Electromagnetics in deterministic and stochastic bianisotropic media

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November 18, 2011

Optical Activity is the ability of a material to rotate the plane of polarisation of a beam of light passing through it.

Original experimental studies by F. Arago (1811), J.-B. Biot (1812), A. Fresnel (1822).

A. L. Cauchy (1842 – first mathematical work on the laws of circular polarisation).

Explanation given by L. Pasteur (1848)[1]: introduction of Geometry into Chemistry (origin of the branch nowadays called Stereochemistry).[2]

[1] A notable historical coincidence: at the same time that Pasteur brought *left* and *right* into Chemistry, K. Marx and F. Engels "officially" introduced them into Politics (publication of the Communist Manifesto)!

[2] The Nobel Prize Lecture by V.Prelog (1975 Nobel Prize winner in Chemistry (together with Sir J. W. Cornforth)) is very interesting on the rôle of chirality in Chemistry.

Handedness is a characteristic of natural and manufactured objects (e.g., DNA, certain bacteria, shells, winding plants, spiral galaxies / cork-screws, doors, cookers, computer mice, keyboards, guitars, a variety of construction tools / Möbius strips, irregular tetrahedra).

The mirror image of a right-handed object is otherwise the same as the original, but it is left-handed (the original object cannot be superposed upon its mirror image).

A handed object is called chiral[3].

[3] From the Greek word $\chi\varepsilon\iota\varrho$ meaning hand. This term was introduced by Lord Kelvin in 1888 (first time in print in his famous 1904 Baltimore Lectures).

- It was the relation between the chiral (micro)structure and the (macroscopic) optical rotation that was discovered by Pasteur: he noticed that that two substances which where chemically identical (in the classification scheme of that time), but which had molecules being mirror images of each other, exhibited different physical properties[4].
- I. Kant[5] was the first eminent scholar to point out the philosophical significance of mirror operation: *"Hence the difference between similar and equal things, which are yet not congruent (for instance, two symmetric helices), cannot be made intelligible by any concept, but only by the relation to the right and the left hands which immediately refers to intuition".*

[4] E.g., one enantiomer of *thalidomide* may be used to cure morning sickness in pregnant women, but its mirror image induces fetal malformation - a big problem in the U.K. in the late 1950s.)

[5] In his celebrated book "Prolegomena To Any Future Metaphysics" (1783).

In the last part of the 19th century, after Maxwell's unification of optics with electricity and magnetism, it became possible to establish the connection between optical activity and the electromagnetic parameters of materials.

In 1914, Karl F. Lindman was the first to demonstrate the effect of a chiral medium on electromagnetic waves (his work in this field was about 40 years ahead of that of other scientists); he devised a macroscopic model for the phenomenon of "optical" activity that used microwaves instead of light and wire spirals instead of chiral molecules. His related work was published in 1920 and 1922.

The revival in the interest of complex media in Electromagnetics emerged in the mid 1980s, motivated and assisted by vast technological progress, especially at microwave frequencies.

Already in the beginning of the 21st century, the related publications within the Applied Physics and Electrical Engineering communities were calculated in more than 3000 papers.

Related books:

- A. Lakhtakia, V. K. Varadan, V. V. Varadan, 1989.
- A. Lakhtakia, 1994.
- I. Lindell, A. H. Sihvola, S. A. Tretyakov, A. J. Viitanen, 1994.
- A. Serdyukov, I. Semchenko, S. Tretyakov, A. Sihvola, 2001.
- S. Zouhdi, A. H. Sihvola, A. P. Vinogradov (eds.), 2009.
- G. Kristensson (in progress).

Mathematical Work – 1: Frequency domain problems (time-harmonic fields)

- (As far as I know) the first publication is by Petri Ola (1994).
- Simultaneous/independent work from the mid 1990s by the groups at
  - CMAP, École Polytechnique (Palaiseau): Jean-Claude Nédélec, Habib Ammari, and their collaborators.
  - the Department of Mathematics of the National and Kapodistrian University of Athens: Christos Athanasiadis, $\mathbb{S}$, and later on collaborators in various places.
- From the late 1990s, in addition to the above, many researchers enter the field; indicatively (in alphabetical, non-chronological, order) some names: A. Boutet de Monvel, G. Costakis, P. Courilleau, T. Gerlach, S. Heumann, T. Horsin, H. Kiili, V. Kravchenko, S. Li, P. A. Martin, S. R. McDowall, M. Mitrea, R. Potthast, D. Shepelsky, S. Vänskä, ...

Mathematical Work – 2: Time domain problems

From the early 2000s attention is focussed on the time domain, as well. Problems on the solvability, the homogenisation, and the controllability of IBVPs for the Maxwell equations, supplemented with nonlocal in time, linear constitutive relations (describing the so-called *bianisotropic* media), are studied.

Again a representative (yet incomplete) list of researchers:
C. E. Athanasiadis, G. Barbatis, A. Bossavit, P. Ciarlet jr., P. Courilleau, G. Griso, S. Halkos, T. Horsin, A. Ioannidis, A. Karlsson, G. Kristensson, G. Legendre, K. Liaskos, B. Miara, S. Nicaise, G. F. Roach, D. Sjöberg, $\mathbb{S}$., N. Wellander, A. N. Yannacopoulos, ...

- The biggest part of this work deals with deterministic bianisotropic media.
- But problems regarding stochastic bianisotropic media are also studied.

A new book will be published in spring 2012, by Princeton U. P.:
`G. F. Roach, I. G. Stratis, A. N. Yannacopoulos: Mathematical Analysis of Deterministic and Stochastic Problems in Complex Media Electromagnetics`

The Maxwell system

Electromagnetic phenomena are specified by 4 (vector) quantities: the *electric field* E, the *magnetic field* H, the *electric flux density* D and the *magnetic flux density* B. The inter-dependence between these quantities is given by the celebrated Maxwell system,

$$\begin{aligned} \operatorname{curl} H(t,x) &= \partial_t D(t,x) + J(t,x), \\ \operatorname{curl} E(t,x) &= -\partial_t B(t,x), \end{aligned} \tag{1}$$

where J is the electric current density. All fields are considered for $x \in \mathcal{O} \subset \mathbb{R}^3$ and $t \in \mathbb{R}$, $\mathcal{O}$ being a domain with appropriately smooth boundary. These equations are the so called Ampère's law and Faraday's law, respectively. In addition to the above, we have the two laws of Gauss

$$\begin{aligned} \operatorname{div} D(t,x) &= \rho(t,x), \\ \operatorname{div} B(t,x) &= 0, \end{aligned} \tag{2}$$

where ρ is the density of the (externally impressed) electric charge.

Initial and Boundary conditions

The initial conditions are considered to be of the form

$$E(0,x) = E_0(x)\,,\;\; H(0,x) = H_0(x)\,,\;\; x \in \mathcal{O}\,. \tag{3}$$

We consider the "perfect conductor" boundary condition

$$n(x) \times E(t,x) = 0\,,\;\; x \in \partial\mathcal{O}\,,\;\; t \in I\,, \tag{4}$$

where I is a time interval, and $n(x)$ denotes the outward normal on $\partial\mathcal{O}$.

From (1) and (2) we wish to determine the quadruplet (B, D, E, H), assuming that the vector J and the scalar ρ are known.
Need to calculate 12 scalar functions from a system of 8 scalar equations.
So, constitutive relations must be introduced

$$D = D(E, H)\,,\;\; B = B(E, H)\,. \tag{5}$$

It can be seen that we may consider as "the Maxwell system" the set of equations (1) ($\mathrm{curl} H = \partial_t D + J$ and $\mathrm{curl} E = -\partial_t B$), plus the constitutive relations (5), plus the equation of continuity

$$\partial_t \rho + \mathrm{div} J = 0. \tag{6}$$

The six vector notation

To express the system in more compact form, we use the six-vector notation:

- the electromagnetic flux density $\mathrm{d} := (D, B)^{tr}$,
- the electromagnetic field $u := (u_1, u_2)^{tr} := (E, H)^{tr}$,
- the current $j := (-J, 0)^{tr}$,
- the initial state $u_0 := (E_0, H_0)^{tr}$,

where the superscript tr denotes transposition.

A linear operator acting on u is written as a 2×2 (block) matrix with linear operators as its entries.

An important case is the Maxwell operator

$$\mathrm{M} := \begin{pmatrix} 0 & \mathrm{curl} \\ -\mathrm{curl} & 0 \end{pmatrix}. \tag{7}$$

The Maxwell system as an IVP

The constitutive relations are now modelled by an operator $\mathcal{L}$ and are understood as the functional equation

$$\mathrm{d} = \mathcal{L}u.$$

The properties of this operator reflect the physical properties of the medium in question.
So the Maxwell system can be written as an IVP for an abstract evolution equation

$$\begin{aligned} &(\mathcal{L}u)'(t) = \mathrm{M}u(t) + j(t) \quad , \quad \text{for } t > 0\,, \\ &u(0) = u_0\,. \end{aligned} \tag{8}$$

The prime stands for the time derivative.
The equation in the IVP (28) is an inhomogeneous *neutral functional* differential equation.

Postulates

To state the postulates that govern the evolution of the e/m field in a complex medium we follow a system-theoretic approach (Ioannidis (PhD; 2006) / Ioannidis, Kristensson, $\mathbb{S}$), in the sense that we consider the e/m field u as the cause, and the e/m flux density d as the effect.

Postulates (plausible physical hypotheses)

- Determinism: For every cause there exists exactly one effect.
- Linearity: The effect is produced linearly by its cause.
- Causality: The effect cannot precede its cause.
- Locality in space: A cause at any particular spatial point produces an effect only at this point and not elsewhere.
- Time–translation invariance: If the cause is advanced (or delayed) by some time interval, the same time-shift occurs for the effect.

Compliance with these postulates dictates the form of the operator $\mathcal{L}$.

Mathematical interpretation in terms of $\mathcal{L}$

- Determinism: $\mathcal{L}$ exists and is a single-valued nontrivial operator.
- Linearity: $\mathcal{L}$ is a linear operator.
- Causality: If $u(t,x) = 0$ for $t \leq \tau$, then $(\mathcal{L}u)(t,x) = 0$, for $t \leq \tau$.
- Locality in space: $\mathcal{L}$ is a local operator with respect to the spatial variables, i.e., $\mathcal{L}(u(\cdot,x))(\cdot,x) = \mathfrak{s}(\cdot,x)$ where $\mathfrak{s}$ is a local functional, allowing spatial derivatives of the electromagnetic fields, but not integrals with respect to the spatial variables.
  Locality with respect to temporal variables is not assumed, on the contrary memory effects are allowed.
- Time–translation invariance: For all $\varkappa \geq 0$, $\mathcal{L}$ commutes with the right $\varkappa$-shift operator $\tau_{\varkappa}$. Therefore, the time instant at which the observation starts does not play any significant rôle; the "present" can be chosen arbitrarily.

We do not assume continuity: it follows by linearity and time–translation invariance.
Note that continuity is not ascertained in the case where the left shift replaces the right shift.

The constitutive relations for bianisotropic media

The general form of $\mathcal{L}$, consistent with the above physical postulates, turns to be a continuous operator having the convolution form

$$\mathrm{d}(t,x) = (\mathcal{L}u)(t,x) = \mathrm{A}_{\mathrm{or}}(x)u(t,x) + \int_0^t \mathrm{G}_{\mathrm{d}}(t-s,x)u(s,x)\,ds \qquad (9)$$

$$\mathrm{A}_{\mathrm{or}}(x) := \begin{pmatrix} \varepsilon(x) & \xi(x) \\ \zeta(x) & \mu(x) \end{pmatrix}, \quad \mathrm{G}_{\mathrm{d}}(t,x) := \begin{pmatrix} \varepsilon_{\mathrm{d}}(t,x) & \xi_{\mathrm{d}}(t,x) \\ \zeta_{\mathrm{d}}(t,x) & \mu_{\mathrm{d}}(t,x) \end{pmatrix}. \qquad (10)$$

Each $\mathrm{A}_{\mathrm{or}}(\cdot)$, $\mathrm{G}_{\mathrm{d}}(t,\cdot)$ defines a multiplication operator in the state space. The above constitutive equation is abbreviated as

$$\mathrm{d} = \mathrm{A}_{\mathrm{or}}u + \mathrm{G}_{\mathrm{d}} \star u. \qquad (11)$$

The local in space part A_{or} (optical response operator) of $\mathcal{L}$ models the instantaneous response of the medium. The nonlocal in space part $\mathrm{G}_{\mathrm{d}}\star$ of $\mathcal{L}$ models the dispersion phenomena; G_{d} is called the susceptibility kernel.

