SIMPLE QUASICRYSTALS ARE SETS OF STABLE SAMPLING

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Version française abrégée

Soit $K \subset \mathbb{R}^n$ un ensemble compact et soit $E_K \subset L^2(\mathbb{R}^n)$ le sous-espace de $L^2(\mathbb{R}^n)$ composé de toutes les fonctions $f \in L^2(\mathbb{R}^n)$ dont la transformée de Fourier $\hat{f}(\xi) = \int e^{-2\pi i x \cdot \xi} f(x)\, dx$ est nulle hors de K. En utilisant la terminologie introduite dans [3], un ensemble $\Lambda \subset \mathbb{R}^n$ est un "ensemble d'échantillonnage stable" pour E_K s'il existe une constante C telle que pour toute $f \in E_K$ on ait

$$\|f\|_2^2 \leq C \sum_{\lambda \in \Lambda} |f(\lambda)|^2. \tag{0.1}$$

Si $n = 1$ et si K est un intervalle de $\mathbb{R}$, le problème a été résolu par Albert Ingham en 1936. L'énoncé d'Ingham a été ensuite généralisé par Beurling et c'est sous cette forme que nous l'utiliserons.

Pour $n \geq 1$ et K arbitraire, H.J. Landau [3] a démontré que (0.1) implique $\underline{\text{dens}}\, \Lambda \geq |K|$. Mais la réciproque n'est pas vraie et $|K| < \underline{\text{dens}}\, \Lambda$ n'implique pas (0.1) même dans le cas le plus simple où $\Lambda = \mathbb{Z}^n$. Nous allons prouver le résultat suivant : *Pour tout quasicrystal simple* $\Lambda \subset \mathbb{R}^n$ *et tout ensemble compact* $K \subset \mathbb{R}^n$, *la condition* $|K| < \text{dens}\, \Lambda$ *entraîne* (0.1).

1. Introduction

This paper is motivated by some recent advances on what is now called "compressed sensing". Let us begin with a theorem by Terence Tao. Let p be a prime number and $\mathbb{F}_p$ be the finite field with p elements. We denote by $\#E$ the cardinality of $E \subset \mathbb{F}_p$. The Fourier transform of a complex valued function f defined on $\mathbb{F}_p$ is denoted by $\hat{f}$. Let M_q be the collection of all $f : \mathbb{F}_p \mapsto \mathbb{C}$ such that the cardinality of the support of f does not exceed q. Then Terence Tao proved that for $q < p/2$ and for any set Ω of frequencies such that $\#\Omega \geq 2q$, the mapping $\Phi : M_q \mapsto l^2(\Omega)$ defined by $f \mapsto \mathbf{1}_\Omega \hat{f}$ is injective. Here and in what follows, $\mathbf{1}_E$ will denote the indicator function of the set E. This property is no longer true if $\mathbb{F}_p$ is replaced by $\mathbb{Z}/N\mathbb{Z}$ and if

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N is not a prime.

We want to generalize this fact to functions f of several real variables with applications to image processing. The Fourier transform of $f \in L^1(\mathbb{R}^n)$ will be defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} \exp(-2\pi i \xi \cdot x) f(x) dx, \ \ \xi \in \mathbb{R}^n. \tag{1.1}$$

