On the long-time asymptotics for degenerate kinetic equations

Francesco Salvarani

University of Pavia, Italy

Séminaire de mathématiques appliquées
Collège de France

February 15th, 2013

Structure of the talk

1. The problem
2. Convergence to equilibrium for non degenerate cross sections
3. Convergence to equilibrium for degenerate cross sections
   - Degeneracy in isolated points
   - A counterexample
   - The geometrical condition
4. References

Table of contents

1. The problem
2. Convergence to equilibrium for non degenerate cross sections
3. Convergence to equilibrium for degenerate cross sections
   - Degeneracy in isolated points
   - A counterexample
   - The geometrical condition
4. References

The linear Boltzmann equation in the d-dimensional torus $\mathbb{T}^d$, $d \geq 2$

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \sigma\,(f - Kf) = 0 & (t, x, v) \in \mathbb{R}_+ \times \mathbb{T}^d \times V \\ f(0, x, v) = f^{in}(x, v) \in L^1\left(\mathbb{T}^d \times V\right) & (x, v) \in \mathbb{T}^d \times V \end{cases}$$

Velocity space: $V = \{v \in \mathbb{R}^d \,:\, 0 < v_m \leq |v| \leq v_M\}$ or $V = \mathbb{S}^{d-1}$

Normalization on $\mathbb{T}^d \times V$: $\displaystyle\int_{\mathbb{T}^d} dx = \int_V dv = 1$

Scattering operator $Kf := \displaystyle\int_V k(v, w) f(t, x, w)\, dw$ with

$k \in L^\infty(V \times V)$, $\displaystyle\int_V k(v, w)\, dw = 1$ and $k(v, w) > 0$ a.e. on $V \times V$

Cross section $\sigma \in L^\infty(\mathbb{T}^d)$, with $\sigma \geq 0$ a.e. and $\displaystyle\int_{\mathbb{T}^d} \sigma(x) dx > 0$

Taxonomy

Non degenerate cross section:

$\sigma \in L^\infty(\mathbb{T}^d)$ and there exists $m > 0$ such that $\sigma \geq m$ a.e. in $\mathbb{T}^d$

Degenerate cross section:

$\sigma \in L^\infty(\mathbb{T}^d)$, $\sigma \geq 0$ a.e. in $\mathbb{T}^d$, $\int_{\mathbb{T}^d} \sigma(x)dx > 0$ but it does not exists $m > 0$ such that $\sigma \geq m$ for a.e. x belonging to $\mathbb{T}^d$

Taxonomy

Non degenerate cross section:

$\sigma \in L^\infty(\mathbb{T}^d)$ and there exists $m > 0$ such that $\sigma \geq m$ a.e. in $\mathbb{T}^d$

Degenerate cross section:

$\sigma \in L^\infty(\mathbb{T}^d)$, $\sigma \geq 0$ a.e. in $\mathbb{T}^d$, $\int_{\mathbb{T}^d} \sigma(x)dx > 0$ but it does not exists $m > 0$ such that $\sigma \geq m$ for a.e. x belonging to $\mathbb{T}^d$

Taxonomy

Non degenerate cross section:

$\sigma \in L^\infty(\mathbb{T}^d)$ and there exists $m > 0$ such that $\sigma \geq m$ a.e. in $\mathbb{T}^d$

Degenerate cross section:

$\sigma \in L^\infty(\mathbb{T}^d)$, $\sigma \geq 0$ a.e. in $\mathbb{T}^d$, $\displaystyle\int_{\mathbb{T}^d} \sigma(x)dx > 0$ but it does not exists $m > 0$ such that $\sigma \geq m$ for a.e. x belonging to $\mathbb{T}^d$

Table of contents

1 The problem

2 Convergence to equilibrium for non degenerate cross sections

3 Convergence to equilibrium for degenerate cross sections
- Degeneracy in isolated points
- A counterexample
- The geometrical condition

4 References

Convergence to equilibrium: the non degenerate case

Theorem (Ukai, Point, Ghidouche - 1978)

If $\sigma(x)$ is non degenerate, there exist $C, \gamma > 0$ such that the solution of the transport equation satisfies the estimate

$$\|f(t,\cdot,\cdot) - f_\infty\|_{L^2(\mathbb{T}^d \times \mathbb{S}^{d-1})} \leq C e^{-\gamma t} \|f^{in}\|_{L^2(\mathbb{T}^d \times \mathbb{S}^{d-1})}.$$

Theorem (Mouhot, Neumann - 2006)

If $\sigma(x)$ is non degenerate, there exist two explicit, strictly positive constants C and γ, such that the solution of the transport equation satisfies the estimate

$$\|f(t,\cdot,\cdot) - f_\infty\|_{H^1(\mathbb{T}^d \times \mathbb{S}^{d-1})} \leq C e^{-\gamma t} \|f^{in}\|_{H^1(\mathbb{T}^d \times \mathbb{S}^{d-1})}.$$

Convergence to equilibrium: the non degenerate case

Theorem (Ukai, Point, Ghidouche - 1978)

If $\sigma(x)$ is non degenerate, there exist C, $\gamma > 0$ such that the solution of the transport equation satisfies the estimate

$$\|f(t,\cdot,\cdot) - f_\infty\|_{L^2(\mathbb{T}^d\times\mathbb{S}^{d-1})} \le Ce^{-\gamma t}\|f^{in}\|_{L^2(\mathbb{T}^d\times\mathbb{S}^{d-1})}.$$

Theorem (Mouhot, Neumann - 2006)

If $\sigma(x)$ is non degenerate, there exist two explicit, strictly positive constants C and γ, such that the solution of the transport equation satisfies the estimate

$$\|f(t,\cdot,\cdot) - f_\infty\|_{H^1(\mathbb{T}^d\times\mathbb{S}^{d-1})} \le Ce^{-\gamma t}\|f^{in}\|_{H^1(\mathbb{T}^d\times\mathbb{S}^{d-1})}.$$

Convergence to equilibrium: the non degenerate case

Theorem (Ukai, Point, Ghidouche - 1978)

If $\sigma(x)$ is non degenerate, there exist $C, \gamma > 0$ such that the solution of the transport equation satisfies the estimate

$$\|f(t,\cdot,\cdot) - f_\infty\|_{L^2(\mathbb{T}^d\times\mathbb{S}^{d-1})} \le Ce^{-\gamma t}\|f^{in}\|_{L^2(\mathbb{T}^d\times\mathbb{S}^{d-1})}.$$

Theorem (Mouhot, Neumann - 2006)

If $\sigma(x)$ is non degenerate, there exist two explicit, strictly positive constants C and γ, such that the solution of the transport equation satisfies the estimate

$$\|f(t,\cdot,\cdot) - f_\infty\|_{H^1(\mathbb{T}^d\times\mathbb{S}^{d-1})} \le Ce^{-\gamma t}\|f^{in}\|_{H^1(\mathbb{T}^d\times\mathbb{S}^{d-1})}.$$