Media Classification

A material is called

- **Isotropic:** if ε, μ, ε_d, μ_d are scalar multiples of $I_{3\times3}$ and $\xi = \zeta = \xi_d = \zeta_d = 0$.
- **Anisotropic:** if the members of at least one of the pairs $\varepsilon, \varepsilon_d$ or μ, μ_d are not scalar multiples of $I_{3\times3}$ and $\xi = \zeta = \xi_d = \zeta_d = 0$.
- **Biisotropic:** if all the blocks of the matrices A_{or}, G_{d} are scalar multiples of $I_{3\times3}$.
- **Bianisotropic:** in all other cases.

Assumptions

Assumption (A_{or})

The optical response matrix A_{or} *has*

- *essentially bounded entries*

and is

- almost everywhere symmetric,
  i.e., $\mathsf{A}_{or}(x) = \mathsf{A}_{or}(x)^{tr}$, *for almost all* $x \in \mathcal{O}$,
- almost everywhere uniformly coercive,
  i.e., there exists a constant C, *such that* $|\mathsf{y} \cdot \mathsf{A}_{or}(x)\,\mathsf{y}| \geq C\,|\mathsf{y}|^2$, *for almost all* $x \in \mathcal{O}$ *and all* $(0 \neq)\ \mathsf{y} \in \mathbb{R}^6$.

Assumption (G_d at $t = 0$)

The dispersion matrix $\mathsf{G}_d(0, x)$ *is*

- almost everywhere non-negative definite,
  i.e., $\mathsf{G}_d(0, x)\,\mathsf{y} \cdot \mathsf{y} \geq 0$, *for almost all* $x \in \mathcal{O}$ *and all* $(0 \neq)\ \mathsf{y} \in \mathbb{R}^6$.

The time-harmonic case

Taking the Fourier transform[6], we obtain

$$\widetilde{\mathsf{d}} = \mathsf{A}_{\mathrm{or}}\widetilde{u} + \widetilde{\mathsf{G}_{\mathrm{d}}}\widetilde{u}\,, \tag{12}$$

and letting

$$\widetilde{\mathsf{A}_{\mathrm{or}}} := \mathsf{A}_{\mathrm{or}} + \widetilde{\mathsf{G}_{\mathrm{d}}} = \begin{pmatrix} \varepsilon + \widetilde{\varepsilon_d} & \xi + \widetilde{\xi_d} \\ \zeta + \widetilde{\zeta_d} & \mu + \widetilde{\mu_d} \end{pmatrix} =: \begin{pmatrix} \varepsilon_{\mathfrak{F}} & \xi_{\mathfrak{F}} \\ \zeta_{\mathfrak{F}} & \mu_{\mathfrak{F}} \end{pmatrix}\,, \tag{13}$$

we get the frequency domain constitutive relations

$$\widetilde{\mathsf{d}} = \widetilde{\mathsf{A}_{\mathrm{or}}}\widetilde{u}\,. \tag{14}$$

[6]Let $\mathfrak{s}$ be a proxy for the vector fields d, u, j, and ϖ be the angular frequency. Then $\mathfrak{s}(t,x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\varpi t}\,\widetilde{\mathfrak{s}}(\varpi,x)\,d\varpi$.

Assumptions

Assumption $(\widetilde{\mathsf{A}_{\mathrm{or}}})$

Let c, C be positive constants. For any fixed frequency ϖ, the matrix $\widetilde{\mathsf{A}_{\mathrm{or}}} = \widetilde{\mathsf{A}_{\mathrm{or}}}(x;\varpi)$ satisfies the following properties

(i) $\widetilde{\mathsf{A}_{\mathrm{or}}} \in L^\infty(\mathcal{O}, \mathbb{C}^{6\times 6})$.

(ii) $\bar{\mathsf{z}}^{tr} \cdot (\mathcal{I}\mathrm{m}\ \widetilde{\mathsf{A}_{\mathrm{or}}})\,\mathsf{z} \geq c\,||\mathsf{z}||^2$, for all $\mathsf{z} \in \mathbb{C}^6$.

(iii) $|\mathsf{z}_1 \cdot \widetilde{\mathsf{A}_{\mathrm{or}}}\,\mathsf{z}_2| \leq C\,||\mathsf{z}_1||\,||\mathsf{z}_2||$, for all $\mathsf{z}_1, \mathsf{z}_2 \in \mathbb{C}^6$.

We now elaborate further on the classification of biisotropic media in the frequency domain. Let

$$\xi_{\mathfrak{F}} = \kappa + i\chi\,,\ \zeta_{\mathfrak{F}} = \kappa - i\chi\,. \tag{15}$$

The *chirality parameter* χ measures the degree of handedness of the material; a change in the sign of χ corresponds to the consideration of the mirror image of the material. The other parameter κ describes the magnetoelectric effect; materials with $\kappa \neq 0$ are nonreciprocal.

Media classification

In the time-harmonic case a medium is called:

- **Isotropic**, if $\kappa = 0$ and $\chi = 0$, i.e., when $\xi_{\mathfrak{F}} = \zeta_{\mathfrak{F}} = 0$.
- **Nonreciprocal Nonchiral**, or **Tellegen**, if $\kappa \neq 0$ and $\chi = 0$, i.e., when $\xi_{\mathfrak{F}} = \zeta_{\mathfrak{F}}$.
- **Reciprocal Chiral**, or **Pasteur**, if $\kappa = 0$ and $\chi \neq 0$, i.e., when $\xi_{\mathfrak{F}} = -\zeta_{\mathfrak{F}}$.
- **Nonreciprocal Chiral** or **General Biisotropic**, if $\kappa \neq 0$ and $\chi \neq 0$, i.e., when $\xi_{\mathfrak{F}} \neq \zeta_{\mathfrak{F}}, -\zeta_{\mathfrak{F}}$.

Reciprocal chiral media will be simply referred to as **chiral** media.
In the case of chiral media the constitutive relations for time-harmonic fields are usually written as

$$\widetilde{D} = \varepsilon_{\mathsf{T}} \widetilde{E} + \beta_{\mathsf{T}} \widetilde{H}, \quad \widetilde{B} = \mu_{\mathsf{T}} \widetilde{H} - \beta_{\mathsf{T}} \widetilde{E}, \tag{16}$$

(where $\varepsilon_{\mathsf{T}} := \varepsilon_{\mathfrak{F}}$, $\mu_{\mathsf{T}} := \mu_{\mathfrak{F}}$, $\beta_{\mathsf{T}} := \xi_{\mathfrak{F}} = -\zeta_{\mathfrak{F}}$).

The DBF constitutive relations

The *Drude-Born-Fedorov* (DBF) constitutive relations were introduced in 1959 by F. I. Fedorov as a modification of constitutive relations used in 1900 by P. K. L. Drude and in 1915 by M. Born. These read

$$\widetilde{D} = \varepsilon_{\mathsf{DBF}}(\widetilde{E} + \beta_{\mathsf{DBF}}\,\mathrm{curl}\widetilde{E}), \quad \widetilde{B} = \mu_{\mathsf{DBF}}(\widetilde{H} + \beta_{\mathsf{DBF}}\,\mathrm{curl}\widetilde{H}). \tag{17}$$

The medium is characterised by three[7] (in general, complex) parameters, the electric permittivity $\varepsilon_{\mathsf{DBF}}$, the magnetic permeability μ_{DBF}, and the chirality measure β_{DBF}.

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[7]For source-free regions the constitutive parameters ε_{T}, μ_{T} and β_{T} of (16) are connected to the constitutive parameters $\varepsilon_{\mathsf{DBF}}$, μ_{DBF} and β_{DBF} of (17) via

$$\varepsilon_{\mathsf{T}} = \frac{\varepsilon_{\mathsf{DBF}}}{1 - \varpi^2 \varepsilon_{\mathsf{DBF}}\mu_{\mathsf{DBF}}\beta_{\mathsf{DBF}}^2}, \quad \mu_{\mathsf{T}} = \frac{\mu_{\mathsf{DBF}}}{1 - \varpi^2 \varepsilon_{\mathsf{DBF}}\mu_{\mathsf{DBF}}\beta_{\mathsf{DBF}}^2},$$

$$\beta_{\mathsf{T}} = i\varpi\varepsilon_{\mathsf{DBF}}\mu_{\mathsf{DBF}}\frac{\beta_{\mathsf{DBF}}}{1 - \varpi^2 \varepsilon_{\mathsf{DBF}}\mu_{\mathsf{DBF}}\beta_{\mathsf{DBF}}^2}.$$

Time-harmonic problems

The study of time-harmonic problems for the Maxwell equations supplemented with the DBF constitutive relations is a well-developed area. The solvability of interior and exterior BVPs is established by variational techniques, and their discretised versions are studied as well.
The interior problem reads

$$\begin{aligned} \operatorname{curl} E &= \beta\gamma^2 E + i\varpi\mu\left(\tfrac{\gamma}{k}\right)^2 H, \\ \operatorname{curl} H &= \beta\gamma^2 H - i\varpi\varepsilon\left(\tfrac{\gamma}{k}\right)^2 E, \end{aligned} \qquad \text{in } \mathcal{O}, \tag{18}$$

where $\varpi > 0$ is the angular frequency and

$$k^2 := \varpi^2\varepsilon\mu \quad , \quad \gamma^2 := k^2(1-\beta^2k^2)^{-1}. \tag{19}$$

These equations are complemented with the boundary condition

$$n \times E = f, \quad \text{on } \partial\mathcal{O}, \tag{20}$$

where $f \in H^{-1/2}(\mathrm{div}, \partial\mathcal{O})$ is a prescribed electric field on $\partial\mathcal{O}$.

Assumptions

A typical assumption on the data is

Assumption

(i) $\mathcal{O}$ is a bounded domain and $\partial\mathcal{O}$ is of class $C^{1,1}$.

(ii) The coefficients ε, μ and β are real valued and positive $C^2(\overline{\mathcal{O}})$ functions.

(iii) The function $\mu^{-1}(1 - \varpi^2\varepsilon\mu\beta^2)$ is positive in $\overline{\mathcal{O}}$.

The solvability of the interior problem ((18), (20)) is established by Ammari and Nédélec, while the study of the discretised version is by $\mathbb{S}$ and Yannacopoulos.

The Exterior Problem – Radiation Conditions

As for the exterior problem, it consists of the equations

$$\left.\begin{aligned} \operatorname{curl} E_e &= \beta_e \gamma_e^2 E_e + i\varpi\mu_e \left(\frac{\gamma_e}{k_e}\right)^2 H, \\ \operatorname{curl} H_e &= \beta_e \gamma_e^2 H_e - i\varpi\varepsilon_e \left(\frac{\gamma_e}{k_e}\right)^2 E, \end{aligned}\right\} \quad \text{in } \mathcal{O}_e, \tag{21}$$

with boundary condition

$$n \times E_e = f_e, \quad \text{on } \partial\mathcal{O}_e$$

and with one of the two Silver-Müller radiation conditions

$$\lim_{|x|\to\infty} |x| \left(\sqrt{\mu} H_e \times \widehat{x} - \sqrt{\varepsilon} E_e\right) = 0\,, \tag{22}$$

or

$$\lim_{|x|\to\infty} |x| \left(\sqrt{\varepsilon} E_e \times \widehat{x} + \sqrt{\mu} H_e\right) = 0\,, \tag{23}$$

uniformly over all directions $\widehat{x}$.

