The Willmore functional in geometry and the calculus of variations

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For an immersed closed surface $f : \Sigma \to \mathbb{R}^n$ the Willmore functional is defined by

$$\mathcal{W}(f) = \frac{1}{4} \int\limits_\Sigma |\vec{\mathbf{H}}|^2 \, \mathrm{d}\mu_g,$$

where $\vec{\mathbf{H}}$ denotes the mean curvature vector of $f, g = f^* g_{euc}$ the pull-back metric and μ_g the induced area measure on Σ. The Gauß equations and the Gauß-Bonnet theorem give rise to equivalent expressions

$$\mathcal{W}(f) = \frac{1}{4} \int\limits_\Sigma |A|^2 \, \mathrm{d}\mu_g + 2\pi(1 - p(\Sigma)) = \frac{1}{2} \int\limits_\Sigma |A^\circ|^2 \, \mathrm{d}\mu_g + 4\pi(1 - p(\Sigma)) \tag{1}$$

where A denotes the second fundamental form, $A^\circ = A - \frac{1}{2} g \otimes H$ its tracefree part and $p(\Sigma)$ is the genus of Σ.

The Willmore functional is scale invariant and moreover invariant under the full Möbius group of $\mathbb{R}^n$. As the Möbius group is non-compact, minimizers of the Willmore energy cannot be found via the direct method. Nevertheless

$$\mathcal{W}(f) \geq 4\pi = \mathcal{W}(S^2), \tag{2}$$

and the round spheres are absolut minimizers. We give here the argument in codimension one for an embedded surface $\Sigma \subseteq \mathbb{R}^3$ due to Willmore in [Wil82].

Let $\nu : \Sigma \to S^2$ be the unique smooth outer normal at Σ. For any unit vector ν_0, we choose $x_0 \in \Sigma$ with

$$\langle x_0, \nu_0 \rangle := \max_{x \in \Sigma} \langle x, \nu_0 \rangle$$

and see

$$\Sigma \subseteq \{ y \in \mathbb{R}^3 \mid \langle y - x_0, \nu_0 \rangle \leq 0 \}.$$

Therefore $\{\nu_0\}^\perp$ is a supporting hyperplane of Σ at x_0 and

$$\nu(x_0) = \nu_0 \quad \text{and} \quad K(x_0) \geq 0,$$

where K denotes the Gauß curvature. As $\nu_0 \in S^2$ was arbitrary, we get

$$\nu(K \geq 0) = S^2. \tag{3}$$

Observing for the principal curvature κ_1, κ_2 that

$$|\vec{\mathbf{H}}|^2 = (\kappa_1 + \kappa_2)^2 = (\kappa_1 - \kappa_2)^2 + 4\kappa_1\kappa_2 \geq 4K$$

and recalling that the Jacobian of the Gauß map ν is given as the modulus of the Gauß curvature

$$J_g\nu = |K|,$$

we get by the area formula

$$\mathcal{W}(\Sigma) = \frac{1}{4}\int_{\Sigma} |\vec{\mathbf{H}}|^2 \, \mathrm{d}\mu_g \geq \int_{[K\geq 0]} K \, \mathrm{d}\mu_g = \int_{[K\geq 0]} J_g\nu \, \mathrm{d}\mu_g \geq$$

$$\geq Area(\nu(K \geq 0)) = Area(S^2) = 4\pi.$$

Checking the case of equality, one gets $A^0 \equiv 0$, and hence Σ is a round sphere.