Table of contents

1. The problem
2. Convergence to equilibrium for non degenerate cross sections
3. Convergence to equilibrium for degenerate cross sections
   - Degeneracy in isolated points
   - A counterexample
   - The geometrical condition
4. References

Table of contents

1. The problem
2. Convergence to equilibrium for non degenerate cross sections
3. Convergence to equilibrium for degenerate cross sections
   - Degeneracy in isolated points
   - A counterexample
   - The geometrical condition
4. References

Degeneracy in isolated points

First suppose that the cross section $\sigma : \mathbb{T}^d \to \mathbb{R}_+$ is degenerate and satisfies, moreover, the following property:

Assumption

There exist $x_i \in \mathbb{T}^d$, $i = 1, \ldots, N$, $C_\sigma > 0$ and $\lambda_\sigma > 0$ such that

$$\text{for a.e. } x \in \mathbb{T}^d, \qquad \sigma(x) \geq C_\sigma \inf_{i=1,\ldots,N} |x - x_i|^{\lambda_\sigma}.$$

Assumption on the scattering kernel

$$k \equiv 1, \qquad \bar{f} := \int_V f(t,x,w)\, dw.$$

Degeneracy in isolated points

First suppose that the cross section $\sigma : \mathbb{T}^d \to \mathbb{R}_+$ is degenerate and satisfies, moreover, the following property:

Assumption

There exist $x_i \in \mathbb{T}^d$, $i = 1, \ldots, N$, $C_\sigma > 0$ and $\lambda_\sigma > 0$ such that

$$\text{for a.e. } x \in \mathbb{T}^d, \qquad \sigma(x) \geq C_\sigma \inf_{i=1,\ldots,N} |x - x_i|^{\lambda_\sigma}.$$

Assumption on the scattering kernel

$$k \equiv 1, \qquad \bar{f} := \int_V f(t,x,w)\, dw.$$

Degeneracy in isolated points

First suppose that the cross section $\sigma : \mathbb{T}^d \to \mathbb{R}_+$ is degenerate and satisfies, moreover, the following property:

Assumption

There exist $x_i \in \mathbb{T}^d$, $i = 1, \ldots, N$, $C_\sigma > 0$ and $\lambda_\sigma > 0$ such that

$$\text{for a.e. } x \in \mathbb{T}^d, \qquad \sigma(x) \geq C_\sigma \inf_{i=1,\ldots,N} |x - x_i|^{\lambda_\sigma}.$$

Assumption on the scattering kernel

$$k \equiv 1, \qquad \bar{f} := \int_V f(t,x,w)\, dw.$$

Theorem (Desvillettes, S. - 2009)

Consider the linear transport equation with a cross section $\sigma \in L^\infty(\mathbb{T}^d) \cap H^1(\mathbb{T}^d)$ satisfying the previous assumption, $k \equiv 1$, and with an initial condition $f^{in} \geq 0$ a.e. such that $f^{in} \in L^\infty(\mathbb{T}^d \times V)$, $\nabla_x \bar{f}^{in} \in L^2(\mathbb{T}^d)$, and $v \otimes v : \nabla_x \nabla_x f^{in} \in L^2(\mathbb{T}^d \times V)$.

Then there exists a unique nonnegative solution $f := f(t,x,v)$ to this system in $C(\mathbb{R}_+; L^2(\mathbb{T} \times V))$.

The solution f converges when $t \to +\infty$ to its asymptotic profile

$$f_\infty(x,v) := \int_{\mathbb{T}^d} \int_V f^{in}(y,w)\, dwdy$$

and

$$||f(t,\cdot,\cdot) - f_\infty||^2_{L^2(\mathbb{T}\times V)} \leq C_1\, t^{-\frac{1}{1+2\,\lambda_\sigma}}.$$

The explicit constant C_1 depends on C_σ, λ_σ, $||\sigma||_{H^1(\mathbb{T}^d)\cap L^\infty(\mathbb{T}^d)}$, and f^{in}.

Strategy of proof

Proposition (Desvillettes, Villani - 2001)

Let z and y be two nonnegative C^2 functions defined on $\mathbb{R}_+$ and satisfying (for all $t > 0$)

$$\begin{cases} -z'(t) \geq \alpha_1 \, y^{1+\delta}(t), \\ y''(t) \geq \alpha_3 \, z(t) - \alpha_2 y^{1-\varepsilon}(t), \end{cases}$$

for some constants $\delta \geq 0$, $\varepsilon \in]0, 1[$ and α_1, α_2, $\alpha_3 > 0$.

Then there exists a constant $\alpha_4 > 0$ depending only on $x(0)$, α_1, α_2, α_3, δ and ε such that (for all $t > 0$)

$$z(t) \leq \alpha_4 \, t^{-\frac{1-\varepsilon}{\delta+\varepsilon}}.$$

The entropy/entropy production pair

$$H(f) = \int_{\mathbb{T}^d \times V} |f - f_\infty|^2 \, dvdx, \qquad D(f) = \int_{\mathbb{T}^d \times V} |f - \bar{f}|^2 \, dvdx.$$

Relationship between entropy production and D:

$$\int \sigma \, |f - \bar{f}|^2 \, dvdx \leq \|\sigma\|_{L^\infty(\mathbb{T}^d)} D(f).$$

By interpolation:

$$D(f)^{1+\lambda_\sigma} \leq \beta_1 \int \sigma \, |f - \bar{f}|^2 \, dvdx, \quad \beta_1 > 0$$

We deduce

$$\begin{cases} -\dfrac{dH(f)}{dt} \geq 2\beta_1 \, D(f)^{1+\lambda_\sigma} \\ \dfrac{d^2}{dt^2} D(f) \geq \beta_2 \, H(f) - \beta_3 \, D(f)^{1/2} \end{cases}$$

Table of contents

1. The problem

2. Convergence to equilibrium for non degenerate cross sections

3. Convergence to equilibrium for degenerate cross sections
   - Degeneracy in isolated points
   - A counterexample
   - The geometrical condition

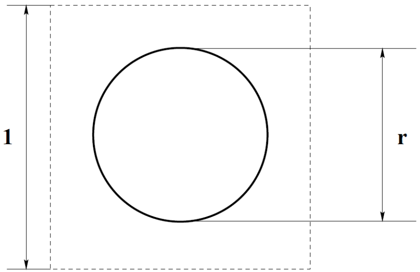
4. References

The domain

For all $r \in (0, 1/2)$ consider the periodic open set

$$Z_r = \{x \in \mathbb{R}^d \ : \ dist(x, \mathbb{Z}^d) > r\}$$

together with the associated fundamental domain $Y_r = Z_r/\mathbb{Z}^d$.