- It is known (Ammari and Nédélec) that the "standard" (achiral) Silver-Müller radiation conditions (written above) are adequate to cover the chiral case, too.
- Another important property is a "continuity result" in terms of the chirality measure β. It is known (Ammari and Nédélec) that if β is assumed to be a non-negative constant, then the limit of the solution of the chiral problem as $\beta \to 0$ coincides with the solution of the corresponding achiral ($\beta = 0$) problem.

Scattering problems – BIEs

Another very well developed area of research regarding chiral media in the time-harmonic regime deals with scattering problems.
Consider that an electromagnetic wave propagating in a chiral (or achiral) homogeneous environment is incident upon a chiral (or achiral) obstacle.
Depending on the materials with which the surrounding space and the obstacle are filled, and on the boundary condition(s) on the obstacle's surface, a variety of scattering problems (exterior BVPs, transmission problems) is treated. For example,

- Using classical potentials, BIEs are employed to study the solvability, the determination of uniquely solvable equations, and the "low-chirality" approximation:
  Ammari, Nédélec / Athanasiadis, $\mathbb{S}$ / Athanasiadis, Costakis, $\mathbb{S}$ / Athanasiadis, Martin, $\mathbb{S}$ / Mitrea / Ola.

- Typical scattering results (e.g., the reciprocity principle, the optical theorem, the general scattering theorem) are extended to the chiral case for plane and spherical incident waves:
  Athanasiadis, Martin, $\mathbb{S}$ / Athanasiadis, Giotopoulos / Athanasiadis, Tsitsas.
- A Low-Frequency theory is developed:
  Ammari, Laouadi, Nédélec / Athanasiadis, Costakis, $\mathbb{S}$.
- Herglotz functions and pairs are introduced and studied:
  Athanasiadis, Kardasi.
- Infinite Fréchet differentiability of the mapping from the boundary of the scatterer onto the far-field patterns is established, along with a characterisation of the Fréchet derivative as a solution to an appropriate boundary value problem:
  Potthast, $\mathbb{S}$.

- Periodic structures, gratings:
  Ammari, Bao / Zhang, Ma.
- Quaternionic methods:
  Kravchenko.
- Inverse scattering problems:
  Athanasiadis, $\mathbb{S}$ / Boutet de Monvel, Shepelsky / Gao, Ma, Zhang / Gerlach / Heumann / Li / Mc Dowall / Rikte, Sauviac, Kristensson, Mariotte.

The Bohren decomposition – Beltrami fields

In view of the DBF constitutive relations, the Maxwell equations can be written as

$$\begin{pmatrix} \operatorname{curl} E \\ \operatorname{curl} H \end{pmatrix} = \frac{\gamma^2}{k^2} \begin{pmatrix} \beta k^2 & \mathrm{i}\,\varpi\mu \\ -\mathrm{i}\,\varpi\varepsilon & \beta k^2 \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} . \tag{24}$$

Diagonalising the matrix in (24), we obtain

$$\begin{pmatrix} \operatorname{curl}(\mathrm{i}\,\eta^{-1}E + H) \\ \operatorname{curl}(E + \mathrm{i}\,\eta\, H) \end{pmatrix} = \begin{pmatrix} \frac{k}{1-k\beta} & 0 \\ 0 & -\frac{k}{1+k\beta} \end{pmatrix} \begin{pmatrix} \mathrm{i}\,\eta^{-1}E + H \\ E + \mathrm{i}\,\eta\, H \end{pmatrix} , \tag{25}$$

where $\eta = \mu^{1/2}\varepsilon^{-1/2}$ is the intrinsic impedance of the medium.

The Bohren decomposition – Beltrami fields

Introducing the fields

$$Q_{\mathrm{L}} := \mathrm{i}\,\eta^{-1} E + H\,,\ \ Q_{\mathrm{R}} := E + \mathrm{i}\,\eta\, H\,,$$

we note that (25) is written as

$$\begin{aligned} \mathrm{curl} Q_{\mathrm{L}} &= \gamma_{\mathrm{L}}\, Q_{\mathrm{L}}, \\ \mathrm{curl} Q_{\mathrm{R}} &= -\gamma_{\mathrm{R}}\, Q_{\mathrm{R}}, \end{aligned} \tag{26}$$

where

$$\gamma_{\mathrm{L}} := k(1 - k\beta)^{-1}\,,\ \gamma_{\mathrm{R}} := k(1 + k\beta)^{-1}\,.$$

Note that $\gamma^2 = \gamma_{\mathrm{L}}\gamma_{\mathrm{R}}$. Q_{L} and Q_{R} satisfy the vector Helmholtz equation

$$\Delta Q_\lambda + \gamma_\lambda^2 Q_\lambda = 0,\ \ \lambda = L, R\,,$$

and $\gamma_{\mathrm{L}}, \gamma_{\mathrm{R}}$ are the wave numbers of the Beltrami fields Q_{L}, Q_{R}, respectively. If E, H are divergence free, the same holds for Q_{L}, Q_{R}, as well.

The Bohren decomposition – Beltrami fields

Thus we obtain the *Bohren decomposition* of E, H into Q_{L}, Q_{R}

$$\begin{aligned} E &= Q_{\mathrm{L}} - \mathrm{i}\,\eta\, Q_{\mathrm{R}}, \\ H &= Q_{\mathrm{R}} - \mathrm{i}\,\eta^{-1}\, Q_{\mathrm{L}}. \end{aligned} \tag{27}$$

From (26), and if the two complex-valued wave numbers γ_{L} and γ_{R} have positive real parts, we note that while Q_{L} is a left-handed Beltrami field, Q_{R} is a right-handed one.

This decomposition is very useful in the study of chiral media, since representation formulae for the fields in chiral media - that constitute the first basic step in developing BIE methods - can easily be deduced from the corresponding ones of metaharmonic fields.

Time-domain problems: Well-posedness

Several alternative approaches to the solvability of the IVP for the Maxwell system

$$\begin{aligned} &(\mathcal{L}u)'(t) = \mathrm{M}u(t) + j(t) \quad , \quad \text{for } t > 0\,, \\ &u(0) = u_0\,, \end{aligned} \tag{28}$$

supplemented with the constitutive relations for dissipative bianisotropic media

$$(\mathcal{L}u)(t,x) = \mathrm{A}_{\mathrm{or}}(x)u(t,x) + \int_0^t \mathrm{G}_{\mathrm{d}}(t-s,x)u(s,x)\,ds\,, \tag{29}$$

can be considered, e.g., semigroups, evolution families, the Faedo–Galerkin method.
We adopt the former, based on the semigroup generated by the Maxwell operator. Then the convolution terms are treated as perturbations of this semigroup.

The choice of the semigroup approach is plausible since

- the semigroup (group actually) generated by the Maxwell operator is very well studied,
- the kernels in the convolution terms are known to be physically small, therefore, it is plausible to consider them as perturbations.

These approaches have been used in different variations by Bossavit, Griso and Miara / Ciarlet jr. and Legendre / Ioannidis, Kristensson and $\mathbb{S}$ / Liaskos, $\mathbb{S}$ and Yannacopoulos.

For the integrodifferential equation (28) a variety of different types of solutions can be defined, regarding *spatial* - or *temporal* - regularity. We totally skip their technical descriptions here.

So a well-posedness result can be stated

Well-posedness

Theorem

Under suitable regularity assumptions on the data, (28) is weakly / mildly / strongly / classically well-posed.

The underlying space is $H_0(\mathrm{curl}, \mathcal{O}) \times H(\mathrm{curl}, \mathcal{O})$, where

- $H(\mathrm{curl}, \mathcal{O}) := \{u \in (L^2(\mathcal{O}))^3 \ : \ \mathrm{curl}\, u \in (L^2(\mathcal{O}))^3\}$.
- For bounded $\mathcal{O}$, $H_0(\mathrm{curl}, \mathcal{O})$ is the space $\{u \in H(\mathrm{curl}, \mathcal{O}) \ : \ n \times u|_{\partial\mathcal{O}} = 0\}$.

The first component of the underlying space incorporates the perfect conductor boundary condition for the electric field.

Other constitutive relations in the time-domain

Instead of using the (general) non-local in time constitutive relations

$$\mathsf{d} = \mathsf{A}_{\mathrm{or}} u + \mathsf{G}_{\mathrm{d}} \star u \,,$$

employed so far, a variety of problems on the class of complex media described by adopting the DBF-like (local in time) constitutive relations in the time-domain

$$\mathsf{d} = \mathsf{A}_0 u + \beta\, \mathsf{C}\, u \,,$$

where

$$\mathsf{A}_0 := \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \;,\; \mathsf{C} := \begin{pmatrix} \varepsilon\, \mathrm{curl} & 0 \\ 0 & \mu\, \mathrm{curl} \end{pmatrix},$$

has been studied by a number of authors, e.g., Ciarlet jr., Legendre / Ciarlet jr., Legendre, Nicaise / Courilleau, Horsin / Courilleau, Horsin, $\mathbb{S}$ / Liaskos, $\mathbb{S}$, Yannacopoulos.

In these studies there are subtleties involved regarding the spectrum of the operator curl.

Controllability Problems

The governing equation

$$(\mathsf{A}_{\mathsf{or}} u + \mathsf{G}_{\mathsf{d}} \star u)' = \mathsf{M} u + j, \tag{30}$$

can be simplified if we assume that $\mathsf{G}_{\mathsf{d}}(t, x)$ is weakly differentiable with respect to the temporal variable. Then we may differentiate the convolution integral, and by multiplying to the right by $\mathsf{A}_{\mathsf{or}}^{-1}$ we get

$$u' = \mathsf{M}_{\mathsf{A}} u + \mathsf{G}_{\mathsf{A}} \star u + \mathsf{J}_{\mathsf{A}}, \tag{31}$$

where

$$\mathsf{G}_{\mathsf{A}} := -\mathsf{A}_{\mathsf{or}}^{-1} \mathsf{G}_{\mathsf{d}}', \ \mathsf{M}_{\mathsf{A}} := \mathsf{A}_{\mathsf{or}}^{-1} \mathsf{M}, \ \mathsf{J}_{\mathsf{A}} := \mathsf{A}_{\mathsf{or}}^{-1} j,$$

and we have assumed that $\mathsf{G}_{\mathsf{d}}(0, x) = 0$.
The boundary conditions, as well as the divergence free character of the electromagnetic field, can be included in the definition of the operator M in appropriately selected function spaces.

We now assume that we have access to an *internal control* v, that acts on the system. The action of the control v on the state of the system is modelled, via the so-called *control to state operator* $\mathcal{B}$, by the evolution equation

$$u' = \mathsf{M}_{\mathsf{A}} u + \mathsf{J}_{\mathsf{A}} + \mathsf{G}_{\mathsf{A}} \star u + \mathcal{B} v. \tag{32}$$

The problem of controllability can now be stated as follows:

`Given` $T > 0$`, an initial condition` $u(0) = U_0$ `and a final condition` $u(T) = U_T$`, can we find a control procedure` $v^*(\cdot)$ `such that the solution of the system (32) with` $v(\cdot) = v^*(\cdot)$ `satisfies` $u(0) = U_0$ `and` $u(T) = U_T$`?`

By a perturbative (fixed-point scheme) approach based on J.-L. Lions' **Hilbert Uniqueness Method** the internal controllability of (32) has been studied ($\mathbb{S}$ and Yannacopoulos).

The study of boundary controllability problems and optimal control problems is in progress.

In the case of time-harmonic fields, the approximate controllability problem has been studied (Horsin and $\mathbb{S}$).

Homogenisation

Within the electromagnetic (applied physics/electrical engineering) community, homogenisation of composites has a huge literature, the major part of which is devoted to dielectrics.

