^d(y) = \int_{(\pi^d)^{-1}(y)} \pi^d(v_t(x)) d\mu_{t,y}(x) \quad \text{for a.e. } y \in \mathbb{R}^d,$$

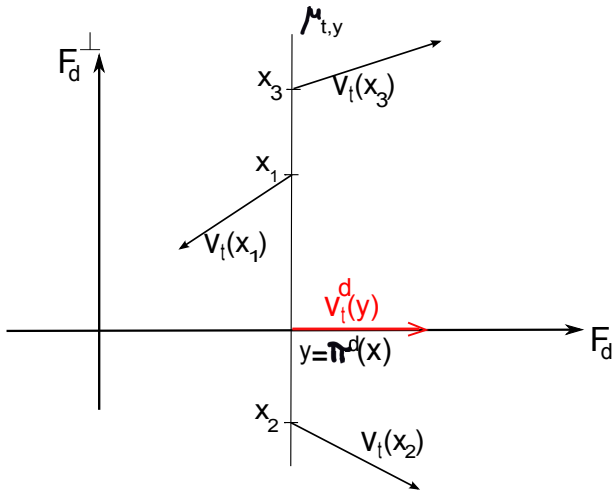
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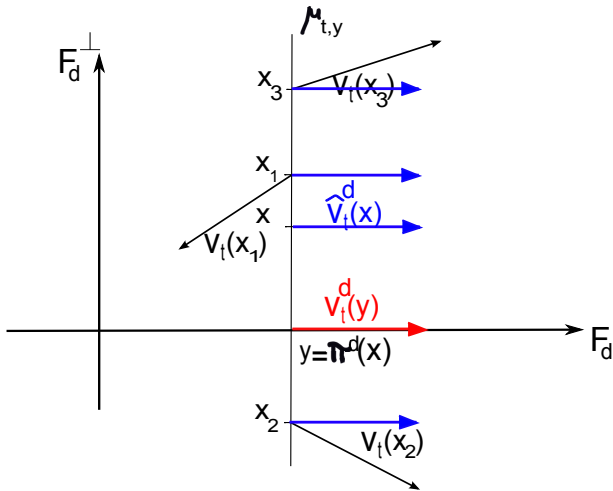
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# Why a probabilistic trajectory picture ?

Properties:

$$|\hat{v}_t^d|_{L^2(E, \hat{\mu}_t^d)} = |v_t^d|_{L^2(\mathbb{R}^d, \mu_t^d)} \leq |v_t|_{L^2(E, \mu_t)}.$$

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$$W_2(\mu_{t_1}^d, \mu_{t_2}^d) \leq \int_{t_1}^{t_2} |v_t^d|_{L^2(\mathbb{R}^d, \mu_t^d)} dt \leq \int_{t_1}^{t_2} |v_t|_{L^2(E, \mu_t)} dt.$$



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The sequence  $(\hat{\mu}_t^d)_{d \in \mathbb{N}^*}$  converges weakly narrowly to  $\mu_t$  with the estimate

$$W_2(\mu_{t_2}, \mu_{t_1}) \leq \liminf_{d \rightarrow \infty} W_2(\hat{\mu}_{t_2}^d, \hat{\mu}_{t_1}^d).$$

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Similar properties hold for the families  $\eta^d$ ,  $\hat{\eta}^d$  after projecting the trajectories.

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In this approximation process, the finite dimensional vector field  $v_t^d$  may be (is) singular  $\rightarrow$  no uniqueness of trajectories.