Convergence of a large time-step scheme for Mean Curvature Motion

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Outline

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Level Set Method

The Level Set method has had a great success for the analysis of front propagation problems for its capability to handle many different physical phenomena within the same theoretical framework.
One can use it for isotropic and anisotropic front propagation, for merging different fronts, for Mean Curvature Motion (MCM) and other situations when the velocity depends on some geometrical properties of the front.
It allows to develop the analysis also after the on set of singularities.

Level Set Method

Our unknown is the "representation" function $u : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ and looking at the 0-level set of u we can back to the front, i.e.

$$\Gamma_t \equiv \{x : u(x, t) = 0\}$$

The model equation corresponding to the LS method is

$$\begin{cases} u_t + c(x)|\nabla u(x)| = 0 & x \in \mathbb{R}^n \times [0, T] \\ u(x) = u_0(x) & x \in \mathbb{R}^n \end{cases}$$

where u_0 must be a representation function for the front (i.e.

$$\begin{cases} u_0 > 0, & x \in \mathbb{R}^n \setminus \Omega_0 \\ u(x) = 0 & x \in \mathbb{R}^n \\ u(x) < 0 & x \in \Omega_0 \end{cases} \quad (1)$$

The front $\Gamma_0 = \partial\Omega_0$.

Front propagation and minimum time problem

Let the velocity of the front in the normal direction is a given $c : \mathbb{R}^n \to \mathbb{R}$

Minimum time problem

Target set $=\Omega_0$

Dynamics

$$\begin{cases} \dot{y}(t) = -c(y)a, \quad a \in B(0,1) \\ y(0) = x \end{cases} \tag{2}$$

Minimum time function

$$T(x) \equiv \inf\{t \in \mathbb{R}_+ : y_x(t; a(t)) \in \Omega_0\}$$

Monotone evolution

$T(\cdot)$ is the unique viscosity solution of

$$\max_{a \in B(0,1)} \{c(y) a \cdot \nabla T(x)\} = 1 \quad \in \mathbb{R}^n \setminus \Omega_0$$

with the Dirichlet condition

$$T(x) = 0 \text{ on } \partial\Omega_0$$

If c does not change sign the evolution is monotone (increasing or decreasing) and we have the following link

$$u(x,t) = T(x) - t$$

So we can solve the stationary problem and get any front Γ_t, for $t > 0$.

More General Models

In the standard model the normal velocity $c : \mathbb{R}^n \to \mathbb{R}$ is given, but the same approach applies to other scalar velocities

$$\begin{cases} c(x,t) & \text{isotropic growth with time varying velocity} \\ c(x,\eta) & \text{anisotropic growth, cristal growth} \\ c(x,k(x)) & \text{Mean Curvature Motion} \\ c(x) & \text{obtained by convolution (dislocation dynamics)} \end{cases}$$

Outline

SL schemes for MCM

$$\begin{cases} u_t(x,t) & = div\left(\frac{Du(x,t)}{|Du(x,t)|}\right)|Du(x,t)| \\ u(x,0) & = u_0(x) \end{cases}$$

Representation formula (Soner–Touzi):

$$u(x,t) = E\{u_0(y(x,t,t))\}, \quad Du \neq 0$$

$$\begin{cases} dy(x,t,s) = \sqrt{2}P(y,t,s)dW(s) \\ y(x,t,0) = x \end{cases}$$

$$P(y,t,s) = \frac{1}{|Du|^2}\begin{pmatrix} u_{x_2}^2 & -u_{x_1}u_{x_2} \\ -u_{x_1}u_{x_2} & u_{x_1}^2 \end{pmatrix}$$