In this community, the related literature on bianisotropic composites is much smaller. Among the recent developments are the work on Maxwell Garnett and Bruggeman formalisms for different classes of bianisotropic inclusions and the work on the Strong Property Fluctuation Theory for bianisotropic composites.

There are many important applications, e.g., in biomedical engineering and optics (optical waveguides, high-dielectric thin-film capacitors, captive video disc units, novel antennas and design of complementary split-ring resonators).

For isotropic media there is important rigorous mathematical work by many authors: see, in particular, the contribution by Artola and Cessenat / Bensoussan, J.-L. Lions and Papanicolaou / Jikov, Kozlov and Oleinik / Markowich and Poupaud / Sanchez-Hubert / Sanchez-Palencia / Visintin / Wellander.

For dissipative bianisotropic media the problem was originally studied by Barbatis and $\mathbb{S}$ (2003) and further developed by $\mathbb{S}$ and Yannacopoulos.

Related work by Bossavit, Griso and Miara (2005) / Sjöberg (2005) / J.S. Jiang, C.K. Lin and C.H. Liu (2008) / L.Cao, Y. Zhang, Allegretto and Y. Lin (2010).

Let $\mathcal{O}$ be a domain in $\mathbb{R}^3$, filled by a complex electromagnetic medium modelled by constitutive relations of the general form

$$\mathrm{d} = \mathrm{A}_{\mathrm{or}} u + \mathrm{G}_{\mathrm{d}} \star u. \tag{33}$$

The material is spatially inhomogeneous i.e., $\mathrm{A}_{\mathrm{or}} = \mathrm{A}_{\mathrm{or}}(x)$, $\mathrm{G}_{\mathrm{d}} = \mathrm{G}_{\mathrm{d}}(x)$. The evolution of the field $u = (u_1, u_2)^{tr}$ in $\mathcal{O}$ is governed by the Maxwell equations,

$$(\mathrm{A}_{\mathrm{or}} u + \mathrm{G}_{\mathrm{d}} \star u)' = \mathrm{M} u + j, \tag{34}$$

complemented with the initial condition

$$u = 0, \qquad x \in \mathcal{O},\ t = 0, \tag{35}$$

and the perfect conductor boundary condition,

$$n \times u_1 = 0, \qquad t > 0,\ x \in \partial\mathcal{O}\,, \tag{36}$$

where n is the outward unit normal on $\partial\mathcal{O}$.

Assumption

The medium exhibits small scale periodicity, i.e.,

$$\begin{aligned} \mathsf{A}_{\mathrm{or}} &= \mathsf{A}^{\epsilon}_{\mathrm{or}}(x) = \mathsf{A}^{\mathrm{per}}_{\mathrm{or}}\left(\frac{x}{\epsilon}\right) , \\ \mathsf{G}_{\mathrm{d}} &= \mathsf{G}^{\epsilon}_{\mathrm{d}}(x) = \mathsf{G}^{\mathrm{per}}_{\mathrm{d}}\left(\frac{x}{\epsilon}\right) , \end{aligned} \tag{37}$$

where $\mathsf{A}^{\mathrm{per}}_{\mathrm{or}}(\cdot)$, $\mathsf{G}^{\mathrm{per}}_{\mathrm{d}}(\cdot)$ are periodic matrix-valued functions on the parallelepiped $Y = [0, \ell_1] \times [0, \ell_2] \times [0, \ell_3] \subset \mathbb{R}^3$ and $0 < \epsilon \ll 1$.

The set Y may be considered as the fundamental cell of the medium; the whole medium structure can be generated by repeating the structure in Y using translations.

To ease notation, we drop the superscript "per" from $\mathsf{A}^{\mathrm{per}}_{\mathrm{or}}$ and $\mathsf{G}^{\mathrm{per}}_{\mathrm{d}}$ and use $\mathsf{A}^{\epsilon}_{\mathrm{or}}(x) = \mathsf{A}_{\mathrm{or}}\left(\frac{x}{\epsilon}\right)$ and $\mathsf{G}^{\epsilon}_{\mathrm{d}}(x) = \mathsf{G}_{\mathrm{d}}\left(\frac{x}{\epsilon}\right)$ instead.

We also need the following definition:
If $a : Y \to \mathbb{R}$ is a periodic function then the periodic averaging operator is

$$\langle a \rangle := \frac{1}{|Y|} \int_Y a(y)\, dy,$$

where $|Y| = \ell_1\, \ell_2\, \ell_3$ is the Lebesgue measure of Y.
In order to be able to model the small scale periodic microstructure, we must let ϵ vary over a range of arbitrarily small values. Since $\mathrm{A}_{\mathrm{or}}^{\mathrm{per}}(x)$ is periodic with period Y, it follows that $\mathrm{A}_{\mathrm{or}}^{\mathrm{per}}\left(\frac{x}{\epsilon}\right)$ is periodic with period ϵY.
We are therefore led to a sequence of boundary value problems,

$$(\mathrm{A}_{\mathrm{or}}^{\epsilon} u^{\epsilon} + \mathrm{G}_{\mathrm{d}}^{\epsilon} \star u^{\epsilon})' = \mathrm{M} u^{\epsilon} + j \tag{38}$$

with initial condition $u^{\epsilon} = 0$, and the perfect conductor boundary condition

$$n \times u_1^{\epsilon} = 0, \;\; \text{on } \mathcal{O}. \tag{39}$$

The solution of the above sequence of boundary value problems exists for all $\epsilon > 0$ by the following result. This generates a sequence of functions $u^\epsilon = u^\epsilon(x,t)$.

Theorem (Existence and uniform bounds)

Assume that j is locally Hölder continuous, and further that $j \in L^1([0,T],\mathbb{X})$. Then the Maxwell system (38)–(39) has a unique solution $u^\epsilon = (u_1^\epsilon, u_2^\epsilon)^{tr} = (E^\epsilon, H^\epsilon)^{tr}$ in $C([0,T],\mathbb{X})$ satisfying the uniform bounds $||u^\epsilon(t)||_{\mathbb{X}} < C$, $\epsilon > 0$, where $\mathbb{X} := (L^2(\mathcal{O}))^3 \times (L^2(\mathcal{O}))^3$.

The claim follows either by semigroup theory, or by the Faedo-Galerkin approach.

The question of interest to homogenisation theory is what happens in the limit of very small scale microstructures.

There are alternative ways that can be used to treat the homogenisation problem. Two principal ones are based on

- The reduction to a properly selected elliptic homogenisation problem,
- A rigourisation of the double scale expansion method, the so-called periodic unfolding method.

Reduction to Elliptic Homogenisation – A two-scale Expansion

Taking the Laplace transform of (38) (with respect to t) and dropping the explicit dependence on the Laplace variable p we obtain

$$\begin{aligned} \mathrm{curl}\widehat{H}^\epsilon &= p(\varepsilon^\epsilon_{\mathfrak{L}}\,\widehat{E}^\epsilon + \xi^\epsilon_{\mathfrak{L}}\,\widehat{H}^\epsilon) + \widehat{J}, \\ -\mathrm{curl}\widehat{E}^\epsilon &= p(\mu^\epsilon_{\mathfrak{L}}\,\widehat{H}^\epsilon + \zeta^\epsilon_{\mathfrak{L}}\,\widehat{E}^\epsilon). \end{aligned} \tag{40}$$

where $\varepsilon^\epsilon_{\mathfrak{L}}, \xi^\epsilon_{\mathfrak{L}}, \mu^\epsilon_{\mathfrak{L}}, \zeta^\epsilon_{\mathfrak{L}}$ are given by

$$\begin{pmatrix} \varepsilon^\epsilon_{\mathfrak{L}} & \xi^\epsilon_{\mathfrak{L}} \\ \zeta^\epsilon_{\mathfrak{L}} & \mu^\epsilon_{\mathfrak{L}} \end{pmatrix} := \begin{pmatrix} \varepsilon^\epsilon + \widehat{\varepsilon}^\epsilon_d & \xi + \widehat{\xi}^\epsilon_d \\ \zeta^\epsilon + \widehat{\zeta}^\epsilon_d & \mu^\epsilon + \widehat{\mu}^\epsilon_d \end{pmatrix} =: \mathsf{A}^\epsilon_{\mathfrak{L}}(y,p)\,, \tag{41}$$

where $\widehat{\mathfrak{s}}$ denotes the Laplace transform of $\mathfrak{s}$.

Assume that the electromagnetic field has an expansion in power series in ϵ

$$\begin{aligned}\widehat{E}^{\epsilon}(x) &= \widehat{E}^{(0)}(x) + \epsilon\,\widehat{E}^{(1)}(x) + \epsilon^2\,\widehat{E}^{(2)}(x) + \cdots\,, \\ \widehat{H}^{\epsilon}(x) &= \widehat{H}^{(0)}(x) + \epsilon\,\widehat{H}^{(1)}(x) + \epsilon^2\,\widehat{H}^{(2)}(x) + \cdots\,,\end{aligned}$$

where

$$\widehat{E}^{(j)}(x) = \widehat{E}^{(j)}\left(x, \frac{x}{\epsilon}\right)\,,\ \widehat{H}^{(j)}(x) = \widehat{H}^{(j)}\left(x, \frac{x}{\epsilon}\right)\,,$$

that may be considered as functions of the variables x and $y = \frac{x}{\epsilon}$ (considered as independent variables).

Substituting these expressions in (40) and working with orders $O(\epsilon^{-1})$, $O(\epsilon^{0})$ we can formally see that the homogenised system has coefficients obtained by averaging the original coefficients and then multiplying the outcome by solutions of the deduced cell equations.

This formal two-scale expansion motivates the following rigorous result.

Theorem

Assume that the family of matrices $\mathbb{A}(y,p)$ *satisfies*

$$(\mathcal{R}e\,\mathbb{A}(y,p))u\cdot u \geq c\|u\|^2, \quad y\in Y,\ p\in\mathbb{C}_+,\ u\in\mathbb{R}^6. \tag{42}$$

for $\mathbb{A} = \mathrm{A}_{\mathfrak{L}}, \mathrm{A}_{\mathfrak{L}}^{-1}$ *(defined in* (41)*).*
The solution $u^\epsilon = (E^\epsilon, H^\epsilon)^{tr}$ *of* $(\mathrm{A}^\epsilon_{or}u^\epsilon + \mathrm{G}^\epsilon_d \star u^\epsilon)' = \mathrm{M}u^\epsilon + j$ *with zero initial conditions and the perfect conductor boundary condition satisfies*

$$u^\epsilon \overset{*}{\rightharpoonup} u^*, \quad in\ L^\infty([0,T],\mathbb{X}),$$

where $u^* = (E^*, H^*)^{tr}$ *is the unique solution of the homogeneous Maxwell system*

$$(\mathrm{d}^*)' = \mathrm{M}u^* + j, \quad in\ (0,T]\times\mathcal{O}, \tag{43}$$

with zero initial conditions and the perfect conductor boundary condition, and subject to the constitutive relations

$$\mathrm{d}^* = \mathrm{A}^h_{or}u^* + \mathrm{G}^h_d \star u^* \tag{44}$$

such that $\mathrm{A}^h_{or} + \widehat{\mathrm{G}^h_d} = \mathrm{A}^h_{\mathfrak{L}}$*, where*

$$\mathrm{A}^h_{\mathfrak{L}} = \begin{pmatrix} \varepsilon^h_{\mathfrak{L}} & \xi^h_{\mathfrak{L}} \\ \zeta^h_{\mathfrak{L}} & \mu^h_{\mathfrak{L}} \end{pmatrix},$$

