Semilinear elliptic problems with measure data.

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We discuss boundary value problems of the form

$$-\Delta u + g(u) = 0, \qquad \text{in } D \tag{1}$$

$$u = \mu, \qquad \text{on } \partial D, \tag{2}$$

D a domain in $\mathbb{R}^N$, μ a Borel measure on ∂D,

$$g \in C(\mathbb{R}), \quad g \uparrow, \quad g(0) = 0, \quad \lim_{t \to \infty} g(t)/t = \infty.$$

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For μ bounded, a solution of (1)-(2) means:

$$\begin{aligned} &u \in L^1(D), \quad g(u) \in L^1_\rho(D), \\ -&\int_D u \Delta \phi \, dx + \int_D g(u) \phi \, dx = -\int_{\partial D} \partial_{\mathbf{n}} \phi \, d\mu, \end{aligned} \tag{3}$$

for every $\phi \in C^2(\bar{D})$ such that $\phi = 0$ on ∂D.

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HISTORY.

Beginnings

Emden (1897), Fowler (1931): Radial solutions in the case $g(t) = t^q$.

Bieberbach (1916): Equation $-\Delta u + e^u = 0$.

Keller (1957): Equations with general nonlinearity, motivated by a model in astrophysics introduced by Chandrasekhar.

The geometric connection.

The equation $-\Delta u + k(x)u^q = 0$, $q = (N+2)/(N-2)$ and the Yamabe problem.

In this context the problem

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A solution of $-\Delta u + k(x)g(u) = 0$, $k > 0$ in D, blowing up on ∂D is called a large solution. The existence, asymptotic behavior and uniqueness of large solutions was first studied by *Loewner and Nirenberg (1972)*, for $-\Delta u + u^q = 0$, $q = (N+2)/(N-2)$ in smooth domains. In the 90's the subject of large solutions received much attention. The questions of existence and uniqueness have been studied in various contexts:

General non-linearities, non-smooth domains, the problem on manifolds etc.

The probabilistic connection.

Branching processes, superdiffusions are related to equations

$$u_t - \Delta u + u^\alpha = 0, \quad \text{and} \quad -\Delta u + u^\alpha = 0,$$

$1 < \alpha \leq 2$.

These have been central subjects of study in probability for almost five decades, *Watanabe (1965, 68, 69), Dawson (1975, 77, 89 ...), Perkins (1988-1991, 2001), Dynkin (1990 and on)), Le Gall (1990 and on)* and others. In particular a paper of Dynkin from 1991 focused attention on the PDE connection. This gave a strong impetus to the study of boundary value problems with measure data.

A solution of

$$-\Delta u + u^{\alpha} = 0 \quad \text{in } D, \qquad u \to \infty \text{ at } \partial D,$$

corresponds to a branching process in D which never crosses the boundary, i.e. becomes extinct in D.

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If F is a closed subset of ∂D, a solution of

$$-\Delta u + u^\alpha = 0 \quad \text{in } D, \quad u \xrightarrow{s} \infty \text{ at } F, \quad u = 0 \text{ on } \partial D \setminus F,$$

corresponds to a branching process in D which is barred from crossing ∂D at F.

The notation $u \xrightarrow{s} \infty$ (i.e. u tends **strongly** to ∞) at a point $y \in \partial D$ means that, for every neighborhood A of y, $\int_{A\cap D} |u|^q \rho dx = \infty$. If $1 < \alpha < (N+1)/(N-1)$ the 'neighborhood' is in the Euclidean topology. If $\alpha \geq (N+1)/(N-1)$, the 'neighborhood' is in another (q dependent) topology.

Main questions.

For which sets F does such a solution exist? Is the solution unique?
For which sets is a barrier at F removable?
What is the rate of blow up at F?

The study of these questions depends on the study of boundary value problems with measure boundary data.

Two basic features of the problem.

A. The absorption effect.

If $g(t) \to \infty$ sufficiently fast as $t \to \infty$ then the absorption effect balances the diffusion effect:

for every compact $K \subset D$, $\exists C_K$ such that
$\sup_K u \le C_K$ for every solution u of (1).

A sharp criterion was supplied by Keller and Osserman (separately, 1955). It is satisfied for example by

$$g(t) = |t|^{q-1}t, \quad q > 1, \quad g(t) = max(e^t - 1, 0).$$

B. The comparison principle.

Let u_1, u_2 be solutions of (1)-(2) with $\mu = \mu_1$, $\mu = \mu_2$ respectively. Then

$$\mu_1 \le \mu_2 \Longrightarrow u_1 \le u_2.$$

I. Classical results for general nonlinearities.

(i) Uniqueness: If $\partial D \in C^2$

$$-\Delta u + g(u) = \tau \ \text{ in } D, \qquad u = \mu \ \text{ on } \partial D, \tag{5}$$

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(ii) Existence for L^1 data:
If $D \in C^2$, $\tau = f dx$, $\mu = h dS$, $f \in L^1_\rho(D)$, $h \in L^1(\partial D)$ then (5) has a solution.