To generalize Tao's theorem to the continuous setting we begin with a parameter $\beta \in (0, 1/2)$ which will play the role of q and define a collection M_β of functions $f \in L^2(\mathbb{R}^n)$ as follows: we write $f \in M_\beta$ if $\hat{f}$ is supported by a compact set $K \subset [0, 1]^2$ whose measure $|K|$ does not exceed β. This compact set K depends on f and M_β is not a vector space. If f, g belong to M_β, then $f + g$ belongs to $M_{2\beta}$, a situation which is classical in nonlinear approximation. It will be proved below that for every $\alpha \in (0, 1/2)$ there exists a set $\Lambda_\alpha \subset \mathbb{Z}^2$ with the following properties: (a) density $\Lambda_\alpha = 2\alpha$ and (b) the mapping $\Phi : M_\beta \mapsto \ell^2(\Lambda_\alpha)$ defined by $\Phi(f) = (f(\lambda))_{\lambda \in \Lambda_\alpha}$ is injective when $0 < \beta < \alpha$. This set Λ_α plays the role of Ω in Tao's work and the density of Λ_α is then playing the role of the cardinality of Ω. Any $f \in M_\beta$ can be retrieved from the information given by the "irregular sampling" $f(\lambda) = a(\lambda), \ \lambda \in \Lambda_\alpha$, and one would like to do it by some fast algorithm. Tao's theorem can be decomposed into two statements. In the first one Ω and T are fixed with the same cardinality. We denote by $l^2(T)$ the vector space consisting of all functions f supported by T. Then the first theorem by Tao says that the mapping $\Phi : l^2(T) \mapsto l^2(\Omega)$ is an isomorphism. The second theorem easily follows form this first statement. We now generalize this first theorem to the continuous case. Let $K \subset \mathbb{R}^n$ be a compact set and $E_K \subset L^2(\mathbb{R}^n)$ be the translation invariant subspace of $L^2(\mathbb{R}^n)$ consisting of all $f \in L^2(\mathbb{R}^n)$ whose Fourier transform $\hat{f}(\xi) = \int e^{-2\pi i x \cdot \xi} f(x)\, dx$ is supported by K. We now follow [3].

Definition 1.1. *A set $\Lambda \subset \mathbb{R}^n$ has the property of stable sampling for E_K if there exists a constant C such that*

$$f \in E_K \Rightarrow \|f\|_2^2 \leq C \sum_{\lambda \in \Lambda} |f(\lambda)|^2. \tag{1.2}$$

In other words any " band-limited " $f \in E_K$ can be reconstructed from its sampling $f(\lambda), \ \lambda \in \Lambda$. Here is an equivalent definition. Let $L^2(K)$ be the space of all restrictions to K of functions in $L^2(\mathbb{R}^n)$. Then $\Lambda \subset \mathbb{R}^n$ is a set of stable sampling for E_K if and only if the collection of functions $\exp(2\pi i \lambda \cdot x), \lambda \in \Lambda$, is a frame of $L^2(K)$.

Definition 1.2. *A set Λ has the property of stable interpolation for E_K if there exists a constant C such that*

$$\sum_{\lambda\in\Lambda} |c(\lambda)|^2 \le C\|f\|^2_{L^2(K)} \tag{1.3}$$

for every finite trigonometric sum $f(x) = \sum_{\lambda\in\Lambda} c(\lambda)\exp(2\pi i\lambda\cdot x)$.

In the one dimensional case and when K is an interval, A. Ingham (1936) proved the following estimate:

Proposition 1.1. *Let Λ be an increasing sequence $\lambda_j,\ j\in\mathbb{Z}$, of real numbers such that $\lambda_{j+1}-\lambda_j \ge \beta,\ j\in\mathbb{Z}$, where β is a positive constant. Let I be any interval with length $|I| > 1/\beta$. Then we have $C\sum|c_j|^2 \le \int_I |\sum c_j \exp(2\pi i\lambda_j t)|^2\,dt$ where $C = \frac{2}{\pi}(1-\frac{1}{|I|^2\beta^2})$.*

This constant C is not optimal. The condition $|I| > 1/\beta$ cannot be replaced by $|I| < 1/\beta$ and Ingham's inequality does not tell anything in the limiting case $|I| = 1/\beta$. This was generalized A. Beurling (see [2]) who proved the following:

Proposition 1.2. *Let Λ be an increasing sequence $\lambda_j,\ j\in\mathbb{Z}$, of real numbers fulfilling the following two conditions*

(a) $\lambda_{j+1}-\lambda_j \ge \beta' > 0$
(b) *if T large enough we have $\lambda_{j+T}-\lambda_j \ge \beta T,\ j\in\mathbb{Z},\ \ \beta>0$.*

Let I be any interval with length $|I| > 1/\beta$. Then we have $C\sum|c_j|^2 \le \int_I |\sum c_j\exp(2\pi i\lambda_j t)|^2\,dt$ where $C = C(\beta,\beta',T,|I|)$.