In general, one has by an inequality of Li and Yau in [LY82] that

$$\# f^{-1}(x) \leq \mathcal{W}(f)/4\pi \quad \text{for all } x \in \mathbb{R}^n,$$

in particular (2) and moreover

$$\mathcal{W}(f) < 8\pi \Longrightarrow f \text{ is an embedding.}$$

Critical points of $\mathcal{W}$ are called Willmore surfaces or more precisely Willmore immersions. They satisfy the Euler-Lagrange equation which is the fourth order, quasilinear geometric equation

$$\Delta_g \vec{\mathbf{H}} + Q(A^0)\vec{\mathbf{H}} = 0$$

where the Laplacian of the normal bundle along f is used and $Q(A^0)$ acts linearly on normal vectors along f by

$$Q(A^0)\phi := g^{ik}g^{jl}A^0_{ij}\langle A^0_{kl}, \phi\rangle.$$

The non-linearity is cubic in A and is hence critical due to the conformal invariance of the Willmore functional. We see that minimal surfaces, that is when $\vec{\mathbf{H}} \equiv 0$, are Willmore surfaces. Now by maximum principle there are no closed minimal surfaces in $\mathbb{R}^3$. Instead, the Clifford torus

$$T_{Cliff} := \frac{1}{\sqrt{2}}(S^1 \times S^1) \subseteq S^3$$

is a minimal surface in S^3 , and by a similar argument is a Willmore surface. Applying an appropriate stereographic projection, this yields a rotational round torus whose ratio of the radi is $\sqrt{2}$. By conformal invariance this is a Willmore surface in $\mathbb{R}^3$.

In 1965, Willmore conjectured that

$$\mathcal{W}(T^2) \geq W(T_{Cliff}) = Area(T_{Cliff}) = 2\pi^2$$

for any torus T^2 in $\mathbb{R}^3$.

In 1993, Simon proved in [Sim93] that there exists a torus which minimizes the Willmore energy under all tori in $\mathbb{R}^n$.

In 2005, Kuwert and S. gave an estimate on the conformal factor in the following setting. By Poincarè's theorem, the pull-back metric $g = f^*g_{euc}$ is conformally equivalent

$$g = e^{2u}g_0$$

to a metric g_0 of constant Gauß curvature $K_{g_0} \equiv const$. If Σ is not a sphere, g_0 respectively u are unique up to a multiplicative respectively additive constant. Applying a Möbius transformation Φ , we get

$$\tilde{g} := (\Phi \circ f)^* g_{euc} = f^* \Phi^* g_{euc} = e^{2\tilde{u}} g_0,$$

as Φ is conformal. Deforming f by an appropriate Möbius transformation Φ to a sphere with small handles keeping the Wilmore energy fixed by conformal invariance, we see that $\tilde{u}$ degenerates, and hence no estimation of u is possible without taking account of the invariance group of $\mathcal{W}$. The theorem reads as follows.

Theorem [KuSch08]: Let $\Sigma \subseteq \mathbb{R}^n$ be an embedded closed surface of genus $p \geq 1, n = 3, 4$ with

$$\mathcal{W}(\Sigma) \leq \mathcal{W}_{n,p} - \delta,$$

where

$$\mathcal{W}_{n,p} := \min(8\pi,\ \text{Douglas condition},\ \text{technical condition for } n = 4).$$

Then the exists a Möbius transformation Φ satisfying

$$\Phi^* g_{euc|\Sigma} = e^{2u} g_0$$

for some constant curvature metric g_0 and

$$\| u \|_{L^\infty(\Sigma)} \leq C(p, \delta).$$

□

Here some comments and consequences:

1. The Douglas condition is neccessary to ensure that all handles stay together at same size and do not split. The technical condition is due to a possibly not optimal model of the Grassmannian $G_{4,2}$ in higher codimension. Anyway we have $\mathcal{W}_{3,1} = 8\pi$.
2. Connecting two concentric spheres by two necks which are approximate catenoids, one gets conformally degenerate tori with Willmore energy $\leq 8\pi + \varepsilon$. Möbius transformations can enlarge one of these necks, but not simultanously both. Therefore the above estimate cannot be obtained on a energy level $> 8\pi$ even after dividing out the Möbius group.
3. Combining the estimate of the above theorem with Mumford's compactness lemma, one can prove that the conformal structures induced by immersions $f : \Sigma \to \mathbb{R}^n$ of a closed surface Σ with genus $p \geq 1, n = 3, 4,$ and $\mathcal{W}(f) \leq \mathcal{W}_{n,p} - \delta$ are compactly contained in the moduli space.
4. For a an immersion $f : \Sigma \to \mathbb{R}^n$ of a closed surface Σ with genus $p \geq 1, n = 3, 4,$ and $\mathcal{W}(f) \leq \mathcal{W}_{n,p} - \delta$ which is conformal to a given metric g_0 on Σ , the above theorem yields a Möbius transformation Φ such that