These results are due partly to Brezis and Strauss (1970) and partly to Brezis in the 70's (mostly unpublished).

(iii) If g satisfies OK criterion: Existence of maximal solution of

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(iv) If g satisfies OK criterion, D is Lipschitz, $F \subset \partial D$ closed:

Existence of maximal solution of

$$-\Delta u + g(u) = 0, \text{ in } D, \quad u \to \infty \text{ at } F, \quad u = 0 \text{ on } \partial D \setminus F. \tag{6}$$

II. Results for equation $(*)$ $-\Delta u + u^q = \tau,\ q > 1.$

(i) If $\tau = \delta_P,\ P \in D,$ equation $(*)$ has a solution iff $q < N/(N-2)$.

Corollary. If $q < N/(N-2)$ then $(*)$ has a solution for every finite measure τ.
(Benilan – Brezis 197–)

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For the supercritical case, $q \geq q_{int}$:
(ii) Equation (*) has a solution if and only if:

$$C_{2,q'}(E) = 0 \Longrightarrow \tau(E) = 0.$$

(Baras and Pierre, 1984)

Here $C_{2,q'}$ denotes **Bessel capacity.** For compact sets $K \subset \mathbb{R}^N$:

$$C_{2,q'}(K) = \inf\{\|\varphi\|_{W^{2,q'}} : \varphi \geq 0,\ \varphi \geq 1 \text{ on } K.\}$$

III. Bounded measure data on ∂D: subcritical case.

Assume $\partial D \in C^2$.
(i) For $P \in \partial D$:

$$-\Delta u + |u|^{q-1}u = 0 \text{ in } D, \quad u = \delta_P \text{ on } \partial D \tag{7}$$

has a solution iff $q < (N+1)/(N-1)$.

Corollary. If $q < (N+1)/(N-1)$ then

$$-\Delta u + |u|^{q-1}u = 0 \text{ in } D, \quad u = \mu \text{ on } \partial D \tag{8}$$

has a solution for every finite measure μ. (Gmira – Veron , 1991)

$q_{bnd} = (N+1)/(N-1)$ is the critical exponent for (7) .

(ii) *Let* μ *be a finite measure on* ∂D *and let* V_μ *denote the harmonic function in* D *with boundary trace* μ. *Assume that* g *is odd. If*

$$\textbf{(Ad)} \qquad \int_D g(V_{|\mu|})\rho\, dx < \infty, \qquad \rho(x) = \operatorname{dist}(x, \partial D)$$

then

$$-\Delta u + g(u) = 0 \text{ in } D, \quad u = \mu \text{ on } \partial D \tag{9}$$

has a solution. (M+Veron 1998)

The result follows from the fact that, if **(AD)** holds, $V_{|\mu|}$ is a supersolution and $-V_{|\mu|}$ is a subsolution of the boundary value problem (9).

If μ satisfies condition **(Ad)** we say that μ is admissible relative to g.

Let K be the Poisson kernel for $-\Delta$ in D. Then, for $y \in \partial D$, $V_{\delta_P}(x) = K(x, y)$. We note that, if $q < (N+1)/(N-1)$ then

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Therefore (ii) implies (i).

A relation between the homogeneous and nonhomogeneous problems.

Under fairly general conditions on g (e.g. convexity):

If (9) *has a solution then the boundary value problem*

$$-\Delta u + g(u) = \tau \text{ in } D, \quad u = \mu \text{ on } \partial D \tag{10}$$

has a solution, provided that the equation has some solution *in D.*

IV. Bounded measure data on ∂D: supercritical case.

We assume that $\partial D \in C^2$ and consider

$$\textbf{(BVPq)} \qquad -\Delta u + |u|^{q-1}u = 0 \text{ in } D, \quad u = \mu \text{ on } \partial D$$

for $q \geq q_{bnd} = (N+1)/(N-1)$.

A closed set $F \subset \partial D$ is removable for **(BVPq)** if:
the only non-negative solution of

$$-\Delta u + |u|^{q-1}u = 0 \text{ in } D, \quad u = 0 \text{ on } \partial D \setminus F$$

is $u \equiv 0$.
A set $E \subset \partial D$ is removable if every closed subset is removable.

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A set $E \subset \partial D$ is removable if every closed subset is removable.

Here a 'solution' means: For every neighborhood A of F

$$\begin{gathered} u \in L^1(D \setminus A), \quad g(u) \in L^1_\rho(D \setminus A), \\ -\int_D u\Delta\phi \, dx + \int_D g(u)\phi \, dx = -\int_{\partial D} \partial_{\mathbf{n}}\phi \, d\mu, \end{gathered} \tag{11}$$

for every $\phi \in C^2(\bar{D})$ such that $\phi = 0$ on $\partial D \cup (D \cap A)$.

We say that a finite measure μ on ∂D is q-good if

$$\textbf{(BVPq)} \qquad -\Delta u + |u|^{q-1}u = 0 \text{ in } D, \quad u = \mu \text{ on } \partial D$$

possesses a solution.