Here the length of I only depends on the averaged distance between λ_{j+1} and λ_j. The final result easily follows from the preceding one:

Proposition 1.3. *Let Λ be an increasing sequence $\lambda_j,\ j\in\mathbb{Z}$, of real numbers such that $\lambda_{j+1}-\lambda_j \ge \beta > 0$ and let $\overline{\mathrm{dens}}\,\Lambda = \lim_{R\to\infty} R^{-1}\sup_{x\in\mathbb{R}} \mathrm{card}\{\Lambda\cap[x,\,x+R]\}$ be the upper density of Λ. The lower density is defined by replacing upper bounds by lower bounds. Then for any interval I, $|I| < \underline{\mathrm{dens}}\,\Lambda$ implies* (1.2) *and $|I| > \overline{\mathrm{dens}}\,\Lambda$ implies* (1.3).

Returning to the general case $K\subset\mathbb{R}^n$ H.J. Landau proved in [3] that (1.2) implies $\underline{\mathrm{dens}}\,\Lambda \ge |K|$ and (1.3) implies $\overline{\mathrm{dens}}\,\Lambda \le |K|$. These necessary conditions are not sufficient. Indeed $|K| < \underline{\mathrm{dens}}\,\Lambda$ does not even imply (1.2) when $\Lambda = \mathbb{Z}^n$. The following result shows that Landau's necessary conditions are sufficient for some sets Λ.

Theorem 1.1. *Let $\Lambda\subset\mathbb{R}^n$ be a simple quasicrystal and $K\subset\mathbb{R}^n$ be a compact set. Then $|K| < \mathrm{dens}\,\Lambda$ implies* (1.2). *If K is Riemann integrable, then $|K| > \mathrm{dens}\,\Lambda$ implies* (1.3).

A compact $K \subset \mathbb{R}^n$ is a Riemann integrable if the Lebesgue measure of its boundary is 0.

We now define a simple quasicrystal as in [2] or [3]. Let $\Gamma \subset \mathbb{R}^n \times \mathbb{R}$ be a lattice and if $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, let us write $p_1(x,t) = x$, $p_2(x,t) = t$. We now assume that p_1 once restricted to Γ is an injective mapping onto $p_1(\Gamma) = \Gamma_1$. We make the same assumption on p_2. We furthermore assume that $p_1(\Gamma)$ is dense in $\mathbb{R}^n$ and $p_2(\Gamma)$ is dense in $\mathbb{R}$. The dual lattice of of Γ is denoted Γ^* and is defined by $x \cdot y \in \mathbb{Z}$, $x \in \Gamma$, $y \in \Gamma^*$. We use the following notations. For $\gamma = (x,t) \in \Gamma$ we write $t = \tilde{x}, \tilde{t} = x$. Note that t is uniquely defined by x. The same notations are used for the two components of $\gamma^* \in \Gamma^*$. If $I = [-\alpha, \alpha]$, the simple quasicrystal $\Lambda_I \subset \mathbb{R}^n$ is defined by

$$\Lambda_I = \{p_1(\gamma); \gamma \in \Gamma,\, p_2(\gamma) \in I\}. \tag{1.4}$$

2. Proof of Theorem 1.4.

If $K \subset \mathbb{R}^n$ is a compact set, $M_K \subset \mathbb{R}$ is defined by

$$M_K = \{p_2(\gamma^*); \gamma^* \in \Gamma^*,\, p_1(\gamma^*) \in K\}. \tag{2.1}$$