The forward exit time

Forward exit time for a particle starting from $x \in Z_r$ in the direction $v \in \mathbb{S}^{d-1}$

$$\tau_r(x, v) = \inf\{t > 0 \;:\; x + tv \in \partial Z_r\}$$

Definition of the forward exit time on the quotient space $Y_r \times \mathbb{S}^{d-1}$

$$\tau_r(x + k, v) = \tau_r(x, v) \textit{ for all } (x, v) \in Z_r \times \mathbb{S}^{d-1} \textit{ and } k \in \mathbb{Z}^d$$

On $Y_r \times \mathbb{S}^{d-1}$, equipped with its Borel σ-algebra, define μ_r as the probability measure proportional to the Lebesgue measure on $Y_r \times \mathbb{S}^{d-1}$:

$$d\mu_r(y, v) = \frac{dydv}{|Y_r|\,|\mathbb{S}^{d-1}|}$$

Distribution of τ_r under μ_r:

$$\Phi_r(t) := \mu_r\left(\{(x, v) \in Y_r \times \mathbb{S}^{d-1} \;:\; \tau_r(y, v) > t\}\right)$$

The distribution of forward exit time

Theorem (Bourgain, Golse, Wennberg - 1998, 2000)

Let $d \geq 2$. Then there exist two positive constants C_1 and C_2 such that, for all $r \in (0, 1/2)$ and each $t > 1/r^{d-1}$

$$\frac{C_1}{r^{d-1}}\, t^{-1} \leq \Phi_r(t) \leq \frac{C_2}{r^{d-1}}\, t^{-1}.$$

The counterexample

A particular choice of σ and f^{in}

Choose

$$\sigma(x) = \mathbb{1}_{\mathbb{T}^d \setminus Y_r}$$

and

$$f^{in}(x, v) = f^{in}(x) = \mathbb{1}_{Y_r}$$

Remarks:

- The only steady solution with the same mass as the initial condition f^{in} is the constant function $f_\infty = |Y_r|$.
- Some particles never meet the scattering region, i.e. $\{x \in \mathbb{T}^d : \sigma(x) > 0\}$, because of the presence of infinite channels.

The counterexample

A particular choice of σ and f^{in}

Choose

$$\sigma(x) = \mathbb{1}_{\mathbb{T}^d \setminus Y_r}$$

and

$$f^{in}(x, v) = f^{in}(x) = \mathbb{1}_{Y_r}$$

Remarks:

- The only steady solution with the same mass as the initial condition f^{in} is the constant function $f_\infty = |Y_r|$.
- Some particles never meet the scattering region, i.e. $\{x \in \mathbb{T}^d : \sigma(x) > 0\}$, because of the presence of infinite channels.

The counterexample

A particular choice of σ and f^{in}

Choose

$$\sigma(x) = \mathbb{1}_{\mathbb{T}^d \setminus Y_r}$$

and

$$f^{in}(x, v) = f^{in}(x) = \mathbb{1}_{Y_r}$$

Remarks:

- The only steady solution with the same mass as the initial condition f^{in} is the constant function $f_\infty = |Y_r|$.
- Some particles never meet the scattering region, i.e. $\{x \in \mathbb{T}^d \,:\, \sigma(x) > 0\}$, because of the presence of infinite channels.

An upper bound on the convergence speed to equilibrium

The only equilibrium solution to which f can converge in $L^2(\mathbb{T}^d \times \mathbb{S}^{d-1})$ as $t \to +\infty$ is

$$f_\infty = \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{T}^d \times \mathbb{S}^{d-1}} f^{in}(x, v)\, dxdv = |Y_r|.$$

Study of the L^2-norm

$$\begin{aligned}
\int_{\mathbb{T}^d \times \mathbb{S}^{d-1}} (f - f_\infty)^2\, dxdv &\geq \int_{Y_r \times \mathbb{S}^{d-1}} (f - f_\infty)^2\, dxdv \\
&= \int_{Y_r \times \mathbb{S}^{d-1}} \mathbb{1}_{\tau_r(x,-v)>t} (f - f_\infty)^2\, dxdv \\
&+ \int_{Y_r \times \mathbb{S}^{d-1}} \mathbb{1}_{\tau_r(x,-v)\leq t} (f - f_\infty)^2\, dxdv \\
&= I + J.
\end{aligned}$$

Duhamel's formula

$$
\begin{aligned}
f(t,x,v) &= f^{in}(x-tv,v)\exp\left(-\int_0^t \sigma(x-sv)\,ds\right)\\
&\quad + \int_0^t \exp\left(-\int_0^s \sigma(x-\tau v)\,d\tau\right)\sigma(x-sv)\bar{f}(s,x-sv)\,ds\\
&\geq f^{in}(x-tv,v)\exp\left(-\int_0^t \sigma(x-sv)\,ds\right)
\end{aligned}
$$

Since $\tau_r(x,-v) > t \Longrightarrow \sigma(x-sv) = 0$ for all $s \in [0,t]$:

$$
f(t,x,v)\mathbb{1}_{\tau_r(x,-v)>t} \geq f^{in}(x-tv,v)\mathbb{1}_{\tau_r(x,-v)>t}.
$$

From $\tau_r(x,-v) > t \Longrightarrow x - tv \in Y_r \Longrightarrow f^{in}(x-tv,v) = 1$:

$$
f(t,x,v)\mathbb{1}_{\tau_r(x,-v)>t} \geq \mathbb{1}_{\tau_r(x,-v)>t}
$$

Since $f_\infty < 1$: $\mathbb{1}_{\tau_r(x,-v)>t} f_\infty \leq \mathbb{1}_{\tau_r(x,-v)>t} \leq \mathbb{1}_{\tau_r(x,-v)>t} f(t,x,v)$.
Hence

$$
\begin{aligned}
I &= \int_{Y_r \times \mathbb{S}^{d-1}} (\mathbb{1}_{\tau_r(x,-v)>t} f - \mathbb{1}_{\tau_r(x,-v)>t} f_\infty)^2 \, dxdv \\
&\geq \int_{Y_r \times \mathbb{S}^{d-1}} \mathbb{1}_{\tau_r(x,-v)>t} (1 - f_\infty)^2 \, dxdv \\
&= (1 - |Y_r|)^2 \int_{Y_r \times \mathbb{S}^{d-1}} \mathbb{1}_{\tau_r(x,-v)>t} \, dxdv \\
&= (1 - |Y_r|)^2 |Y_r| \, |\mathbb{S}^{d-1}| \Phi_r(t).
\end{aligned}
$$

Therefore

$$
I \geq (1 - |Y_r|)^2 |Y_r| \, |\mathbb{S}^{d-1}| \frac{C_1}{r^{d-1}} \, t^{-1}
$$

for all $t > r^{1-d}$.