Construction of the scheme

Soner–Touzi formula between t and $t + \Delta t$:

$$u(x, t + \Delta t) = E\{u(y(x, t + \Delta t, \Delta t), t)\}$$

Brownian dimension reduction

$$\sqrt{2} P dW = \frac{\sqrt{2}}{|Du|} \begin{pmatrix} u_{x_2} \\ -u_{x_1} \end{pmatrix} \left(\frac{u_{x_2} dW_1}{|Du|} - \frac{u_{x_1} dW_2}{|Du|} \right) = \sigma d\widehat{W}$$

where

$$\sigma(x, t) = \frac{\sqrt{2}}{|Du|} \begin{pmatrix} u_{x_2} \\ -u_{x_1} \end{pmatrix}$$

we can replace the stochastic Cauchy problem by

$$\begin{cases} dy(x,t,s) = \sigma(y,t,s)\,d\widehat{W}(s) \\ y(x,t,0) = x. \end{cases}$$

i.e. with a 1-dimensional brownian motion in the direction tangent to the curve.

Euler scheme for SDE

$$\begin{cases} y_{k+1} = y_k + \sqrt{2}\sigma(y, t_k, 0)\Delta \widehat{W}_k \\ y_0 = x. \end{cases}$$

with

$$P(\Delta \widehat{W}_k = \pm\sqrt{\Delta t}) = \frac{1}{2}.$$

Time-discretization

$$\begin{aligned} u_{\Delta t}(x, t_{n+1}) = \quad & \tfrac{1}{2} u_{\Delta t}(x + \sqrt{2}\sigma(x, t_n, 0)\sqrt{\Delta t}, t_n) + \\ & + \tfrac{1}{2} u_{\Delta t}(x - \sqrt{2}\sigma(x, t_n, 0)\sqrt{\Delta t}, t_n). \end{aligned}$$

Fully discrete scheme $Du \neq 0$

$$u_j^{n+1} = \frac{1}{2}\left(I[u^n](x_j + \sigma_j^n \sqrt{\Delta t}) + I[u^n](x_j - \sigma_j^n \sqrt{\Delta t}) \right)$$

Some references

MCM via viscosity solution techniques

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Crandall-Lions, Numer. Math., 1996

Barles-Georgelin, SIAM J. Num. Anal., 1995

Catté-Dibos-Koepfler, SIAM J. Num. Anal., 1995

F. Ferretti, ENUMATH 2001

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Outline

Modified scheme with threshold $C\Delta x^s$

$$\begin{cases} u_j^{n+1} = \frac{1}{2}\left[I[u^n](x_j + \sigma_j^n\sqrt{\Delta t}) + I[u^n](x_j - \sigma_j^n\sqrt{\Delta t})\right] & \text{if } |D_j^n| > C\Delta x^s \\ u_j^{n+1} = \frac{1}{4}\sum_{i\in\mathcal{D}(j)} u_i^n & \text{if } |D_j^n| \le C\Delta x^s \end{cases}$$

Consistency error (case $|D_j^n| > C\Delta x^s$)

$$\tau_{\Delta x,\Delta t} = O\left(\frac{\Delta x^r}{\Delta t}\right) + O\left(\frac{\Delta x^{q-s}}{\Delta t^{\frac{1}{2}}}\right) + O(\Delta t^{\frac{1}{2}}) + O(\Delta t)$$

Consistency for small gradients

Let us consider the case $|D_j^n| \leq C\Delta x^s$.
Case a: (x_j, t_n) *is such that* $Du(x_j, t_n) = 0$.
This is a stardad computation based on the consistency with the heat equation.
Case b : (x_j, t_n) *is such that* $|Du(x_j, t_n)| \neq 0$ *and* $|D_j[w]| \leq C\Delta x^s$.
By the lower semicontinuity of $\underline{F}$ we have that for any $\varepsilon_1 > 0$ there exists a $\delta_1(\varepsilon_1)$ such that

$$\underline{F}(Du, D^2u)(y, s) \geq \underline{F}(Du, D^2u)(x, t) - \varepsilon_1 \text{ for any } (y, s) \in B_{\delta_1}(x, t). \tag{3}$$