The density of Λ_I is uniform and is given by $c|I|$ where $c = c(\Gamma)$ and similarly the density of M_K is $|K|/c$ when K is Riemann integrable ([5], [6]). Therefore $|K| < \operatorname{dens} \Lambda_I$ implies $|I| > \operatorname{dens} M_K$ which will be crucial in what follows. We sort the elements of M_K in increasing order and denote the corresponding sequence by $\{m_k; k \in \mathbb{Z}\}$. Then we have ([5], [6])

Lemma 2.1. *The sequence $\{\tilde{m}_k; k \in \mathbb{Z}\}$ is equidistributed on K.*

We now prove our main result.
We replace K by a larger compact set still denoted by K which is Riemann integrable and still satisfies $|K| < \operatorname{dens} \Lambda$. By a standard density argument we can assume $\hat{f} \in \mathcal{C}_0^\infty(K)$. Lemma 2.1 implies

$$\frac{1}{|K|}\|\hat{f}\|_2^2 = \lim_{T\to\infty} \frac{1}{2T} \sum_{k=-T}^{T} |\hat{f}(\tilde{m}_k)|^2. \tag{2.2}$$

The right-hand side in(2.2) is given by

$$c_K \lim_{\varepsilon \downarrow 0} \varepsilon \sum_{k \in \mathbb{Z}} |\varphi(\varepsilon m_k)|^2 |\hat{f}(\tilde{m}_k)|^2 \tag{2.3}$$

where φ is any function in the Schwartz class $\mathcal{S}(\mathbb{R})$ normalized by $\|\varphi\|_2 = 1$. The constant $c_K = \frac{C}{|K|}$ is taking care of the density of the sequence $m_k, k \in \mathbb{Z}$ and C only depends on the lattice Γ. At this stage we use the auxiliary function of the real variable t defined as

$$F_\varepsilon(t) = \sqrt{\varepsilon} \sum_{k \in \mathbb{Z}} \varphi(\varepsilon m_k) \hat{f}(\tilde{m}_k) \exp(2\pi i m_k t). \tag{2.4}$$

We denote by ϕ the Fourier transform of φ. We will suppose that $\phi \in \mathcal{C}_0^\infty([-1,1])$ is a positive and even function. Since $|I| > \operatorname{dens} M_K$, Beurling's theorem applies to the interval I, to the set of frequencies M_K and to the trigonometric sum defined in (2.4). Then one has

$$\varepsilon \sum_{k\in\mathbb{Z}} |\varphi(\varepsilon m_k)|^2 |\hat{f}(\tilde{m}_k)|^2 \leq C \int_I |F_\varepsilon(t)|^2 dt. \tag{2.5}$$

Let us compute the lim sup as $\varepsilon \to 0$ of the right-hand side of (2.5). To this aim, we use the definition of M_K and write

$$F_\varepsilon(t) = \sqrt{\varepsilon} \sum_{\gamma^*\in\Gamma^*} \varphi(\varepsilon p_2(\gamma^*)) \hat{f}(p_1(\gamma^*)) \exp{(2\pi i p_2(\gamma^*) t)}. \tag{2.6}$$

Poisson identity says that this sum can be computed on the dual lattice. We then have

$$F_\varepsilon(t) = c(\Gamma) \frac{1}{\sqrt{\varepsilon}} \sum_{\gamma\in\Gamma} \phi(\frac{t - p_2(\gamma)}{\varepsilon}) f(p_1(\gamma)). \tag{2.7}$$

We then return to the estimation of

$$\limsup_{\varepsilon\downarrow 0} \int_I |F_\varepsilon(t)|^2 dt, \tag{2.8}$$

where F_ε is given by (2.7). To this end, we notice that all terms in the right-hand side of (2.7) for which $|p_1(\gamma)| \geq \alpha + \varepsilon$ vanish on $I = [-\alpha, \alpha]$. Indeed the support of ϕ is contained in $[-1, 1]$. We can restrict the summation to the set $\Lambda_{I,\varepsilon} = \{p_1(\gamma); \gamma \in \Gamma, |p_2(\gamma)| \leq \alpha + \varepsilon\}$. For $0 \leq \varepsilon \leq 1$ we have

$$\lim_{\varepsilon\to 0} \Lambda_{I,\varepsilon} = \Lambda_I \text{ and } \Lambda_{I,\varepsilon} \subset \Lambda_{I,1}. \tag{2.9}$$