The following results were obtained, during the 90's:
by *probabilistic techniques* – Le Gall ($q = 2$), Dynkin and Kuznetsov ($q_{bnd} \leq q \leq 2$);
by analytic methods – M+Veron (all $q \geq q_{bnd}$).

Theorem IV.1

$$\text{A set } E \subset \partial D \text{ is removable} \\ \Updownarrow \\ C_{2/q,q'}(E) = 0.$$

Theorem IV.2

$$\text{A finite measure } \mu \text{ is q-good} \\ \Updownarrow \\ \mu \text{ vanishes on sets of } C_{2/q,q'}\text{-capacity zero.}$$

Main ingredients in the proof of **IV.2** ; μ denotes a finite measure on ∂D.

(a) If μ is q-good then it vanishes on removable sets.

(b) Assume $\mu \geq 0$. Let V_μ denote the harmonic function with boundary trace μ.

$$V_\mu \in L^q_\rho(D) \iff \mu \in W^{-2/q,q}.$$

(c) $\mu \in W^{-2/q,q} \Longrightarrow \mu$ is q-good.

(d) Remark: If $\mu \in W^{-2/q,q}$ then μ vanishes on sets of $C_{2/q,q'}$-capacity zero.

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(e) A theorem of Feyel - de la Pradelle: Assume $\mu \geq 0$. Then

$$\mu \text{ vanishes on sets of } C_{2/q,q'}\text{-capacity zero} \iff$$
$$\mu \text{ is the limit of an increasing sequence of measures in } W^{-2/q,q}.$$

(f) If μ vanishes on sets of $C_{2/q,q'}$-capacity zero then μ is q-good.

V. Solutions blowing up on a subset of ∂D: subcritical case.

We consider the problem

$$-\Delta u + u^\alpha = 0 \quad \text{in } D, \quad u \xrightarrow{\text{s}} \infty \text{ at } F, \quad u = 0 \text{ on } \partial D \setminus F, \qquad (12)$$

where $F \subset \partial D$ assuming $q < (N+1)/(N-1)$.

Theorem V.1 The problem possesses a solution if and only if F is closed. Furthermore the solution is unique. (M+Veron 1996)

Overview of proof:

(a) If $y \in \partial D$ there exists a unique solution of (12) such that $F = \{y\}$. Denote this solution by U_y.

Remark: There exist infinitely many solutions of the problem

$$-\Delta u + u^\alpha = 0 \quad \text{in } D, \quad u \to \infty \text{ at } y, \quad u = 0 \text{ on } \partial D \setminus \{y\}.$$

(b) If u is a solution of (12) then

$$u \geq U_y \quad \forall y \in F.$$

(c) Let $\{y_n\}$ be a dense sequence in F and let $A_n := \{y_1, \cdots, y_n\}$. Then $\{V_{A_n}\}$ is an increasing sequence of solutions whose limit W_F is the *minimal solution* of (12).

(d) $W_F \xrightarrow{\text{s}} \infty$ at $\bar{F}$. Therefore $'F$ closed$'$ is a necessary condition for existence.

(e) If F is closed there exists a *maximal solution* U_F of (12).

(f) There exists a constant c such that

$$W_F \leq U_F \leq cW_F.$$

(g) The above implies that $U_F = W_F$.

VI. Solutions blowing up on a subset of ∂D: supercritical case.

When $q \geq (N+1)/(N-1)$ the problem (12) is not well posed; it may have infinitely many solutions. (This was shown by Le Gall in 1997.) Therefore one must interpret 'strong blow-up' in a more refined way. It turns out that the correct topology in this context is the $C_{2/q,q'}$-fine topology and the appropriate definition of 'strong blow-up' is:

$u \xrightarrow{\text{sq}} \infty$ (i.e. u tends **q-strongly** to ∞) at a point $y \in \partial D$ means that, for every $C_{2/q,q'}$-fine neighborhood A of y, $\int_{A\cap D} |u|^q \rho dx = \infty$.

Thus we consider the problem

$$-\Delta u + u^\alpha = 0 \quad \text{in } D, \quad u \xrightarrow{\text{sq}} \infty \text{ at } F, \quad u = 0 \text{ on } \partial D \setminus F. \qquad (13)$$

For $q < (N+1)/(N-1)$, the fine topology is the same as the Euclidean topology so 'q-strong blow up' reduces to 'strong blow up'.

We need an additional definition:
A solution u of the equation is moderate if $\int_D |u|^q \rho dx < \infty$; it is σ-moderate if it is the limit of an increasing sequence of moderate solutions.

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Theorem VI.1 Problem (13) possesses a solution if and only if F is $C_{2/q,q'}$-finely closed. Furthermore the solution is unique in the class of σ-moderate solutions. (M+Veron 2007)

For $q = 2$ Mselati (2001) proved that every positive solution is σ-moderate. This was extended by Dynkin (2004) to $1 < q \leq 2$. Therefore:

Theorem VI.2 If $1 < q \leq 2$ and F is $C_{2/q,q'}$-finely closed, problem (13) possesses a unique solution.