Bound on J:

$$J = \int_{Y_r \times \mathbb{S}^{d-1}} \mathbb{1}_{\tau_r(x,-v) \leq t} (f - f_\infty)^2 \, dxdv \geq 0,$$

Hence

$$\int_{\mathbb{T}^d \times \mathbb{S}^{d-1}} (f - f_\infty)^2 \, dxdv \geq \frac{C_1}{r^{d-1}} (1 - |Y_r|)^2 |Y_r| \, |\mathbb{S}^{d-1}| \, t^{-1}$$

or, equivalently,

$$\|f - f_\infty\|_{L^2(\mathbb{T}^d \times \mathbb{S}^{d-1})} \geq \frac{C}{\sqrt{t}}.$$

Theorem (Bernard, S. - 2012)

For all $r \in (0, 1/2)$, there exists an initial condition $f^{in} \in L^\infty(\mathbb{T}^d \times \mathbb{S}^{d-1})$ satisfying $f^{in}(x, v) \geq 0$ for a.e. $(x, v) \in \mathbb{T}^d \times \mathbb{S}^{d-1}$ and such that, for each cross section $\sigma \in L^\infty(\mathbb{T}^d)$ satisfying $\sigma(x) \geq 0$ for a.e. $x \in \mathbb{T}^d$ and $\sigma(x) = 0$ for a.e. $x \in Y_r$, the solution f of the transport problem satisfies

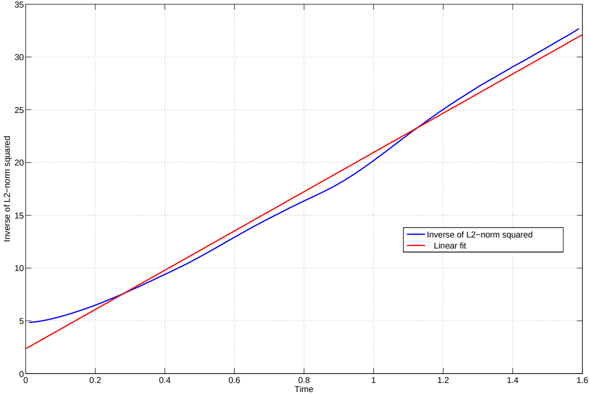
$$\|f - f_\infty\|_{L^2(\mathbb{T}^d \times \mathbb{S}^{d-1})} \geq \frac{C}{\sqrt{t}}$$

for each $t > r^{1-d}$, where

$$f_\infty = \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{T}^d \times \mathbb{S}^{d-1}} f^{in}(x, v)\, dxdv$$

and C is a positive constant.

Numerical simulation of the long-time decay (De Vuyst, S.)



Particle method 10^9 numerical particles, $r = 0.3$, $\sigma = 3$
Uniform mesh $100 \times 100 \times 100$ on $\mathbb{T}^2 \times (0, 2\pi)$

Table of contents

1. The problem
2. Convergence to equilibrium for non degenerate cross sections
3. Convergence to equilibrium for degenerate cross sections
   - Degeneracy in isolated points
   - A counterexample
   - The geometrical condition
4. References

Definition

The cross section $\sigma \equiv \sigma(x)$ is said to verify the geometrical condition if there exist T_0 and $C > 0$ such that

$$\int_0^{T_0} \sigma\left(\phi_{x,v}(s)\right) ds \geq C \text{ a.e. in } (x, v) \in \mathbb{T}^d \times V,$$

where $\phi_{x,v}$ designates the linear flow starting at $x \in \mathbb{T}^d$ in the direction $-v \in V$:

$$\phi_{x,v} : t \mapsto x - tv.$$

- The geometrical condition entails that, for a.e. $(x, v) \in \mathbb{T}^d \times V$, there exists $t \in (0, T_0)$ such that $\phi_{x,v}(t) \in \left\{x \in \mathbb{T}^d \,|\, \sigma(x) > 0\right\}$.
- In 1D: geometrical condition always fulfilled for cross sections that are strictly positive on a sub-domain of the interval $(0, 1)$ with positive Lebesgue measure, since $|v| \geq v_m > 0$.

Definition

The cross section $\sigma \equiv \sigma(x)$ is said to verify the geometrical condition if there exist T_0 and $C > 0$ such that

$$\int_0^{T_0} \sigma\left(\phi_{x,v}(s)\right) ds \geq C \text{ a.e. in } (x,v) \in \mathbb{T}^d \times V,$$

where $\phi_{x,v}$ designates the linear flow starting at $x \in \mathbb{T}^d$ in the direction $-v \in V$:

$$\phi_{x,v} : t \mapsto x - tv.$$

- The geometrical condition entails that, for a.e. $(x,v) \in \mathbb{T}^d \times V$, there exists $t \in (0, T_0)$ such that $\phi_{x,v}(t) \in \left\{x \in \mathbb{T}^d \,|\, \sigma(x) > 0\right\}$.
- In 1D: geometrical condition always fulfilled for cross sections that are strictly positive on a sub-domain of the interval $(0,1)$ with positive Lebesgue measure, since $|v| \geq v_m > 0$.

Definition

The cross section $\sigma \equiv \sigma(x)$ is said to verify the geometrical condition if there exist T_0 and $C > 0$ such that

$$\int_0^{T_0} \sigma\left(\phi_{x,v}(s)\right) ds \geq C \text{ a.e. in } (x,v) \in \mathbb{T}^d \times V,$$

where $\phi_{x,v}$ designates the linear flow starting at $x \in \mathbb{T}^d$ in the direction $-v \in V$:

$$\phi_{x,v} : t \mapsto x - tv.$$

- The geometrical condition entails that, for a.e. $(x,v) \in \mathbb{T}^d \times V$, there exists $t \in (0, T_0)$ such that $\phi_{x,v}(t) \in \left\{x \in \mathbb{T}^d \,|\, \sigma(x) > 0\right\}$.
- In 1D: geometrical condition always fulfilled for cross sections that are strictly positive on a sub-domain of the interval $(0,1)$ with positive Lebesgue measure, since $|v| \geq v_m > 0$.

Theorem (Bernard, S. - 2012)

Let $\sigma \in L^\infty\left(\mathbb{T}^d\right)$ be a non-negative cross section satisfying the geometrical condition. Then there exist two constants $M > 0$ and $\alpha > 0$ such that the solution f of the transport problem satisfies the inequality

$$\left\| f - \int_{\mathbb{T}^d \times V} f^{in}(x, v)\, dxdv \right\|_{L^1\left(\mathbb{T}^d \times V\right)} \leq M e^{-\alpha t} \left\| f^{in} \right\|_{L^1\left(\mathbb{T}^d \times V\right)}$$

for all $t \in \mathbb{R}_+$.
Conversely, if the solution of the linear Boltzmann equation converges uniformly in L^1 to its equilibrium state at an exponential rate, then σ must satisfy the geometrical condition.

The semigroup formulation of the problem

Define the transport operator $B := A_0 - M_\sigma + K_\sigma$ with domain

$$D(B) = \left\{ f \in L^1\left(\mathbb{T}^d \times V\right) \,\middle|\, v \cdot \nabla_x f \in L^1\left(\mathbb{T}^d \times V\right) \right\}.$$

The collisionless transport operator is

$$(A_0 f)(x, v) := -v \cdot \nabla_x f \text{ for each } f \in D(A_0),$$

with domain $D(A_0) = D(B)$.