We split F_ε into a sum $F_\varepsilon = F_\varepsilon^N + R_N$ where

$$F_\varepsilon^N(t) = \frac{1}{\sqrt{\varepsilon}} \sum_{\gamma\in\Gamma, |p_1(\gamma)|\leq N, |p_2(\gamma)|\leq\alpha+\varepsilon} \phi(\frac{t - p_2(\gamma)}{\varepsilon}) f(p_1(\gamma)), \tag{2.10}$$

and

$$R_N(t) = \frac{1}{\sqrt{\varepsilon}} \sum_{\gamma\in\Gamma, |p_1(\gamma)|>N, |p_2(\gamma)|\leq\alpha+\varepsilon} \phi(\frac{t - p_2(\gamma)}{\varepsilon}) f(p_1(\gamma)). \tag{2.11}$$

The triangle inequality yields $\|R_N\|_2 \leq \varepsilon_N \|\phi\|_2$ with

$$\varepsilon_N = \sum_{\gamma\in\Gamma, |p_1(\gamma)|>N, |p_2(\gamma)|\leq\alpha+1} |f(p_1(\gamma))|. \tag{2.12}$$

Let us observe that this series converges. Therefore ε_N tends to 0. Indeed f belongs to the Schwartz class and the set $Y = \{p_1(\gamma); |p_2(\gamma)| \leq \alpha + 1\}$ is uniformly sparse in $\mathbb{R}^n$. Using the terminology of [3], Y is a "model set". For the term (2.10) the estimations are more involved. Since $|p_1(\gamma)| \leq N$, the points $p_2(\gamma)$ appearing in (2.10) are separated by a distance $\geq \beta_N > 0$. If $0 < \varepsilon < \beta_N$ the different terms in (2.10) have disjoint supports which implies

$$\|F_\varepsilon^N\|_{L^2(I)} \leq \sigma(N,\varepsilon)\|\phi\|_2 \tag{2.13}$$

where

$$\sigma(N,\varepsilon)^2 = \sum_{\gamma\in\Gamma,|p_1(\gamma)|\leq N,|p_2(\gamma)|\leq\alpha+\varepsilon} |f(p_1(\gamma))|^2.$$

If ε is small enough we have

$$\{\gamma \in \Gamma, |p_1(\gamma)| \leq N, |p_2(\gamma)| \leq \alpha+\varepsilon\} = \{\gamma \in \Gamma, |p_1(\gamma)| \leq N, |p_2(\gamma)| \leq \alpha\}.$$

and $\sigma(N,\varepsilon) = \sigma(N,0)$. Therefore

$$\limsup_{\varepsilon\to 0} \int_I |F_\varepsilon(t)|^2 dt \leq \sum_{\lambda\in\Lambda_I} |f(\lambda)|^2 + \eta_N \tag{2.14}$$

and letting $N \to \infty$ we obtain the first claim. The following lemma clarifies and summarizes our proof:

Lemma 2.2. *Let x_j, $j \in \mathbb{N}$, be a sequence of pairwise distinct points in $\mathbb{R}^n$, let $f(x)$ be a function in $L^2(\mathbb{R}^n)$ with a compact support and, for $\varepsilon > 0$, let $f_\varepsilon(x) = \varepsilon^{-n/2} f(x/\varepsilon)$. Then for any sequence $c_j \in l^1$ we have*

$$\lim_{\varepsilon\to 0} \|\sum_0^\infty c_j f_\varepsilon(x - x_j)\|_2 = (\sum_0^\infty |c_j|^2)^{1/2} \tag{2.15}$$

The proof of the second claim uses the same strategy and notations. The first assertion of Beurling's theorem is used and the details can be found in a forthcoming paper.