Consistency for small gradients

The upper semicontinuity of $\overline{F}$ implies that for any $\varepsilon_2 > 0$ there exists a $\delta_2(\varepsilon_2)$ such that

$$\overline{F}(Du, D^2u)(y,s) \le \overline{F}(Du, D^2u)(x,t) + \varepsilon_2 \text{ for any } (y,s) \in B_{\delta_2}(x,t). \tag{4}$$

The interesting case is when for $\overline{\varepsilon} \equiv \max(\varepsilon_1, \varepsilon_2)$ there exists

$$(y,s) \in B_{\delta_1}(x,t) \cap B_{\delta_2}(x,t)$$

such that $Du(y,s) = 0$. For $(x,t) = (x_j, t_n)$, we can apply both (3) and (4) getting

$$\underline{F}(Du, D^2u)(x_j, t_n) \le \overline{F}(Du, D^2u)(x_j, t_n) + 2\overline{\varepsilon} \tag{5}$$

Consistency for small gradients

In fact, we have

$$\underline{F}(Du, D^2u)(x_j, t_n) \leq -2|D^2u(y,s)| + \overline{\varepsilon} \leq \overline{\varepsilon} =$$
$$= \liminf_{\Delta t \to 0} \frac{u(x_j,t_n) - H(w;j)}{\Delta t} + \overline{\varepsilon} = \limsup_{\Delta t \to 0} \frac{u(x_j,t_n) - H(w;j)}{\Delta t} + \overline{\varepsilon} \leq$$
$$\leq 2|D^2u(y,s)| + \overline{\varepsilon} \leq \overline{F}(Du, D^2u)(x_j, t_n) + 2\overline{\varepsilon}$$

and, since ε_1 and ε_2 are arbitrary, we obtain the result.

Monotonicity

The standard scheme presented at the beginning is not monotone even local monotone interpolation operators (linear, bilinear).
We modify it in order to satisfy a relaxed monotonicity property.

$$\begin{cases} \dfrac{u_j^{n+1} - u_j^n}{\Delta t} = \dfrac{1}{\rho^2}\left(\dfrac{1}{2} I[u^n](x_j + \sigma_j^n \rho) + \quad \frac{1}{2} I[u^n](x_j - \sigma_j^n \rho) - u_j^n\right) \\ u_j^{n+1} = \dfrac{1}{4} \sum_{i \in \mathcal{D}(j)} u_i^n \qquad \text{if } |D_j^n| \le C \Delta x^s. \end{cases}$$

In compact form we write

$$u_j^{n+1} = H_\rho(u^n; j).$$

Monotonicity

Since for this scheme H_ρ we have

$$\frac{\partial H_\rho(u^n;j)}{\partial u_i^n} \geq -\frac{4\Delta t L_{I[u^n],\mathcal{G}}}{C\rho\Delta x^{s+1}}$$

we need to compensate the negative bound and we add a small viscosity

$$\begin{cases} \mathcal{H}_\rho(u^n;j) = H_\rho(u^n;j) + \Delta t \frac{W\Delta x}{\rho\Delta x^s}\frac{\sum_{i\in\mathcal{D}(j)} u_i^n - 4u_j^n}{\Delta x^2}, & \text{if } |D_j^n| > C\Delta x^s \\ u_j^{n+1} = \frac{1}{4}\sum_{i\in\mathcal{D}(j)} u_i^n & \text{if } |D_j^n| \leq C\Delta x^s. \end{cases}$$

where W is a positive constant.

Monotonicity

Differentiating $\mathcal{H}_\rho$ we get:

$$\begin{cases} \dfrac{\partial \mathcal{H}_\rho(u^n;j)}{\partial u_j^n} \geq 1 - \dfrac{\Delta t}{\rho^2} - \dfrac{4W\Delta t}{\rho \Delta x^{1+s}} \\ \dfrac{\partial \mathcal{H}_\rho(u^n;j)}{\partial u_i^n} = \psi_i(x_j + \sigma_j^n \rho) \geq 0 & \text{for } i \in \mathcal{S}(j) \setminus (\mathcal{D}(j) \cup \{j\}) \\ \dfrac{\partial \mathcal{H}_\rho(u^n;j)}{\partial u_i^n} \geq -\dfrac{4\Delta t L_{I[u^n],\mathcal{G}}}{C\rho \Delta x^{s+1}} + \dfrac{W\Delta t}{\rho \Delta x^{1+s}} & \text{for } i \in \mathcal{D}(j). \end{cases} \tag{6}$$