The absorption and the scattering operator are

$$(M_\sigma f)(x, v) := \sigma(x) f(x, v) \text{ for each } f \in L^1\left(\mathbb{T}^d \times V\right)$$

and

$$(K_\sigma f)(x, v) := \sigma(x) \int_V k(v, w) f(x, w) dw \text{ for each } f \in L^1\left(\mathbb{T}^d \times V\right)$$

The abstract Cauchy problem

$$\begin{cases} \dfrac{d}{dt} f = Bf \\ \\ f(0,x,v) = f^{in}(x,v) \in \mathbb{T}^d \times \ V. \end{cases}$$

The operator B generates a strongly continuous positive semigroup on $L^1(\mathbb{T}^d \times \ V)$ $\mathcal{T} \equiv (T_t)_{t\geq 0}$

GOAL: prove the existence of a pair (M, α) of positive constants such that

$$\| T_t - P \|_{\mathcal{L}(L^1(\mathbb{T}^d \times \ V))}(t) \leq M e^{-\alpha t},$$

where

$$P(f) = \int_{\mathbb{T}^d \times \ V} f(x,v) dx dv \text{ for each } f \in L^1 \left(\mathbb{T}^d \times \ V \right).$$

A result concerning postive semigroups

Theorem

Let $(G_t)_{t\geq 0}$ be a bounded, quasi-compact, irreducible, positive C_0-semigroup on $L^1\left(\mathbb{T}^d \times \ V\right)$ with spectral bound zero. Then there exist a positive rank-one projection P and suitable constants $C \geq 1$ and $a > 0$ such that

$$\|G_t - P\|_{\mathcal{L}(L^1(\mathbb{T}^d\times\ V))} \leq Ce^{-at} \text{ for each } t \geq 0.$$

Check, under the assumptions above, that

- the spectral bound of B is zero,
- $\mathcal{T}$ is irreducible,
- the geometrical condition implies that $\mathcal{T}$ is quasi-compact.

A result concerning postive semigroups

Theorem

Let $(G_t)_{t\geq 0}$ be a bounded, quasi-compact, irreducible, positive C_0-semigroup on $L^1\left(\mathbb{T}^d \times \; V\right)$ with spectral bound zero. Then there exist a positive rank-one projection P and suitable constants $C \geq 1$ and $a > 0$ such that

$$\|G_t - P\|_{\mathcal{L}(L^1(\mathbb{T}^d \times \; V))} \leq Ce^{-at} \text{ for each } t \geq 0.$$

Check, under the assumptions above, that

- the spectral bound of B is zero,
- $\mathcal{T}$ is irreducible,
- the geometrical condition implies that $\mathcal{T}$ is quasi-compact.

A result concerning postive semigroups

Theorem

Let $(G_t)_{t\geq 0}$ be a bounded, quasi-compact, irreducible, positive C_0-semigroup on $L^1\left(\mathbb{T}^d \times V\right)$ with spectral bound zero. Then there exist a positive rank-one projection P and suitable constants $C \geq 1$ and $a > 0$ such that

$$\|G_t - P\|_{\mathcal{L}(L^1(\mathbb{T}^d \times V))} \leq Ce^{-at} \text{ for each } t \geq 0.$$

Check, under the assumptions above, that

- the spectral bound of B is zero,
- $\mathcal{T}$ is irreducible,
- the geometrical condition implies that $\mathcal{T}$ is quasi-compact.

A result concerning postive semigroups

Theorem

Let $(G_t)_{t \geq 0}$ be a bounded, quasi-compact, irreducible, positive C_0-semigroup on $L^1\left(\mathbb{T}^d \times V\right)$ with spectral bound zero. Then there exist a positive rank-one projection P and suitable constants $C \geq 1$ and $a > 0$ such that

$$\|G_t - P\|_{\mathcal{L}(L^1(\mathbb{T}^d \times V))} \leq Ce^{-at} \text{ for each } t \geq 0.$$

Check, under the assumptions above, that

- the spectral bound of B is zero,
- $\mathcal{T}$ is irreducible,
- the geometrical condition implies that $\mathcal{T}$ is quasi-compact.

A result concerning postive semigroups

Theorem

Let $(G_t)_{t\geq 0}$ be a bounded, quasi-compact, irreducible, positive C_0-semigroup on $L^1\left(\mathbb{T}^d \times V\right)$ with spectral bound zero. Then there exist a positive rank-one projection P and suitable constants $C \geq 1$ and $a > 0$ such that

$$\|G_t - P\|_{\mathcal{L}(L^1(\mathbb{T}^d \times V))} \leq Ce^{-at} \text{ for each } t \geq 0.$$

Check, under the assumptions above, that

- the spectral bound of B is zero,
- $\mathcal{T}$ is irreducible,
- the geometrical condition implies that $\mathcal{T}$ is quasi-compact.

The spectral bound of $\mathcal{T}$

Proposition

Let B be the transport operator with domain $D(B)$ and let $\mathcal{T}$ be the semigroup generated by B. Then $s(\mathcal{T}) = s(B) = 0$.

$\mathcal{T}$ is a strongly continuous positive semigroup in $L^1(\mathbb{T}^d \times V) \Longrightarrow$ its spectral bound $s(\mathcal{T})$ is equal to its growth bound $\omega_0(\mathcal{T})$:

$$s(B) = \omega_0(\mathcal{T}) := \inf\left\{\omega \in \mathbb{R} \;\middle|\; \exists\, M \geq 1 : \|T_t\|_{\mathcal{L}(L^1(\mathbb{T}^d\times V))} \leq M e^{\omega t} \;\forall t \geq 0\right\}$$

$$\omega_0(\mathcal{T}) = \frac{1}{t}\ln r(T_t) \text{ for each } t > 0, \qquad r(T_t) = \sup\{|\lambda| \,:\, \lambda \in \sigma(T_t)\}$$

For each $t \geq 0$,

$$r(T_t) \leq \|T_t\|_{\mathcal{L}(L^1(\mathbb{T}^d\times V))} = 1 \textit{ and } T_t\left(\mathbb{1}_{\mathbb{T}^d\times V}\right) = \mathbb{1}_{\mathbb{T}^d\times V}$$

$$r(T_t) = 1 \text{ for each } t \geq 0$$

Irreducibility

Definition

Banach lattice (of type L^p): a real Banach space E endowed with an ordering $\geq$ compatible with the vector structure such that, if $f, g \in E$ and $|f| \geq |g|$, then $\|f\|_E \geq \|g\|_E$.