Theorem (continued)

$$\varepsilon_{\mathfrak{L}}^{h} := \langle \varepsilon_{\mathfrak{L}} + \varepsilon_{\mathfrak{L}}\,\mathrm{grad}_y R_1 + \zeta_{\mathfrak{L}}\,\mathrm{grad}_y R_2 \rangle,$$
$$\zeta_{\mathfrak{L}}^{h} := \langle \zeta_{\mathfrak{L}} + \varepsilon_{\mathfrak{L}}\,\mathrm{grad}_y V_1 + \zeta_{\mathfrak{L}}\,\mathrm{grad}_y V_2 \rangle, \qquad (45)$$
$$\xi_{\mathfrak{L}}^{h} := \langle \xi_{\mathfrak{L}} + \xi_{\mathfrak{L}}\,\mathrm{grad}_y R_1 + \mu_{\mathfrak{L}}\,\mathrm{grad}_y R_2 \rangle,$$
$$\mu_{\mathfrak{L}}^{h} := \langle \mu_{\mathfrak{L}} + \xi_{\mathfrak{L}}\,\mathrm{grad}_y V_1 + \mu_{\mathfrak{L}}\,\mathrm{grad}_y V_2 \rangle,$$

$$\mathrm{grad}_y \mathfrak{S}_\ell = \begin{pmatrix} \partial_{y_1}\mathfrak{s}_\ell^{(1)} & \partial_{y_2}\mathfrak{s}_\ell^{(1)} & \partial_{y_3}\mathfrak{s}_\ell^{(1)} \\ \partial_{y_1}\mathfrak{s}_\ell^{(2)} & \partial_{y_2}\mathfrak{s}_\ell^{(2)} & \partial_{y_3}\mathfrak{s}_\ell^{(2)} \\ \partial_{y_1}\mathfrak{s}_\ell^{(3)} & \partial_{y_2}\mathfrak{s}_\ell^{(3)} & \partial_{y_3}\mathfrak{s}_\ell^{(3)} \end{pmatrix}, \quad \ell = 1, 2,$$

$\mathfrak{S}$ and $\mathfrak{s}$ being proxies for R, V and r, v, respectively, and $r^{(j)} = (r_1^{(j)}, r_2^{(j)})^{tr}$, $v^{(j)} = (v_1^{(j)}, v_2^{(j)})^{tr}$, $j = 1, 2, 3$, being the solutions of the elliptic systems[a]

$$L_{\mathfrak{L},per}\begin{pmatrix} r_1^{(j)} \\ r_2^{(j)} \end{pmatrix} = \begin{pmatrix} \mathrm{div}_y(\varepsilon_{\mathfrak{L}})_{j,\sharp} \\ \mathrm{div}_y(\xi_{\mathfrak{L}})_{j,\sharp} \end{pmatrix}, \; L_{\mathfrak{L},per}\begin{pmatrix} v_1^{(j)} \\ v_2^{(j)} \end{pmatrix} = \begin{pmatrix} \mathrm{div}_y(\zeta_{\mathfrak{L}})_{j,\sharp} \\ \mathrm{div}_y(\mu_{\mathfrak{L}})_{j,\sharp} \end{pmatrix}, \qquad (46)$$

$$L_{\mathfrak{L},per} := \mathrm{div}_y(\mathrm{A}_{\mathfrak{L}}^{tr}(y, p)\,\mathrm{grad}_y).$$

[a] Notation: for a 3×3 matrix $\mathfrak{m}$, $\mathfrak{m}_{j,\sharp}$ denotes its j-th row.

The periodic unfolding method and two scale convergence

In 1990, Arbogast, Douglas and Hornung defined a "dilation" operator to study homogenisation for a periodic medium with double porosity.
In 2002, Cioranescu, Damlamian and Griso expanded this idea and presented a general and simple approach for classical or multiscale periodic homogenisation, under the name periodic unfolding method.
This method is essentially based on two ingredients: the unfolding operator (which is similar to the dilation operator and whose effect is to "zoom" the microscopic structure in a periodic manner), and the separation of the characteristic scales by decomposing every function $\phi \in W^{1,p}(\mathcal{O})$ into two parts; this scale-splitting can be either achieved by using the local average, or by a procedure inspired by the Finite Element Method.
The periodic unfolding method simplifies many of the two-scale convergence proofs.

Homogenisation of the Maxwell system for bianisotropic media: the periodic unfolding method

Let $Y = [0, \ell_1] \times [0, \ell_2] \times [0, \ell_3]$ be the reference periodic cell. Define

$$[x]_Y := \sum_{i=1}^{3} k_i \ell_i, \; x \in \mathbb{R}^3$$

to be the unique integer combination of periods such that

$$\{x\}_Y := x - [x]_Y \in Y.$$

Hence

$$x = \epsilon \left(\left[\frac{x}{\epsilon} \right]_Y + \left\{ \frac{x}{\epsilon} \right\}_Y \right) \text{ a.e. } \forall x \in \mathbb{R}^3.$$

Further, define $\widehat{\mathcal{O}}_\epsilon$ as the largest union of translated and rescaled $\epsilon\,(k + Y)$ cells which are included in $\mathcal{O}$.
Finally, let $\Lambda_\epsilon = \mathcal{O} \setminus \widehat{\mathcal{O}}_\epsilon$ be the subset of $\mathcal{O}$ containing the translated and rescaled cells that intersect $\partial\mathcal{O}$.

The periodic unfolding operator $\mathcal{T}^\epsilon : L^2(\mathcal{O}) \to L^2(\mathcal{O} \times Y)$ is defined by

$$\mathcal{T}^\epsilon(\mathfrak{u})(x,y) = \begin{cases} \mathfrak{u}\left(\epsilon\left[\frac{x}{\epsilon}\right]_Y + \epsilon y\right) & \text{for } x \in \widehat{\mathcal{O}}_\epsilon,\ y \in Y \\ 0 & \text{for } x \in \Lambda_\epsilon,\ y \in Y. \end{cases}$$

If $\mathfrak{u} = a^\epsilon(x) = a_{\text{per}}\left(\frac{x}{\epsilon}\right)$ where a_{per} is a periodic function of period Y then $\mathcal{T}^\epsilon(a^\epsilon)(x,y) = a_{\text{per}}(y)$. This shows that the action of the operator $\mathcal{T}^\epsilon$ is to "magnify" the periodic microstructure.
Clearly, for functions of the special type considered above

$$\mathcal{T}^\epsilon(a)(x,y) \to a_{\text{per}}(y), \text{ a.e. in } \mathcal{O} \times Y.$$

This result extends trivially for matrix valued functions of this special type.

Furthermore, the following properties (Bossavit, Griso and Miara (2005)) of $\mathcal{T}^\epsilon$ are very important:

1. $\mathcal{T}^\epsilon$ is a linear and continuous operator.
2. For all $\mathfrak{u}, \mathfrak{v} \in L^2(\mathcal{O})$, we have that $\mathcal{T}^\epsilon(\mathfrak{u}\,\mathfrak{v}) = \mathcal{T}^\epsilon(\mathfrak{u})\,\mathcal{T}^\epsilon(\mathfrak{v})$.
3. For all $\mathfrak{u} \in L^2(\mathcal{O})$

$$\int_{\mathcal{O}} \mathfrak{u}(x)\,dx = \frac{1}{|Y|}\int_{\mathcal{O}\times Y} \mathcal{T}^\epsilon(\mathfrak{u})(x,y)\,dx\,dy + \mathbf{C}(\epsilon)$$

   where $\mathbf{C}(\epsilon)$ is a correction term that may be shown to be negligible in the limit as $\epsilon \to 0$.

We need the space $H^1_{\text{per}}(Y)$ defined as follows: let $C^\infty_{\text{per}}(Y)$ be the subset of $C^\infty(\mathbb{R}^N)$ of Y-periodic functions.
$H^1_{\text{per}}(Y)$ is the closure of $C^\infty_{\text{per}}(Y)$ in the H^1-norm.

The following convergence results hold for $\mathcal{T}^\epsilon$:

1. If $\{\mathfrak{u}^\epsilon\}$ is uniformly bounded in $L^2(\mathcal{O})$ then there exists $\mathfrak{u} \in L^2(\mathcal{O} \times Y)$ such that $\mathcal{T}^\epsilon(\mathfrak{u}^\epsilon) \rightharpoonup \mathfrak{u}$ in $L^2(\mathcal{O} \times Y)$ (up to subsequences).
2. If $\{\mathfrak{u}^\epsilon\}$ is uniformly bounded in $H(\mathrm{curl}, \mathcal{O})$ then there exist a triplet $(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) \in H(curl, \mathcal{O}) \times L^2(\mathcal{O}, H^1_{\text{per}}(Y; \mathbb{R})) \times L^2(\mathcal{O}, H^1_{\text{per}}(Y; \mathbb{R}^3))$, with $\mathrm{div}_y \mathfrak{w} = 0$ so that
$$\begin{aligned} \mathfrak{u}^\epsilon &\rightharpoonup \mathfrak{u}, \quad \text{in } H(\mathrm{curl}, \mathcal{O}), \\ \mathcal{T}^\epsilon(\mathfrak{u}^\epsilon) &\rightharpoonup \mathfrak{u} + \mathrm{grad}_y \mathfrak{v} \ \text{ in } L^2(\mathcal{O} \times Y; \mathbb{R}^3), \\ \mathcal{T}^\epsilon(\mathrm{curl}\mathfrak{u}^\epsilon) &\rightharpoonup \ \mathrm{curl}_x \mathfrak{u} + \mathrm{curl}_y \mathfrak{w} \ \text{ in } L^2(\mathcal{O} \times Y; \mathbb{R}^3). \end{aligned}$$
3. If $\{\mathfrak{u}^\epsilon\}$ is bounded in $L^2(\mathcal{O})$ and such that $\mathcal{T}^\epsilon(\mathfrak{u}^\epsilon) \rightharpoonup \widehat{\mathfrak{u}}$ in $L^2(\mathcal{O} \times Y)$ then
$$\mathfrak{u}^\epsilon \rightharpoonup \mathfrak{u} := \frac{1}{|Y|} \int_Y \widehat{\mathfrak{u}} \, dy$$

- The functions $\mathfrak{v}$, $\mathfrak{w}$ are to be understood as correctors.
- $\mathfrak{v}(x,y)$ is a scalar and can be understood as a function $\mathfrak{v} : \mathcal{O} \to H^1_{\text{per}}(Y)$, such that $\int_{\mathcal{O}} ||\mathfrak{v}||^2_{H^1_{\text{per}}(Y)} dx < \infty$.
- $\mathfrak{w}(x,y)$ is a 3-vector and can be understood as a function $\mathfrak{w} : \mathcal{O} \to H^1_{\text{per}}(Y;\mathbb{R}^3) \simeq (H^1_{\text{per}}(Y))^3$, such that $\int_{\mathcal{O}} ||\mathfrak{w}||^2_{H^1_{\text{per}}(Y;\mathbb{R}^3)} dx < \infty$.

The above weak compactness results allow us to derive a homogenised Maxwell equation. We first define the following auxiliary system:

Definition (Cell equations)

Let $r_k \in H^1_{per}(Y) \times H^1_{per}(Y)$, $m_k \in W^{2,1}([0,T]; H^1_{per}(Y) \times H^1_{per}(Y))$, $h_k \in W^{1,1}([0,T]; H^1_{per}(Y) \times H^1_{per}(Y))$, $k = 1, \ldots, 6$ *be the solutions of the following systems*

$$
\begin{aligned}
&-\mathrm{div}_y(\mathsf{A}_{or}(y)\mathrm{grad}_y r_k) = \mathrm{div}_y(\mathsf{A}_{or}(y)e_k),\\
&-\mathrm{div}_y\left(\mathsf{A}_{or}(y)\mathrm{grad}_y m_k(y,t) + (\mathbf{G}m_k)(y,t)\right) = -\mathrm{div}_y(\mathsf{A}_{or}(y)e_k),\\
&-\mathrm{div}_y\left(\mathsf{A}_{or}(y)\mathrm{grad}_y h_k(y,t) + (\mathbf{G}h_k)(y,t)\right)\\
&= -\mathrm{div}_y(\mathsf{G}_d(y,t)(e_k + \mathrm{grad}_y r_k(y))),
\end{aligned}
$$

where $(\mathbf{G}\mathfrak{s})(y,t) := \int_0^t \mathsf{G}_d(y,t-s)\mathrm{grad}_y \mathfrak{s}(y,s)ds$ *and* e_k *is the canonical basis in* $\mathbb{R}^6$.