Then $\mathcal{H}_\rho(u^n;j)$ is monotone if

$$\begin{cases} 1 - \dfrac{\Delta t}{\rho^2} - \dfrac{4W\Delta t}{\rho \Delta x^{1+s}} \geq 0 \\ \dfrac{4L_{I[u^n],\mathcal{G}}}{C} < W. \end{cases} \tag{7}$$

Weak monotonicity property

Theorem Under the above assumption, the scheme $\mathcal{H}_\rho$ satisfies

$$\mathcal{H}_\rho \leq \widetilde{H}_\rho(\eta; j) + o(\Delta t).$$

Outline

Let us consider now a general scheme that is supposed to approximate our HJ equation. Its abstract form on a lattice will be given by

$$\begin{cases} u_j^{n+1} = S^{\mathcal{D}t}(u^n; j) & \text{for } j \in \mathcal{Z}^2 \quad n = 0, ..., N-1 \\ u_j^0 = u_0(x_j) & \text{for } j \in \mathcal{Z}^2 \end{cases} \qquad (HJ^{\mathcal{D}t})$$

where $S^{\mathcal{D}t} : B(\mathcal{G}_{\Delta x}) \to \mathbb{R}$ and $B(D)$ is the space of the bounded functions defined on D.

The scheme has to satisfy the following conditions:

- *A1 - Invariance with respect to the addition of constants* For any $k \in \mathbb{R}$ and $j \in \mathcal{Z}^2$,

$$S^{\mathcal{D}t}(v+k;j) = S^{\mathcal{D}t}(v;j) + k \qquad \text{(A1)}$$

- *A2 - Weak consistency*
  Let us define

$$\underline{F}(D\phi, D^2\phi)(x,t) = \liminf_{(y,s)\to(x,t)} F(D\phi, D^2\phi)(y,s),$$

$$\overline{F}(D\phi, D^2\phi)(x,t) = \limsup_{(y,s)\to(x,t)} F(D\phi, D^2\phi)(y,s).$$

Consistency

The consistency assumption (A2) requires

$$
\begin{aligned}
\phi_t(x,t) + \underline{F}(D\phi, D^2\phi)(x,t) &\le \liminf_{\substack{(x_j,t_n)\to(x,t)\\ \Delta t\to 0}} \frac{\phi(x_j,t_n) - S^{\mathcal{D}t}(\phi^{n-1};j)}{\Delta t} \\
&\le \limsup_{\substack{(x_j,t_n)\to(x,t)\\ \Delta t\to 0}} \frac{\phi(x_j,t_n) - S^{\mathcal{D}t}(\phi^{n-1};j)}{\Delta t} \\
&\le \phi_t(x,t) + \overline{F}(D\phi, D^2\phi)(x,t)
\end{aligned}
$$

where $\phi \in C^\infty(\mathbb{R}^2 \times (0,T])$ and $\phi^{n-1} = (\phi(x_j, t_{n-1}))_{x_j \in \mathcal{G}_{\Delta x}}$.
Note that $\underline{F}(D\phi, D^2\phi)$, $\overline{F}(D\phi, D^2\phi)(x,t)$ are respectively lower and upper semicontinuous extension of F.

Note that if F is continuous, then the lim inf and the lim sup must coincide, and the definition reduces to the usual definition of consistency.

- *A3 - Generalized monotonicity*

$$v_j \leq \phi_j^{n-1} \quad \text{for} \quad j \in \mathcal{Z}^2 \qquad \text{implies} \qquad S^{\mathcal{D}t}(v;j) \leq \widetilde{S}^{\mathcal{D}t}(\phi^{n-1};j) + o(\mathcal{D}t)$$

where $v \in B(\mathcal{G}_{\Delta x})$ and $\widetilde{S}_{\mathcal{D}t}$ is a (possibly different) scheme weakly consistent.