Example: the space $L^1\left(\mathbb{T}^d \times \; V\right)$, endowed with the standard L^1-norm, with the partial order defined by

$$f \geq 0 \text{ if and only if } f(x, v) \geq 0 \text{ a.e. on } \mathbb{T}^d \times V.$$

Let E be a Banach lattice. The space $\mathcal{L}(E)$ of bounded operators on E can be ordered in the following way: Let $A, B \in \mathcal{L}(E)$ then

$$0 \leq A \leq B \text{ if and only if, for each nonnegative } x \in E, \; 0 \leq Ax \leq Bx.$$

Order ideals

Definition

A closed vector subspace W of a Banach lattice E is called order ideal if, when $x \in W$ and $y \in E$, $|y| \leq |x|$ implies $y \in W$.
Notation: $\mathcal{I}(E)$ is the set of the order ideals of E.

Definition

Let G be a operator in a Banach lattice E and $\mathcal{G} \equiv (G_t)_{t \geq 0}$ be a semigroup.
An order ideal W is a G-invariant if $G(W) \subset W$.
Notation: $\mathcal{I}(G) := \{W \in \mathcal{I}(E) \mid G(W) \subset W\}$ is the set of G-invariants.
We denote

$$\mathcal{I}(\mathcal{G}) := \bigcap_{t \geq 0} \mathcal{I}(G_t)$$

and we say that an order ideal W is a $\mathcal{G}$-invariant if $W \in \mathcal{I}(\mathcal{G})$.

Order ideals

Definition

A closed vector subspace W of a Banach lattice E is called order ideal if, when $x \in W$ and $y \in E$, $|y| \leq |x|$ implies $y \in W$.
Notation: $\mathcal{I}(E)$ is the set of the order ideals of E.

Definition

Let G be a operator in a Banach lattice E and $\mathcal{G} \equiv (G_t)_{t \geq 0}$ be a semigroup.
An order ideal W is a G-invariant if $G(W) \subset W$.
Notation: $\mathcal{I}(G) := \{W \in \mathcal{I}(E) \mid G(W) \subset W\}$ is the set of G-invariants.
We denote

$$\mathcal{I}(\mathcal{G}) := \bigcap_{t \geq 0} \mathcal{I}(G_t)$$

and we say that an order ideal W is a $\mathcal{G}$-invariant if $W \in \mathcal{I}(\mathcal{G})$.

Order ideals

Definition

A closed vector subspace W of a Banach lattice E is called order ideal if, when $x \in W$ and $y \in E$, $|y| \le |x|$ implies $y \in W$.
Notation: $\mathcal{I}(E)$ is the set of the order ideals of E.

Definition

Let G be a operator in a Banach lattice E and $\mathcal{G} \equiv (G_t)_{t \ge 0}$ be a semigroup.
An order ideal W is a G-invariant if $G(W) \subset W$.
Notation: $\mathcal{I}(G) := \{W \in \mathcal{I}(E) \mid G(W) \subset W\}$ is the set of G-invariants.
We denote

$$\mathcal{I}(\mathcal{G}) := \bigcap_{t \ge 0} \mathcal{I}(G_t)$$

and we say that an order ideal W is a $\mathcal{G}$-invariant if $W \in \mathcal{I}(\mathcal{G})$.

Irreducibility of $\mathcal{T}$

Definition

An operator $G \in \mathcal{L}\left(L^1\left(\mathbb{T}^d \times V\right)\right)$ is said to be irreducible if and only if

$$\mathcal{I}(G) = \left\{\{0\}, L^1\left(\mathbb{T}^d \times V\right)\right\}.$$

Likewise, a semigroup $\mathcal{G}$ is irreducible if

$$\mathcal{I}(\mathcal{G}) = \left\{\{0\}, L^1\left(\mathbb{T}^d \times V\right)\right\}.$$

Proposition

The semigroup $\mathcal{T}$ generated by the transport operator B is irreducible in $L^1\left(\mathbb{T}^d \times V\right)$.

Irreducibility of $\mathcal{T}$

Definition

An operator $G \in \mathcal{L}\left(L^1\left(\mathbb{T}^d \times V\right)\right)$ is said to be irreducible if and only if

$$\mathcal{I}(G) = \left\{\{0\}, L^1\left(\mathbb{T}^d \times V\right)\right\}.$$

Likewise, a semigroup $\mathcal{G}$ is irreducible if

$$\mathcal{I}(\mathcal{G}) = \left\{\{0\}, L^1\left(\mathbb{T}^d \times V\right)\right\}.$$

Proposition

The semigroup $\mathcal{T}$ generated by the transport operator B is irreducible in $L^1\left(\mathbb{T}^d \times V\right)$.

Irreducibility of $\mathcal{T}$

Definition

An operator $G \in \mathcal{L}\left(L^1\left(\mathbb{T}^d \times V\right)\right)$ is said to be irreducible if and only if

$$\mathcal{I}(G) = \left\{\{0\}, L^1\left(\mathbb{T}^d \times V\right)\right\}.$$

Likewise, a semigroup $\mathcal{G}$ is irreducible if

$$\mathcal{I}(\mathcal{G}) = \left\{\{0\}, L^1\left(\mathbb{T}^d \times V\right)\right\}.$$

Proposition

The semigroup $\mathcal{T}$ generated by the transport operator B is irreducible in $L^1\left(\mathbb{T}^d \times V\right)$.

Quasi-compactness of $\mathcal{T}$ I

Definition

The essential resolvent of $A \in \mathcal{L}(E)$ is

$$\rho_{ess}(A) := \{\lambda \in \mathbb{C} \,|\, \lambda I - A \;\; \textit{is Fredholm}\},$$

and its essential spectrum is

$$\sigma_{ess}(A) := \mathbb{C} \setminus \rho_{ess}(A).$$

The essential radius of A is

$$r_{ess}(A) := \sup\{|\lambda| \mid \lambda \in \sigma_{ess}(A)\}.$$

Quasi-compactness of $\mathcal{T}$ II

Definition

A semigroup $\mathcal{G} \equiv (G_t)_{t \geq 0}$ is said to be quasi-compact on $L^1(\mathbb{T}^d \times V)$ if and only if there exist a compact operator C on $L^1(\mathbb{T}^d \times V)$ and a constant $t_0 > 0$ such that

$$\|G_{t_0} - C\|_{\mathcal{L}(L^1(\mathbb{T}^d \times V))} < 1.$$

Proposition

The semigroup $\mathcal{T}$ is quasi-compact on $L^1(\mathbb{T}^d \times V)$ if and only if

there exists $t_o > 0$ such that $r_{ess}(T_{t_o}) < 1.$

Quasi-compactness of $\mathcal{T}$ II

Definition

A semigroup $\mathcal{G} \equiv (G_t)_{t \geq 0}$ is said to be quasi-compact on $L^1\left(\mathbb{T}^d \times V\right)$ if and only if there exist a compact operator C on $L^1\left(\mathbb{T}^d \times V\right)$ and a constant $t_0 > 0$ such that

$$\|G_{t_0} - C\|_{\mathcal{L}(L^1(\mathbb{T}^d \times V))} < 1.$$

Proposition

The semigroup $\mathcal{T}$ is quasi-compact on $L^1\left(\mathbb{T}^d \times V\right)$ if and only if

there exists $t_o > 0$ such that $r_{ess}\left(T_{t_o}\right) < 1$.