Definition (Homogenised coefficients)

The homogenised optical response matrix A_{or}^h *and the homogenised dispersion matrix* G_d^h *consist of the columns*

$$(\mathsf{A}_{or}^h)_{\sharp,k} = \int_Y \mathsf{A}_{or}(y)\mathfrak{r}_k(y)dy,$$

$$(\mathsf{G}_d^h)_{\sharp,k} = \int_Y \mathsf{G}_d(y,t)\mathfrak{r}_k(y)dy + \int_Y \mathsf{A}_{or}(y)\mathrm{grad}_y h_k(y,t)dy + \int_Y (\mathbf{G}h_k)(y,t)dy$$

for $k = 1, \ldots, 6$ *where* $\mathfrak{r}_k := e_k + \mathrm{grad}_y r_k(y))$ *and* $(\mathbf{G}h_k)(y,t)$ *is as in the previous definition.*

Notation: for a 3×3 matrix $\mathfrak{m}$, $\mathfrak{m}_{\sharp,k}$ denotes its k-th column.

Theorem

The solution u^ϵ *of*

$$(\mathsf{A}_{or}^\epsilon u^\epsilon + \mathsf{G}_d^\epsilon \star u^\epsilon)' = \mathsf{M} u^\epsilon + j$$

(with initial condition $u^\epsilon = 0$*, and the perfect conductor boundary condition* $n \times u_1^\epsilon = 0$*), is such that*

$$u^\epsilon \overset{*}{\rightharpoonup} u \ \text{ in } L^\infty([0,T], H_0(\mathrm{curl}, \mathcal{O}) \times H(\mathrm{curl}, \mathcal{O}))\,,$$

where u *is the solution of the homogenised Maxwell system*

$$(\mathsf{A}_{or}^h u + \mathsf{G}_d^h \star u)' = \mathsf{M} u + j^h\,,$$

with A_{or}^h*,* G_d^h *given as in the previous definition.*

Nonlinear problems

Regarding chiral media, although third-order nonlinear effects were predicted as early as 1967, nonlinear optical rotation experiments were not undertaken before 1993.

We consider a nonlinear complex electromagnetic medium modelled by

$$\mathrm{d} = \mathcal{L}u = \mathrm{A}_0 u + \mathrm{G}_0 \star u + \mathrm{G}_{0,\mathrm{n}\ell} \star \mathrm{N}(u)u \,,$$

where the linear part is given by

$$\mathrm{A}_0(x) = \begin{bmatrix} \varepsilon(x) & 0 \\ 0 & \mu(x) \end{bmatrix}, \mathrm{G}_0\,(t,x) = \begin{bmatrix} \varepsilon(x)\chi^{\mathrm{e}}(t) & \chi^{\mathrm{em}}(t) \\ \chi^{\mathrm{me}}(t) & \mu(x)\chi^{\mathrm{m}}(t) \end{bmatrix},$$

while the nonlinearity is given by

$$\mathrm{N}(u) := \begin{bmatrix} N_1|u_1|^q & 0 \\ 0 & N_2|u_2|^q \end{bmatrix}, \;\; \mathrm{G}_{0,\mathrm{n}\ell}\,(t,x) := \begin{bmatrix} \chi^{\mathrm{e}}_{\mathrm{n}\ell}(t,x) & 0 \\ 0 & \chi^{\mathrm{m}}_{\mathrm{n}\ell}(t,x) \end{bmatrix},$$

where $q \in \mathbb{N}$, $N_1, N_2 \in \mathbb{R}^{3\times 3}$ are matrices independent of the spatial and temporal variables.

Let $B_A := G_0^{-1} G_{0,n\ell}(0)$, $G_{A,n\ell} := G_0^{-1} G'_{0,n\ell}$, $M_A := G_0^{-1} M$, $G_A := -G_0^{-1} G'_0$, $J_A := G_0^{-1} j$.

Then, under suitable regularity assumptions on G_0 and $G_{0,n\ell}$, the Maxwell system takes the form

$$u' + B_A N(u)u + G_{A,n\ell} \star N(u)u = M_A u + G_A \star u + J_A ,$$

with initial condition

$$u(0,x) = u_0(x), \quad x \in \mathcal{O} ,$$

and the perfect conductor boundary condition

$$n \times u_1 = 0, \quad (t,x) \in [0,T] \times \partial\mathcal{O} .$$

Assumption

1. $\mathrm{B}_A\mathrm{N}(u)u$ *is* monotone, *i.e., there exists a* $p \in \mathbb{N}$ *such that*

$$\mathrm{N}(\mathrm{v})\mathrm{v} \cdot \mathrm{v} \geq \mathrm{c}\,|\mathrm{v}|^p, \ \mathrm{c} > 0, \ \forall \mathrm{v} \in \mathbb{R}^6.$$

2. *There exists a constant* $\alpha > 0$ *such that*

$$|\mathrm{N}(\mathrm{v})\mathrm{v}| \leq \alpha\left(1 + |\mathrm{v}|^{p-1}\right), \ \forall \mathrm{v} \in \mathbb{R}^6.$$

3. $\mathrm{B}_A\mathrm{N}(u)u$ *satisfies*

$$D_{\mathrm{v}}\Big(\mathrm{B}_A\mathrm{N}(\mathrm{v})\mathrm{v}\Big)\mathrm{w}\cdot\mathrm{w} \geq 0\,, \ \forall \mathrm{v}, \mathrm{w} \in \mathbb{R}^6,$$

   where D_{v} *denotes the derivative with respect to the vector* v.

4. *There exists a constant* $C > 0$ *such that*

$$\Big(\mathrm{G}_{A,n\ell} \star \mathrm{N}(u)u\Big)(t)\cdot\, u(t) \geq C\,\Big(\mathrm{B}_A\mathrm{N}(u)u\Big)(t)\cdot u(t)\,,$$

   for every function u *and* $t \in [0, T]$.

Assumption 2 corresponds to the convexity of the energy functional of the medium. Assumptions 1 and 3 (for $p = q + 2$) hold provided that the matrices N_1, N_2 are positive definite. Assumption 4 can be seen as a generalisation, in the framework of nonlinear equations, of the condition that the kernels are functions of positive type, a fact consistent with energy considerations.

Based on the Faedo-Galerkin approximation and monotonicity arguments we have

Theorem

Under the previous assumptions on the nonlinearity and further assuming that

- $\mathrm{G}_0 \in W^{1,\infty}([0,T]; L^\infty(\mathcal{O})^{6\times 6})$,
- $\mathrm{G}_A \in W^{1,1}([0,T], (L^\infty(\mathcal{O}))^{6\times 6})$,
- $u_0 \in (L^{q+2}(\mathcal{O}))^3 \times (L^{q+2}(\mathcal{O}))^3$,
- $\mathrm{J}_A \in W^{1,1}([0,T]; (L^2(\mathcal{O}))^3 \times (L^2(\mathcal{O}))^3)$,

the IBVP has a unique weak solution

$$u \in L^\infty([0,T], H_0(\mathrm{curl}, \mathcal{O}) \times H(\mathrm{curl}, \mathcal{O})) \cap L^\infty([0,T], (L^{q+2}(\mathcal{O}))^6) .$$

Nonlinear systems present interesting types of solutions in the form of travelling waves that propagate with unchanged shape through the medium as an effect of the interplay between dispersion and nonlinearity. This type of behaviour is typical and is very well studied in integrable systems; however, solutions of similar type are often present in nonintegrable systems and find important applications in various branches of science.

A formal approach to the evolution of nonlinear waves in chiral media with weak dispersion and weak nonlinearity of the Kerr type in the low chirality case has been studied by Frantzeskakis, $\mathbb{S}$, Yannacopoulos and by Tsitsas, Frantzeskakis, Lakhtakia.

A set of modulation equations is obtained for the evolution of the slowly varying field envelopes that is in the form of 4 coupled nonlinear Schrödinger equations. This set of equations is nonintegrable; however, with the use of reductive perturbation theory, under certain conditions these equations may be reduced to an integrable system, the Melnikov system. This system is known to possess vector soliton solutions. Thus, by the above reduction, in certain (limiting) cases the existence of vector solitons in chiral media may be shown; these appear in pairs of dark and bright solitons. Depending on the chosen behaviour at infinity, the dark component can be along the right-handed component of the field and the bright component along the left-handed component of the field, or vice versa.

Stochastic problems

In many applications there is uncertainty concerning either the externally imposed sources or the nature of the medium under consideration. In such cases, it is useful to model the uncertain quantities as random variables, with a prescribed probability distribution. These random variables are now functions of the spatial variables and of time, and can be considered as random fields defined on a probability space $(\Omega, \mathbf{F}, P)$. The randomness is assumed to have as an effect that the repetition of different experiments on the medium will generate different outcomes ω, either of the medium parameters or of the external sources. The set Ω contains the outcomes ω of all possible experiments or all possible realisations of the medium, $\mathbf{F}$ is a σ-algebra on Ω and P is a probability measure on $\mathbf{F}$, quantifying the relative frequency of realisations of different outcomes in an (ideally infinite) repetition of experiments under identical experimental conditions. In turn, the randomness in the sources or the medium is reflected in the resulting electromagnetic fields, which have to be modelled as random fields as well.

The constitutive relations for random media are of the form

$$\mathrm{d}_\omega = \mathrm{A}_{\mathrm{or},\omega} u_\omega + \mathrm{G}_\omega \star u_\omega, \tag{47}$$

where now the quantities d_ω, u_ω are considered to be random fields, depending on x and t, with the explicit dependence suppressed for simplicity. We include the explicit dependence on ω to remind us that the values of these quantities depend on the particular realisation ω of the experiment performed (or the particular realisation of the random medium). Furthermore, $\mathrm{A}_{\mathrm{or},\omega}$, G_ω are (in general) random matrices whose elements consist of random fields, which model the random parameters of the medium. The medium may be spatially homogeneous or not, depending on the circumstances. Similarly, we consider the external source J a random field $J = J(t, x, \omega)$.
The description of random media in terms of random fields finds a number of interesting applications in the theory of composites and in scattering problems from rough surfaces.

The Maxwell system

In view of the constitutive relations (47) the Maxwell system becomes

$$(\mathrm{A}_{\mathrm{or},\omega} u + \mathrm{G}_\omega \star u)' = \mathrm{M}u + j_\omega \,, \tag{48}$$

where now $j_\omega = \mathrm{j}(t, x, \omega)$ is a random process defined on $(\Omega, \mathbf{F}, P)$.
An important aspect of the problem is the choice of a convenient model for the randomness.
There are two major classes, presenting qualitatively different behaviour, depending on the choice of j_ω (in particular on its variation with respect to time):

- j_ω is of finite variation $\rightarrowtail$ Random $\rightarrowtail$ evolution equations treated in the usual sense (Riemann-Stieltjes or Bochner integration) $\rightarrowtail$ partial integrodifferential equation with random coefficients.
- j_ω is of infinite variation $\rightarrowtail$ Stochastic $\rightarrowtail$ Banach space valued stochastic processes (Q-Wiener) involved $\rightarrowtail$ Itō stochastic integration $\rightarrowtail$ SPDEs.