Then, consider $u^n = (u_j^n)_{j\in\mathcal{Z}^2}$ with u_j^n solution of $(HJ^{\mathcal{D}t})$ and its piecewise constant (in time) interpolation $u^{\mathcal{D}t}$ defined as:

$$u^{\mathcal{D}t}(x,t) = \begin{cases} I[u^n](x) & \text{if } t \in [t_n, t_{n+1}), \\ u_0(x) & \text{if } t \in [0, \Delta t), \end{cases}$$

where $I[\cdot] : B(\mathcal{G}_{\Delta x}) \to \mathbb{R}$ is a general interpolation operator

$$I[u^n](x) \equiv \sum_{I\in\mathcal{I}(x)} \psi_I(x) u_I^n \qquad (8)$$

where $\psi_I(x)$ are basis functions in $\mathbb{R}^2$ and $\mathcal{I}(x)$ is the set of indices corresponding to the vertices of the cell containing x.

Let us note that we can always couple the choice of $\mathcal{D}t$ and $\mathcal{D}x$ according to $\mathcal{D}t = C\mathcal{D}x^{\gamma}$, with C a positive constant, in order to deal with a unique discretization parameter. We assume $\gamma \geq 1$ (this parameter has to be tuned to guarantee monotonicity).
The interpolation operator $I[\cdot]$ has to verify a relaxed monotonicity property:

$$\text{if } v_j \leq \eta_j \text{ for any } j \in \mathcal{I}(x) \quad \text{then} \quad I[v](x) \leq I[\eta](x) + o(\mathcal{D}t) \quad \text{(A4)}$$

with $v \in B(\mathcal{G}_{\Delta x})$ and $\eta = (f(x_j))_{x_j \in \mathcal{G}_{\Delta x}}$,where $f(x)$ is a smooth function.

Moreover $I[\cdot]$ satisfies

$$|I[\eta](x) - f(x)| = o(\mathcal{D}t) \text{ for any } x \in \mathbb{R}^2. \tag{A5}$$

Theorem

Assume (A1)–(A5) and let $u(x,t)$ be the unique viscosity solution of (HJ). Then $u^{\mathcal{D}t}(x,t) \to u(x,t)$ locally uniformly on $\mathbb{R}^2 \times [0,T]$ as $\mathcal{D}t \to 0$.

Outline

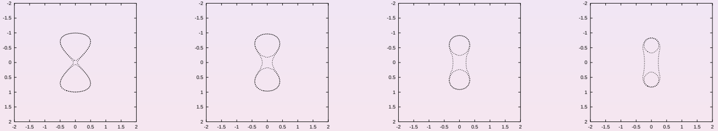
TEST: a shrinking circle

$$\Delta t = O(\Delta x^{\frac{3}{4}}),\ s = \frac{1}{4}.$$

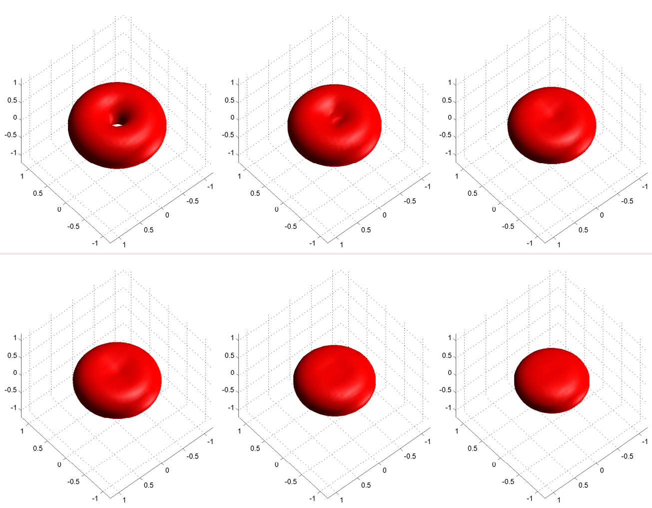
Δx	Δt	$\| \cdot \|_\infty$	$\| \cdot \|_1$	$L^\infty - order$	$L^1 - order$
0.04	0.08	$3.04 \cdot 10^{-4}$	$6.50 \cdot 10^{-6}$		
0.02	0.053	$1.25 \cdot 10^{-4}$	$3.42 \cdot 10^{-6}$	1.2	0.9
0.01	0.032	$5.22 \cdot 10^{-5}$	$1.82 \cdot 10^{-6}$	1.2	1.6
0.005	0.02	$2.09 \cdot 10^{-5}$	$7.75 \cdot 10^{-7}$	1.3	1.2