Quasi-compactness of $\mathcal{T}$ II

Definition

A semigroup $\mathcal{G} \equiv (G_t)_{t \geq 0}$ is said to be quasi-compact on $L^1\left(\mathbb{T}^d \times V\right)$ if and only if there exist a compact operator C on $L^1\left(\mathbb{T}^d \times V\right)$ and a constant $t_0 > 0$ such that

$$\|G_{t_0} - C\|_{\mathcal{L}(L^1(\mathbb{T}^d \times V))} < 1.$$

Proposition

The semigroup $\mathcal{T}$ is quasi-compact on $L^1\left(\mathbb{T}^d \times V\right)$ if and only if

there exists $t_o > 0$ such that $r_{ess}\left(T_{t_o}\right) < 1$.

A control of the essential radius of $\mathcal{T}$

Define $\mathcal{S} \equiv (S_t)_{t \geq 0}$ by the formula

$$S_t g(x,v) := e^{-\int_0^t \sigma(x-vs)ds} g(x-vt,v) \text{ for all } g \in L^1\left(\mathbb{T}^d \times V\right).$$

The semigroup $\mathcal{T}$ can be seen as a perturbation of $\mathcal{S}$ by Duhamel's formula

$$T_t = S_t + \int_0^t S_s K_\sigma T_{t-s} ds. \tag{1}$$

Proposition

Under the assumptions above we have, for each $t > 0$,

$$r_{ess}\left(T_t\right) \leq r\left(S_t\right).$$

A control of the essential radius of $\mathcal{T}$

Define $\mathcal{S} \equiv (S_t)_{t\geq 0}$ by the formula

$$S_t g(x,v) := e^{-\int_0^t \sigma(x-vs)ds} g(x-vt,v) \quad \text{for all } g \in L^1\left(\mathbb{T}^d \times V\right).$$

The semigroup $\mathcal{T}$ can be seen as a perturbation of $\mathcal{S}$ by Duhamel's formula

$$T_t = S_t + \int_0^t S_s K_\sigma T_{t-s} ds. \tag{1}$$

Proposition

Under the assumptions above we have, for each $t > 0$,

$$r_{ess}\left(T_t\right) \leq r\left(S_t\right).$$

The asymptotic behaviour of the essential radius

In order to prove that $\mathcal{T}$ is quasi-compact on $L^1\left(\mathbb{T}^d \times V\right)$, it is enough to prove that for some $t_0 > 0$, $r\left(S_{t_0}\right) < 1$:

Proposition

If σ verifies the geometrical condition, then

$$\lim_{t \to +\infty} r\left(S_t\right) = 0.$$

The geometrical condition means that there exist T_0 and C such that

$$\int_0^{T_0} \sigma(x - sv) ds > C \text{ a.e. in } (x, v) \in \mathbb{T}^d \times \ V.$$

The asymptotic behaviour of the essential radius II

Since $\sigma \geq 0$ we have, for each $t > T_0$ ($\lfloor x \rfloor$: largest integer $\leq x$):

$$\int_0^t \sigma(x - sv)ds \geq \int_0^{\left\lfloor \frac{t}{T_0} \right\rfloor T_0} \sigma(x - sv)ds$$
$$\geq \sum_{n=0}^{\left\lfloor \frac{t}{T_0} \right\rfloor} \int_0^{T_0} \sigma\left((x - nT_0 v) - sv\right) ds \geq \left\lfloor \frac{t}{T_0} \right\rfloor C.$$

Hence

$$\|S_t\|_{\mathcal{L}(L^1(\mathbb{T}^d \times V))} \leq e^{-\left\lfloor \frac{t}{T_0} \right\rfloor C} \text{ for each } t \geq T_0.$$

Since $r(S_t) \leq \|S_t\|_{\mathcal{L}(L^1(\mathbb{T}^d \times V))}$ we deduce

$$r(S_t) \leq e^{-C\left\lfloor \frac{t}{T_0} \right\rfloor} \text{ for each } t \geq T_0 \Longrightarrow \lim_{t \to +\infty} r(S_t) = 0.$$

The characterization of P

Sketch of the proof:

- If σ verifies the geometrical condition, then $\lim_{t\to+\infty} r_{ess}\left(T_t\right) = 0$.
- The spectrum of B is discrete. In particular, $s(A)$ is a pole of the resolvent $R(A)$.
- B is the generator of an irreducible semigroup $\mathcal{T}$: the residue P associated to $s(A) = 0$ is a projection onto $\mathrm{Ker} B$, that is one-dimensional.
- By conservation of the mass, we have, for each $f \in L^1\left(\mathbb{T}^d \times V\right)$,

$$\int_{\mathbb{T}^d \times V} Pf(x,v)\, dxdv = \int_{\mathbb{T}^d \times V} f(x,v)\, dxdv.$$

- By convexity (i.e. Jensen's inequality),

$$Pf = \int_{\mathbb{T}^d \times V} f(x,v)\, dxdv.$$

The characterization of P

Sketch of the proof:

- If σ verifies the geometrical condition, then $\lim_{t\to+\infty} r_{ess}\left(T_t\right) = 0$.
- The spectrum of B is discrete. In particular, $s(A)$ is a pole of the resolvent $R(A)$.
- B is the generator of an irreducible semigroup $\mathcal{T}$: the residue P associated to $s(A) = 0$ is a projection onto $\mathrm{Ker}B$, that is one-dimensional.
- By conservation of the mass, we have, for each $f \in L^1\left(\mathbb{T}^d \times V\right)$,

$$\int_{\mathbb{T}^d \times V} Pf(x,v)\,dxdv = \int_{\mathbb{T}^d \times V} f(x,v)\,dxdv.$$

- By convexity (i.e. Jensen's inequality),

$$Pf = \int_{\mathbb{T}^d \times V} f(x,v)\,dxdv.$$

The characterization of P

Sketch of the proof:

- If σ verifies the geometrical condition, then $\lim_{t\to+\infty} r_{ess}\left(T_t\right) = 0$.
- The spectrum of B is discrete. In particular, $s(A)$ is a pole of the resolvent $R(A)$.
- B is the generator of an irreducible semigroup $\mathcal{T}$: the residue P associated to $s(A) = 0$ is a projection onto $\mathrm{Ker}B$, that is one-dimensional.
- By conservation of the mass, we have, for each $f \in L^1\left(\mathbb{T}^d \times V\right)$,

$$\int_{\mathbb{T}^d\times V} Pf(x,v)\,dxdv = \int_{\mathbb{T}^d\times V} f(x,v)\,dxdv.$$

- By convexity (i.e. Jensen's inequality),

$$Pf = \int_{\mathbb{T}^d\times V} f(x,v)\,dxdv.$$

The characterization of P

Sketch of the proof:

- If σ verifies the geometrical condition, then $\lim_{t\to+\infty} r_{ess}\left(T_t\right) = 0$.
- The spectrum of B is discrete. In particular, $s(A)$ is a pole of the resolvent $R(A)$.
- B is the generator of an irreducible semigroup $\mathcal{T}$: the residue P associated to $s(A) = 0$ is a projection onto $\mathrm{Ker}B$, that is one-dimensional.
- By conservation of the mass, we have, for each $f \in L^1\left(\mathbb{T}^d \times V\right)$,

$$\int_{\mathbb{T}^d\times V} Pf(x,v)\,dxdv = \int_{\mathbb{T}^d\times V} f(x,v)\,dxdv.$$

- By convexity (i.e. Jensen's inequality),

$$Pf = \int_{\mathbb{T}^d\times V} f(x,v)\,dxdv.$$

The characterization of P

Sketch of the proof:

- If σ verifies the geometrical condition, then $\lim_{t\to+\infty} r_{ess}(T_t) = 0$.
- The spectrum of B is discrete. In particular, $s(A)$ is a pole of the resolvent $R(A)$.
- B is the generator of an irreducible semigroup $\mathcal{T}$: the residue P associated to $s(A) = 0$ is a projection onto $\mathrm{Ker}B$, that is one-dimensional.
- By conservation of the mass, we have, for each $f \in L^1\left(\mathbb{T}^d \times V\right)$,

$$\int_{\mathbb{T}^d\times V} Pf(x,v)\,dxdv = \int_{\mathbb{T}^d\times V} f(x,v)\,dxdv.$$

- By convexity (i.e. Jensen's inequality),

$$Pf = \int_{\mathbb{T}^d\times V} f(x,v)\,dxdv.$$

The characterization of P

Sketch of the proof:

- If σ verifies the geometrical condition, then $\lim_{t\to+\infty} r_{ess}\left(T_t\right) = 0$.
- The spectrum of B is discrete. In particular, $s(A)$ is a pole of the resolvent $R(A)$.
- B is the generator of an irreducible semigroup $\mathcal{T}$: the residue P associated to $s(A) = 0$ is a projection onto $\mathrm{Ker}B$, that is one-dimensional.
- By conservation of the mass, we have, for each $f \in L^1\left(\mathbb{T}^d \times V\right)$,

$$\int_{\mathbb{T}^d\times V} Pf(x,v)\,dxdv = \int_{\mathbb{T}^d\times V} f(x,v)\,dxdv.$$

- By convexity (i.e. Jensen's inequality),

$$Pf = \int_{\mathbb{T}^d\times V} f(x,v)\,dxdv.$$

On the sharpness of the geometrical condition

The quasi-compactness of $\mathcal{T}$ in $L^1\left(\mathbb{T}^d \times V\right)$ implies the quasi-compactness of $\mathcal{S}$ in $L^1\left(\mathbb{T}^d \times V\right)$ as a consequence of:

Proposition (Caselles - 1987)

Let E be a Banach lattice. Let $S, T \in \mathcal{L}(E)$ be such that

$$0 \leq S \leq T.$$

If $r(T) \leq 1$ and $r_{\mathrm{ess}}(T) < 1$, then $r_{\mathrm{ess}}(S) < 1$.

Lemma

The semigroup $\mathcal{S}$ is quasi-compact on $L^1\left(\mathbb{T}^d \times V\right)$ if $\mathcal{T}$ is quasi-compact on $L^1\left(\mathbb{T}^d \times V\right)$.

On the sharpness of the geometrical condition

The quasi-compactness of $\mathcal{T}$ in $L^1\left(\mathbb{T}^d \times V\right)$ implies the quasi-compactness of $\mathcal{S}$ in $L^1\left(\mathbb{T}^d \times V\right)$ as a consequence of:

Proposition (Caselles - 1987)

Let E be a Banach lattice. Let $S, T \in \mathcal{L}(E)$ be such that

$$0 \leq S \leq T.$$

If $r(T) \leq 1$ and $r_{\text{ess}}(T) < 1$, then $r_{\text{ess}}(S) < 1$.

Lemma

The semigroup $\mathcal{S}$ is quasi-compact on $L^1\left(\mathbb{T}^d \times V\right)$ if $\mathcal{T}$ is quasi-compact on $L^1\left(\mathbb{T}^d \times V\right)$.

The quasi-compactness of $\mathcal{T}$ and $\mathcal{S}$

By Duhamel's Formula:

$$T_t = S_t + \int_0^t S_s K_\sigma T_{t-s} ds, \text{ for all } t \geq 0.$$

$\mathcal{T}$ and $\mathcal{S}$ are positive semigroups and K_σ is a positive operator, $\Longrightarrow$

$$\int_0^t S_s K_\sigma T_{t-s} ds \geq 0 \text{ for each } t \geq 0.$$

The equality above implies that $T_t \geq S_t$ for each $t \geq 0$. Besides,

$$r\left(T_t\right) = 1 \text{ for each } t \geq 0.$$

Since $\mathcal{T}$ is quasi-compact on $L^1\left(\mathbb{T}^d \times V\right)$, there exists t_0 such that

$$r_{\mathrm{ess}}\left(T_{t_0}\right) < 1.$$

Hence Caselles' Theorem implies that

$$r_{\mathrm{ess}}\left(S_{t_0}\right) < 1.$$

The geometrical condition

Assume that $\|S_t\|_{\mathcal{L}\left(L^1\left(\mathbb{T}^d \times V\right)\right)} \to 0$ as $t \to +\infty$.
$\mathcal{S} \equiv (S_t)_{t \geq 0}$ is defined by the formula

$$S_t g(x,v) := e^{-\int_0^t \sigma(x-vs)ds} g(x-vt, v) \quad \text{for all } g \in L^1\left(\mathbb{T}^d \times V\right).$$

This implies that there exist T_0 and C such that

$$\int_0^{T_0} \sigma(x-sv)ds > C \text{ a.e. in } (x,v) \in \mathbb{T}^d \times V.$$

Table of contents

1. The problem
2. Convergence to equilibrium for non degenerate cross sections
3. Convergence to equilibrium for degenerate cross sections
   - Degeneracy in isolated points
   - A counterexample
   - The geometrical condition
4. References

Bibliography

Laurent Desvillettes, FS, *Asymptotic behavior of degenerate linear transport equations*, Bull. Sci. Mat. 133, 8 (2009) 848-858

Étienne Bernard, FS, *On the convergence to equilibrium for degenerate transport problems*, Arch. Rational Mech. Anal., in press (2013)

Étienne Bernard, FS, *On the exponential decay to equilibrium of the degenerate linear Boltzmann equation*, preprint (2012)

Florian De Vuyst, FS, *Numerical investigation of the long-time behaviour for degenerate neutron transport*, article in preparation (2013)