The Maxwell system

The integral form of (48) is

$$u(t) = u(0) + \int_0^t \left(\mathrm{M}_{\mathrm{A}} u(s) + \int_0^s \mathrm{G}_{\mathrm{A}}(s-r)u(r)\, dr + \mathrm{J}_{\mathrm{A}}(s) \right) ds + \int_0^t \mathrm{Q}_{\mathrm{A}}(s,\omega)\, dW(s), \tag{49}$$

where $\mathrm{G}_{\mathrm{A}} = \mathrm{A}_{\mathrm{or},\omega}^{-1} \mathrm{G}_{\omega}$, $\mathrm{M}_{\mathrm{A}} = \mathrm{A}_{\mathrm{or},\omega}^{-1}\mathrm{M}$, and $\mathrm{J}_{\mathrm{A}} = \mathrm{A}_{\mathrm{or},\omega}^{-1} j_{\omega}$.
The first integral is considered as a Riemann-Stieltjes integral, whereas the second one is considered as an Itō integral w.r.t. the infinite-dimensional Wiener process $W(t)$.
Q_{A} is an operator-valued stochastic process that models the effect of spatial correlations of the fluctuating terms; it may be either independent of, or dependent on, the electromagnetic field. The first case, especially if G_{A} is not a random process, is called the additive noise case; the second case is called the multiplicative noise case.
The boundary conditions are considered to be those of the perfect conductor.

Solvability

Upon properly[8] defining the notions of "mild", "weak" and "strong" solutions[9] for (49) and employing a semigroup approach (based on the semigroup generated by the Maxwell operator and treating the convolution terms as perturbations of this semigroup) we have the following result (Liaskos, $\mathbb{S}$, Yannacopoulos)

Theorem

Under suitable regularity assumptions on the data, (49) *is weakly / mildly / strongly well-posed.*

[8] The electromagnetic field is now a stochastic process on the probability space $(\Omega, \mathbf{F}, P)$ taking values on $(L^2(\mathcal{O}))^3 \times (L^2(\mathcal{O}))^3$.

[9] The temporal regularity is not expected to be as good as the spatial regularity because of the pathological properties of the Wiener process (with respect to temporal regularity) that are inherited by the solution of (49).

The Wiener chaos approach to solvability

Another approach for the solvability of (49) is based on a Wiener chaos expansion, i.e., on an expansion of square-integrable stochastic processes adapted[10] with respect to the filtration[11] generated by the Wiener process ($\mathbb{S}$, Yannacopoulos).

This important approach was initially (2006) introduced by Rozovskii and Lototsky for parabolic SPDEs, and then generalised by Kalpinelli, Frangos and Yannacopoulos for hyperbolic SPDEs (2011) and by Yannacopoulos, Frangos and Karatzas for backward SPDEs (2011).

[10] A stochastic process $\{X(t)\}$ with the property that $X(s)$ is measurable with respect to $\mathbf{F}_s$, for all $s \in \mathcal{I}$, is called *adapted*.

[11] A family of σ-algebras $\{\mathbf{F}_t\}_{t\in\mathcal{I}}$ is called a *filtration* if it has the property $\mathbf{F}_s \subseteq \mathbf{F}_t$ for $s \leq t$.

The Wiener chaos approach to (49) is particularly interesting since:

(i) It reduces the original stochastic problem to an infinite hierarchy of deterministic problems. This hierarchy is decoupled in the case of additive noise and has a lower triangular structure in the case of multiplicative noise.

(ii) It can be seen as a Galerkin-type approach, which separates the effects of randomness from the effects of the spatio-temporal dynamics; it is well suited for numerical analysis and simulation purposes, especially when statistical moments of the solutions are needed.

(iii) It allows an easier treatment of the spatial regularity of the solutions.

Homogenisation for random complex media presenting an ergodic structure

In certain classes of materials the spatial structure is not as regular as to be modelled by periodic functions. Such materials can be modelled as random media having some sort of statistical periodicity (the structure of the material repeats itself in a statistical law); this is expressed mathematically via the notion of ergodicity. This concept is powerful enough to generalise periodicity and allows the construction of a homogenisation theory that bypasses the need for periodic structure. From the applications point of view, this generalisation leads to more realistic models.
Regarding electromagnetics in complex media, according to some authors chirality itself is due to randomly positioned helices in the medium.

Recently, there has been a resurgence of interest in the homogenisation of first- and second-order PDEs in random stationary ergodic media.

This is a very general setting lacking the compactness properties used extensively in various places in the study of the periodic / almost periodic homogenisation.

To overcome this difficulty, it is necessary to resort to the ergodic theorem, coupled with stationarity properties.

Self-averaging environments

- Periodic, quasi-periodic (linear combination of periodic of incommensurate periods), almost periodic (closure of quasi-periodic).
- Random: for each experiment we encounter a different material (lack of information of the exact composition, or true randomness); the frequency of appearance of each particular material configuration is described by a probability measure μ on the measurable space of all possible configurations $(\Omega, \mathbf{F})$.
  For a random environment to be self-averaging, statistical self repetition is required (from a sufficiently large block of a particular realisation of the material, one should be able to reconstruct all possible realisations of the material at a particular point): stationarity and ergodicity.
  This is a kind of a generalisation of (classical) periodicity, in which from a single cell the whole material can be reproduced by translations.

Let $(\Omega, \mathbf{F}, P)$ be a probability space, and let $\mathfrak{G}$ be a group of transformations on Ω.
We say that the probability measure P is preserved under the action τ of the group $\mathfrak{G}$, if $P(\tau A) = P(A)$ for every $A \in \mathbf{F}$.

> A typical example of this setup is the case where $\Omega = \mathbb{R}^3$, i.e., each ω is identified with a point $x \in \mathbb{R}^3$.
> Further, $(\mathbb{R}^3, +)$ is the usual translation group, and $\tau_y = x + y$ the action of a group acting on Ω such that the probability measure is preserved under the action of the group, i.e.,
>
> $$P(\tau_y A) = P(A), \quad \forall A \in \mathbf{F}, \ \forall y \in \mathbb{R}^3 .$$

The probability space $(\Omega, \mathbf{F}, P)$ is to be interpreted as follows: Each realisation ω is to be understood as a particular configuration of the medium. In other words, each experiment we perform on a particular medium corresponds to a particular choice of $\omega \in \Omega$. However, it is neither known beforehand, nor with certainty, which medium is to be realised at the time instant that the experiment is performed. The probability that a particular medium is realised is given by the probability measure P.
A random variable $F : \Omega \to X$, where X is an appropriate metric space will serve as a mathematical model for a medium. For instance, when $X \in \mathbb{R}^{6\times 6}$ we may consider F as a particular outcome of the electromagnetic parameters of a random complex medium (e.g., a particular outcome of $\mathrm{A}_{\mathrm{or},\omega}$ or G_{ω}).
More precisely, if F is a measurable function on Ω, we will call for each fixed $\omega \in \Omega$, $F(\tau_x\omega)$ a *realisation* of F. A measurable function F is called *invariant under the group action* if $F(\tau_x\omega) = F(\omega)$ for every x and ω.

Ergodicity and Stationarity

1. The action τ is called ergodic if for all $A \in \mathbf{F}$

$$\tau_x A = A\,,\ \ \forall x \in \mathbb{R}^3 \implies P(A) = 0,\ \text{or}\ P(A) = 1.$$

2. A random variable F is called stationary if

$$\forall y \in \mathbb{R}^3,\ F(x+y,\omega) = F(x,\tau_y\omega),\ \text{a.e. in x, a.s.}$$

An alternative definition of ergodicity is to say that an action is ergodic if every invariant function under this action is the constant function almost surely in Ω (P-a.s.). One can say that ergodicity implies that when moving along the medium (in one realisation), by the time we reach infinity it is as if we have seen all possible realisations of the medium at a single point. This allows to interchange averaging over the probability measure with averaging over space.

As for stationarity, it means that what is observed at a point x is statistically the same with what is observed at the point $x + y$ (this is the analogue of periodicity).

These properties guarantee that, in a statistical sense, parts of the material located at different positions will present the same properties. This fact allows us to look at average properties of the material at long scales and obtain nice expressions for these quantities.
The ergodic hypothesis implies that instead of looking at an ensemble average of media, and averaging the properties of the medium on the ensemble average, we may consider a single realisation of the medium whose spatial dimensions are large and sample its properties by traversing this single realisation for large enough distances.

Complex Media Electromagnetics

Now, let $\mathcal{O}$ be a domain in $\mathbb{R}^3$, filled with a random complex linear electromagnetic medium. We consider the case where the randomness is assumed to be spatial only. The effect of the random structure of the medium is that the electromagnetic field is a random field, whose evolution is given by the random Maxwell equations

$$\partial_t(\mathrm{A}_{\mathrm{or},\omega} u + \mathrm{G}_\omega \star u) = \mathrm{M}u + j, \text{ in } (0, T] \times \mathcal{O}, \tag{50}$$

subject to the perfect conductor boundary condition

$$n \times u_1 = 0, \text{ in } [0, T] \times \partial\mathcal{O},$$

and for homogeneous initial conditions $u(x, 0) = 0$, $x \in \mathcal{O}$.

The coefficients of the medium are now random variables, with a spatial dependence; the same can also hold for the source term j.
The fields $u = u(t, x; \omega)$ are random fields (vector space valued random variables).
All these random variables are assumed to be defined on a suitable probability space $(\Omega, \mathbf{F}, P)$ related to the nature of the random structure of the medium.

The differential equation (50) is now an equation between random variables and is assumed to hold almost surely in P; it is a random differential equation.

The medium coefficients must be of such form as to allow us to model small scale (fast varying) random microstructure:

1. Consider a probability space $(\Omega, \mathbf{F}, P)$ and let $\Phi(\cdot, \omega) : \mathbb{R}^3 \to \mathbb{R}^3$ be a random diffeomorphism $P-$a.s. with the property that $\mathrm{grad}\Phi$ is stationary under an ergodic group action, i.e.,

$$\forall y \in \mathbb{R}^3, \;\; \mathrm{grad}\Phi(x+y,\omega) = \mathrm{grad}\Phi(x, \tau_y \omega).$$

2. The random medium can be modelled with coefficients of the form

$$\begin{aligned} \mathrm{A}_{\mathrm{or},\omega} &= \mathrm{A}^{\epsilon}_{\mathrm{or}}(x,\omega) = \mathrm{A}_{\mathrm{or}}\left(\Phi^{-1}\left(\frac{x}{\epsilon},\omega\right)\right), \\ \mathrm{G}_{\omega} &= \mathrm{G}^{\epsilon}_{\mathrm{d}}(t,x,\omega) = \mathrm{G}_{\mathrm{d}}\left(t, \Phi^{-1}\left(\frac{x}{\epsilon},\omega\right)\right) \end{aligned} \qquad (51)$$

where $\mathrm{A}_{\mathrm{or}}(y)$ and $\mathrm{G}_{\mathrm{d}}(y,t)$ are deterministic matrix valued functions periodic in $y \in \mathbb{R}^3$ with common period Y, while Φ is as above.

This assumption on the coefficients of the medium is inspired by recent very interesting work by X. Blanc, C. Le Bris and P.-L. Lions (2006, 2007) on stochastic elliptic homogenisation. This type of coefficients models some kind of statistical periodicity of the medium and guarantees ergodicity.

In order to be able to model the small scale periodic microstructure, we must let ϵ vary over a range of arbitrarily small values. We are therefore led to a sequence of random boundary value problems,

$$(\mathsf{A}^{\epsilon}_{\mathsf{or}} u^{\epsilon} + \mathsf{G}^{\epsilon}_{\mathsf{d}} \star u^{\epsilon})' = \mathsf{M} u^{\epsilon} + j \tag{52}$$

with initial condition $u^{\epsilon} = 0$, and the perfect conductor boundary condition

$$n \times u^{\epsilon}_{1} = 0, \ \text{ on } \mathcal{O}. \tag{53}$$

The explicit t, x and ω dependence is omitted for simplicity.

If the solution of the above sequence of random boundary value problems exists for all $\epsilon > 0$, then this will generate a sequence of random fields $\{u^\epsilon\} = \{u^\epsilon(t, x, \omega)\}$. To understand the effects of small scale random microstructure we must go to the limit as $\epsilon \to 0$.