Errors for the SL scheme

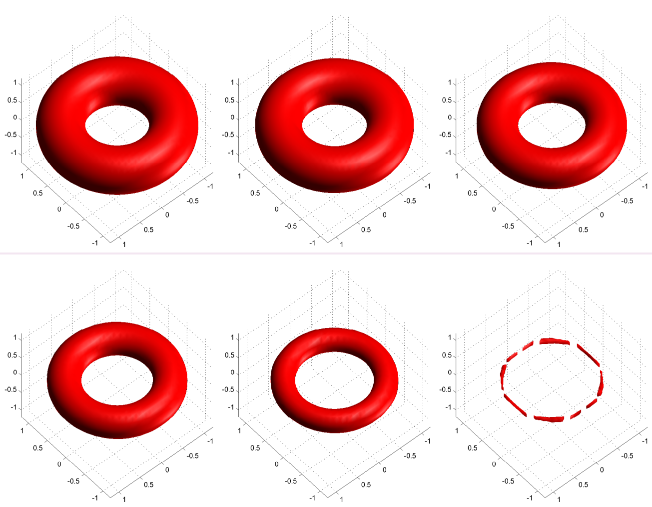
TEST: Development of non empty interior



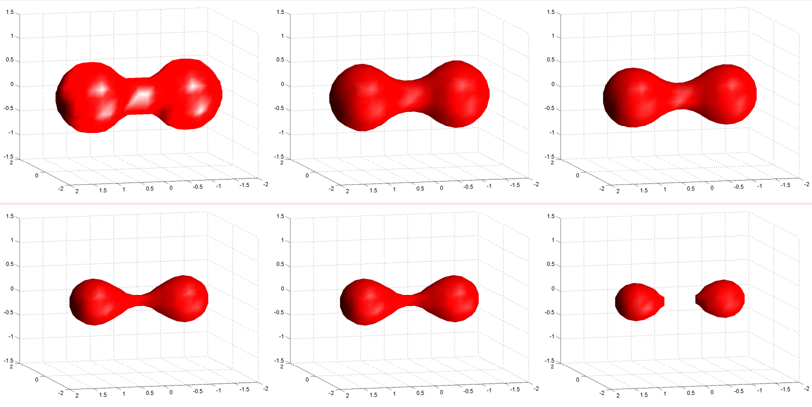
Fattening: evolution of the level curves $u = 0.095, 1, 1.05$



The torus evolving into a sphere, $R \ll r$



The torus collapsing in a circle, $R > r$



Dumb-bell: topology change in $\mathbb{R}^3$

Crandall-Lions scheme

In the scheme proposed by such problems are avoided by replacing the matrix Θ in the equation of MCM by the following one:

$$\Theta_\epsilon(Du) = I - \frac{Du \bigotimes Du}{|Du|^2 + \epsilon},$$

with $\epsilon > 0$.