$$\| \Phi \circ f \|_{W^{2,2}(\Sigma)} \leq C(p, \delta, g_0),$$
$$Area(\Phi \circ f) = 1.$$

□

We add to 3.:
Any torus is conformally equivalent to a quotient $T^2 \cong \mathbb{C}/(\mathbb{Z}+\omega\mathbb{Z})$ with $\omega = x+iy \notin \mathbb{R}$. Actually ω can uniquely be chosen with

$$x^2 + y^2 \geq 1, 0 \leq x \leq 1/2, y > 0.$$

Now it was proved in [LY82]

$$\mathcal{W}(T^2) \geq \frac{2\pi^2}{y},$$

which yields the estimate of the Willmore conjecture

$$\mathcal{W}(T^2) \geq 2\pi^2, \tag{4}$$

if $y \leq 1$. In [MoRo86] this was improved to

$$\mathcal{W}(T^2) \geq \frac{4\pi^2}{1 + y^2 + x^2 - x},$$

which enlarges (4) to the circle with center at $\omega = \frac{1}{2} + i$ and radius $1/2$. Both regions include the square structure, that is $\omega = i$. As the Clifford torus has square structure $T_{Cliff} \cong \mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$ and satisfies $\mathcal{W}(T_{Cliff}) = 2\pi^2$, both estimates imply that the Clifford torus minimizes the Willmore energy in its conformal class.

Critical points of the Willmore functional under fixed conformal class are called constrained Willmore surfaces or immersions. They satisfy the Euler-Lagrange equation

$$\Delta_g \vec{\mathbf{H}} + Q(A^0)\vec{\mathbf{H}} = g^{ik} g^{jl} A^0_{ij} q_{kl},$$

where q is symmetric, traceless and transverse, that is

$$q_{kl} = q_{lk},$$
$$tr_g q = g^{kl} q_{kl} = 0,$$
$$g^{ij} \nabla_i q_{jk} = 0.$$

These tensors are equivalent to holomorphic quadratic differentials, hence q are analytic.

We generalize the Clifford torus to

$$T_r := rS^1 \times \sqrt{1 - r^2} S^1 \subseteq S^3,$$

$T_{1/\sqrt{2}} = T_{Cliff}$, which are surfaces of constant mean curvature. By an easy argument these are in codimension one constrained Willmore surfaces. T_r are of rectangular class with $\omega = (\sqrt{1 - r^2}/r)i$ and

$$\mathcal{W}(T_r) \to \infty \quad \text{for } r \to 0.$$

On the other hand, the examples in 2. are of degenerate rectangular class with Willmore energy $\mathcal{W} \leq 8\pi + \varepsilon$. Therefore T_r do not minimize the Willmore energy in its conformal class for small r . □

We add to 4.:
With the estimate in 4., one can apply the direct method of the calculus of variations and obtain minimizers of the Willmore energy in fixed conformal class.

Theorem [KuSch09]: Let Σ be a closed surface of genus $p \geq 1, n = 3, 4$ and g_0 be a metric on Σ . If

$$\mathcal{W}(\Sigma, g_0, n) := \inf\{\mathcal{W}(f) \mid f : (\Sigma, g_0) \to \mathbb{R}^n \text{ conformal immersion } \} < \mathcal{W}_{n,p},$$

then there exists a minimizer f under all conformal immersions

$$\mathcal{W}(f) = \mathcal{W}(\Sigma, g_0, n).$$

□

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