François Golse raised the following problem. Let us assume $\Lambda = \{\lambda_j,\ j \in \mathbb{Z}\}$ where $\lambda_j = j + r_j,\ j \in \mathbb{Z}$, and r_j are equidistributed mod 1. Is it true that Λ is a set of stable sampling for any compact set K with measure $|K| < 1$? There are some examples where this happens. For instance if α is irrational, the set Λ defined by $\lambda_j = j + \{\alpha j\}$, $j \in \mathbb{Z}$, is a set of stable sampling for every compact set of the real line with a measure less than 1. Indeed this set Λ is a simple quasicrystal. On the other hand we cannot take r_j at random as the following lemma is showing:

Lemma 2.3. *Let r_j, $j \in \mathbb{Z}$, be independent random variables equidistributed in $[0,1]$. Then almost surely the random set $\Lambda = \{\lambda_j = j + r_j,\ j \in \mathbb{Z}\}$ is not a set of stable sampling.*

This lemma answers an issue raised by Jean-Michel Morel. Recently in [1] the problem of random sampling of band-limited functions was studied. More precisely, the authors proved the following

Proposition 2.1. *Let*

$$B = \{f \in L^2(\mathbb{R}^d) : \mathrm{supp}\hat{f} \subset [-1/2, 1/2]^d\},$$

be the space band-limited functions. Let $r \geq 1$ be the number of random samples in each cube $k + [0,1]^d$. With probability one the following holds: For each $k > 0$ there exists a function $f_k \in B$ such that

$$\sum_{x_i \in X(\omega)} |f_k(x_i)|^2 \leq \frac{1}{k}\|f_k\|_2^2. \tag{2.16}$$

Consequently, the sampling inequality is false almost surely.

Exactly the same proof applies to our lemma. Here are the details.

Our probability space Ω is $[0,1]^{\mathbb{N}}$ equipped with the product measure and the elements of Ω are denoted by $\omega = (r_j)_{j\in\mathbb{N}}$. The random set under study is $\Lambda(\omega)$. We now prove a stronger statement. Almost surely (1.1) fails for $\Lambda(\omega)$ when the compact set K in (1.1) is the union between the two intervals $[0, \alpha]$ and $[1, 1+\alpha]$ where α is arbitrarily small. The measure of K is 2α. To prove this statement, it suffices to construct a sequence of random functions $f_{N,\omega}(x),\ N \in \mathbb{N}$ such that $\|f_N\|_2 = \sqrt{2},\ \hat{f}_N$ is supported by K but $\sum_j |f_N(\lambda_j)|^2 \leq CN^{-2}$. We start with a function ϕ belonging to the Schwartz class, normalized by $\|\phi\|_2 = 1$ and such that the Fourier transform of ϕ is supported by $[0, \alpha]$. Then $f_N(x) = \phi(x - n_N)(\exp(2\pi i x) - 1)$ fulfils these requirements when $n_N = n_N(\omega)$ is a random integer which is now defined. Given N there are almost surely infinitely many integers m such that the following holds

$$m - N \leq j \leq m + N \Rightarrow 0 < r_j < 1/N. \tag{2.17}$$

This observation follows form the Borel-Cantelli lemma applied to the independent events $E_{k,N} = \{0 < r_j < 1/N,\ |3Nk - j| \leq N\}$. The probability of $E_{k,N}$ is N^{-2N-1} and the sum over k of these probabilities diverges. Therefore with probability 1, for every N there exists a random integer n_N such that $|n_N - j| \leq N \Rightarrow 0 < r_j < 1/N$. Since $f_N(j) = 0,\ \lambda_j = j + r_j$ and $0 < r_j < 1/N$ we have

$$\sum_{|n_N - j| \leq N} |f_N(\lambda_j)|^2 < CN^{-2}. \tag{2.18}$$

On the other hand the rapid decay of ϕ yields

$$\sum_{|n_N - j| > N} |f_N(\lambda_j)|^2 < CN^{-2}. \tag{2.19}$$

Putting these estimates together we can conclude.

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