Questions similar to those posed in the deterministic case arise, e.g.,

- Does the sequence of random fields $\{u^\epsilon\}$ converge in some weak sense to a limit random field u^* ?
- Is this limit random field the solution of a differential equation

$$(\mathrm{A}^h_{\mathrm{or}} u^* + \mathrm{G}^h_{\mathrm{d}} \star u^*)' = \mathrm{M} u^* + j \,,$$

  similar in type with the original Maxwell system, but now with constant coefficients $\mathrm{A}^h_{\mathrm{or}}$, $\mathrm{G}^h_{\mathrm{d}}$?
- Can these coefficients be specified by those of the original medium?
- Can it be that, in certain circumstances, the limiting field and the limiting differential equation are not random?

The answer to these questions is complicated, since quantities involved are random fields defined on a probability space and not just deterministic functions.

Well posedness of the random Maxwell system

Theorem

The Maxwell system (50) is uniquely solvable for all $\epsilon > 0$ and $\omega \in \Omega$ and the solution satisfies

$$\|u^\epsilon(t)\|_{\mathbb{X}} \leq C, \ \ \textit{for all } \epsilon, \ \ t > 0, \ \ P - a.s.$$

and

$$\|u^\epsilon(t)\|_{L^2(\Omega,\mathbf{F},P;\mathbb{X})} \leq C, \ \ \textit{for all } \epsilon, \ \ t > 0.$$

The existence of a solution P-a.s. follows by the Galerkin method.

An auxiliary random elliptic problem

Motivated by a formal two scale expansion we see that the following random elliptic system is closely related to the Laplace transformed random Maxwell problem.

Consider a 6×6 random matrix $\mathrm{A}_{e\ell}$ expressed in block form as

$$\mathrm{A}_{e\ell} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{54}$$

where a, b, c, d are random matrices of the form assumed in (51).

Definition

For a random matrix $\mathrm{A}_{e\ell}(x,\omega)$ as in equation (54) consider the random elliptic operator $L^{\epsilon} : H_0^1(\mathcal{O}) \times H_0^1(\mathcal{O}) \to H^{-1}(\mathcal{O}) \times H^{-1}(\mathcal{O})$, defined as

$$L^{\epsilon} = \mathrm{div}_x(\mathrm{A}_{e\ell}(x,\omega)\, \mathrm{grad}_x \cdot).$$

Note that it is the quantity $\mathfrak{m}\left(\Phi^{-1}\left(\frac{x}{\epsilon},\omega\right)\right)$ which is random. The matrix $\mathfrak{m}$ itself is not random.

Assumption ($\mathrm{A}_{e\ell}$)

The matrix $\mathrm{A}_{e\ell} \in L^\infty(\mathcal{O}, \mathbb{R}^{6\times 6})$ *is assumed to satisfy the conditions*

1. *There exists a positive constant* c_1 *such that* $|\mathrm{A}_{e\ell}(z)\,\mathrm{y}\cdot\mathrm{y}| \geq c_1\,|\mathrm{y}|$, *for almost all* $z \in \mathcal{O}$ *and all (deterministic)* $\mathrm{y} \in \mathbb{R}^6$.
2. *There exists a positive constant* c_2 *such that* $|\mathrm{A}_{e\ell}^{-1}(z)\,\mathrm{y}\cdot\mathrm{y}| \geq c_2\,|\mathrm{y}|$, *for almost all* $z \in \mathcal{O}$ *and all* $\mathrm{y} \in \mathbb{R}^6$.

When choosing $z = \Phi^{-1}\left(\frac{x}{\epsilon}, \omega\right)$ the above Assumption holds P-a.s. for the family of random matrices $\{\mathrm{A}_{e\ell}^{\epsilon}\}$.

Definition (random averaging operator)

Let $\mathfrak{s}$ be a random field of the form $\mathfrak{s}(x,y,\omega) = \mathfrak{s}\left(x, \Phi^{-1}(y,\omega)\right)$. The random averaging operator is defined as

$$\langle \mathfrak{s} \rangle = \left(\mathbb{E}\left[\int_Y \det(\mathrm{grad}\Phi(y,\omega))\, dy \right] \right)^{-1} \mathbb{E}\left[\int_{\Phi(Y)} \mathfrak{s}(x, \Phi^{-1}(y,\omega)) dy \right].$$

This is the random generalisation of the periodic averaging operator used in the deterministic case.

Definition (random cell systems)

For $j = 1,2,3$, $\ell = 1,2$ the random cell systems are the random elliptic systems

$$L_c \begin{pmatrix} r_1^{(j)} \\ r_2^{(j)} \end{pmatrix} = \begin{pmatrix} \operatorname{div}_y a_{\sharp,j} \\ \operatorname{div}_y c_{\sharp,j} \end{pmatrix}, \quad L_c \begin{pmatrix} v_1^{(j)} \\ v_2^{(j)} \end{pmatrix} = \begin{pmatrix} \operatorname{div}_y b_{\sharp,j} \\ \operatorname{div}_y d_{\sharp,j} \end{pmatrix}, \tag{55}$$

where L_c is the random matrix operator

$$L_c = -\operatorname{div}_y \mathsf{A}_{or} \left(\Phi^{-1}(y,\omega)\ \operatorname{grad}_y \cdot\,\right)$$

and $\mathfrak{m}(y) = \mathfrak{m}(\Phi^{-1}(y,\omega))$, where $\mathfrak{m}$ is a proxy for the matrices a, b, c, d. These equations are supplemented with the conditions (a generalisation of the periodic boundary conditions used in the deterministic case)

$$\operatorname{grad}_y \mathfrak{s} = \operatorname{grad}_y \check{\mathfrak{s}}(\Phi^{-1}(y,\omega)),\ \ \operatorname{grad}_y \check{\mathfrak{s}} \textit{ is stationary},\ \langle \operatorname{grad}_y \mathfrak{s} \rangle = 0 \tag{56}$$

where $\mathfrak{s}$ is a proxy for the random fields $r^{(j)}$, $v^{(j)}$, $j = 1,2,3$.

This system of equations (55) is called the cell system and complemented with the boundary conditions (56) has a unique solution (modulo random constants).
Note that in (55) y is in $\mathbb{R}^3$ rather than in Y; it is $\Phi^{-1}(y,\omega)$ that belongs in Y.
The solvability follows by a proper application of the Lax-Milgram lemma, adapting the approach of Blanc, Le Bris, P.-L. Lions (2007) to elliptic systems.
The solution is in $L^2(\Omega, \mathbf{F}, P; H^1_{loc}(\mathbb{R}^3) \times H^1_{loc}(\mathbb{R}^3))$.

Consider now the 3×3 matrices a^h, b^h, c^h, d^h defined as

$$\begin{aligned}
(a^h)_{ij} &= \langle a_{ij} + \sum_{k=1}^{3} a_{ik}\partial_{y_k} r_1^{(j)} + \sum_{k=1}^{3} b_{ik}\partial_{y_k} r_2^{(j)} \rangle, \\
(b^h)_{ij} &= \langle b_{ij} + \sum_{k=1}^{3} a_{ik}\partial_{y_k} v_1^{(j)} + \sum_{k=1}^{3} b_{ik}\partial_{y_k} v_2^{(j)} \rangle, \\
(c^h)_{ij} &= \langle c_{ij} + \sum_{k=1}^{3} c_{ik}\partial_{y_k} r_1^{(j)} + \sum_{k=1}^{3} d_{ik}\partial_{y_k} r_2^{(j)} \rangle, \\
(d^h)_{ij} &= \langle d_{ij} + \sum_{k=1}^{3} c_{ik}\partial_{y_k} v_1^{(j)} + \sum_{k=1}^{3} d_{ik}\partial_{y_k} v_2^{(j)} \rangle,
\end{aligned} \tag{57}$$

where $r_\ell^{(j)}$, $v_\ell^{(j)}$, $j = 1, 2, 3$, $\ell = 1, 2$ are the solutions of the cell systems (55) and the averaging operation is to be understood in the sense defined above.

Definition (homogenised diffusion matrix)

The constant coefficient matrix

$$\mathrm{A}^h_{e\ell} = \begin{pmatrix} a^h & b^h \\ c^h & d^h \end{pmatrix}, \tag{58}$$

where a^h, b^h, c^h, d^h are defined as in (57) is called the homogenised diffusion matrix.

The following random homogenisation theorem holds for the elliptic problem:

Theorem

Consider the solution u^ϵ of the random elliptic problem $L^\epsilon u^\epsilon = f$. As $\epsilon \to 0$, we have that $u^\epsilon \rightharpoonup u^h$, in $H^1_0(\mathcal{O}) \times H^1_0(\mathcal{O})$, P-a.s. where u^h is the solution of the elliptic problem $L^h u^h = f$. The homogenised matrix $\mathrm{A}^h_{e\ell}$ is given as in (58). Furthermore, $\mathrm{A}^\epsilon_{e\ell}\, u^\epsilon \rightharpoonup \mathrm{A}^h_{e\ell}\, u^h$, in $L^2(\mathcal{O})$.

Homogenisation of the random Maxwell system

We work with the Laplace transform of the Maxwell system and make the following

Assumption

The block matrix

$$\mathrm{A}_{or,\omega}(y,p) := \mathrm{A}_{or,\omega}(y) + \widehat{\mathrm{G}_\omega}(y,p) = \begin{pmatrix} \varepsilon + \widehat{\varepsilon}_d & \xi + \widehat{\xi}_d \\ \xi^{tr} + \widehat{\xi}_d^{tr} & \mu + \widehat{\mu}_d \end{pmatrix} =: \begin{pmatrix} \varepsilon_{\mathfrak{L}} & \xi_{\mathfrak{L}} \\ \zeta_{\mathfrak{L}} & \mu_{\mathfrak{L}} \end{pmatrix}$$

satisfies the conditions of the assumptions on $\mathrm{A}_{e\ell}$.

The following random elliptic operators are needed.

Definition

The auxiliary "microstructure" random elliptic operator associated with the Maxwell system is

$$L_M^{\epsilon} = -\mathrm{div}_x(\mathsf{A}_{or}^{\epsilon,tr}\mathrm{grad}_x\cdot), \tag{59}$$

and the auxiliary "cell" random elliptic operator associated with the Maxwell system is

$$L_{c,M} = -\mathrm{div}_y((\mathsf{A}_{or}^{per})^{tr}\mathrm{grad}_y\cdot) \tag{60}$$

Let $\mathsf{A}^h_{\mathrm{or},\omega}$ be the homogenised matrix for the random elliptic system of the above definition, obtained as in the previous discussion. Then, the Laplace transform of the homogenised constitutive relation is given by

$$\widehat{\mathrm{d}}^h = \mathsf{A}^h_{\mathrm{or},\omega}\widehat{u}$$

We now perform random elliptic homogenisation for the auxiliary elliptic systems and obtain the constant coefficient matrix $\mathrm{A}^h_{\mathrm{or}}$. Note that by the ergodicity of the medium the homogenised coefficients are deterministic.

Theorem

The solution $u^\epsilon = (E^\epsilon, H^\epsilon)^{tr}$ of the random Maxwell system

$$(\mathrm{A}^\epsilon_{or} u^\epsilon + \mathrm{G}^\epsilon_d \star u^\epsilon)' = \mathrm{M} u^\epsilon + j$$

satisfies

$$u^\epsilon \overset{*}{\rightharpoonup} u^*, \quad in\ L^2(\Omega, \mathbf{F}, P; L^\infty([0,T], \mathbb{X})),$$

where $u^* = (E^*, H^*)^{tr}$ is the unique solution of the Maxwell system

$$(\mathrm{A}^h_{or} u + \mathrm{G}^h_d \star u)' = \mathrm{M} u + j$$

with homogeneous initial conditions and perfect conductor boundary conditions, where the homogenised coefficients A^h_{or} and G^h_d are defined as above.