Crandall-Lions scheme cntd

Denoting by e_j ($j = 1, 2$) the canonical base of $\mathbb{R}^2$, we can write the CL scheme as:

$$u_j^{n+1} = H_{CL}(u^n; j)$$

where

$$\begin{aligned} H_{CL}(u^n; j) \quad = \quad & u_j^n + \frac{\Delta t}{\rho^2} \left(I[u^n](x_j + \rho\Theta_\epsilon(D_j[u^n])e_1) + \right. \\ & + I[u^n](x_j + \rho\Theta_\epsilon(D_j[u^n])e_2) + \\ & + I[u^n](x_j - \rho\Theta_\epsilon(D_j[u^n])e_1) + \\ & \left. + \; I[u^n](x_j - \rho\Theta_\epsilon(D_j[u^n])e_2) - u_j^n \right) \end{aligned}$$

The scheme for which convergence is proved is

$$u_j^{n+1} = H_{CL}(u^n; j) + \frac{\Delta t K}{\rho \Delta x} \left(\sum_{i \in \mathcal{D}(j)} u_i^n - 4 u_j^n \right).$$

Kohn-Serfaty scheme

It has been proved that we can write

$$\mathrm{div}\left(\frac{Du(x,t)}{|Du(x,t)|}\right)|Du(x,t)| = \min_{a\in S^1, a\cdot Du=0}\left\{a^T D^2u(x,t)a\right\}, \quad (9)$$

By penalization we can write

$$\min_{a\in S^1} \max\left\{a^T D^2u(x,t)a - \frac{1}{\varepsilon}a\cdot Du, a^T D^2u(x,t)a + \frac{1}{\varepsilon}a\cdot Du\right\}$$

Kohn-Serfaty scheme

We write the scheme as $u_j^{n+1} = H_{KS}(u^n; j)$,
where

$$H_{KS}(w; j) = \min_{a \in S^1} \max \left\{ I[w]\left(x_j + \sqrt{2\Delta t}\, a\right), I[w]\left(x_j - \sqrt{2\Delta t}\, a\right) \right\}$$

The average of the (SL) scheme is replaced by a min-max operator. This scheme is more expensive and difficult to extend to higher dimension.

Test : Evolution of a circle (shrinking)

Errors for the SL scheme

Δx	Δt	$\|\cdot\|_\infty$	$\|\cdot\|_1$	$order_\infty$	$order_1$	CPU
0.08	0.16	$2.96 \cdot 10^{-3}$	$4.76 \cdot 10^{-5}$			$0.15s$
0.04	0.08	$8.45 \cdot 10^{-4}$	$1.88 \cdot 10^{-5}$	1.80	1.34	0.61
0.02	0.04	$3.13 \cdot 10^{-4}$	$8.19 \cdot 10^{-6}$	1.43	1.21	$2.5s$
0.01	0.02	$1.14 \cdot 10^{-4}$	$3.84 \cdot 10^{-6}$	1.45	1.09	$12.38s$

Test : Evolution of a circle (shrinking)

Errors for the min − max *scheme*

Δx	Δt	$\|\cdot\|_\infty$	$\|\cdot\|_1$	$order_\infty$	$order_1$	CPU
0.08	0.16	$2.95 \cdot 10^{-3}$	$4.51 \cdot 10^{-5}$			$0.8s$
0.04	0.08	$1.81 \cdot 10^{-3}$	$1.88 \cdot 10^{-5}$	0.7	1.21	$6s$
0.02	0.04	$9.04 \cdot 10^{-4}$	$7.76 \cdot 10^{-6}$	1.0	1.27	$1m29s$
0.01	0.02	$2.54 \cdot 10^{-4}$	$3.45 \cdot 10^{-6}$	1.83	1.16	$26m11s$

Outline

Cuves evolving in the space

Let us consider the evolution of a curve $\mathcal{C}$ in $\mathbb{R}^3$.
There are two ways to handle the problem:

- The curve is described as the intersection of two surfaces This leads to a system of HJ equation (Osher at alia)
- The curve is replaced by an ε-tube centered at the curve $\mathcal{C}$. We study the evolution of the surface and get back to the curve in the limit for $\varepsilon \longrightarrow 0$.

Following the second characterization, the (SL) scheme has been extended to codimension-2 problems.

Cuves evolving in the space

$$\begin{cases} u_t = F(D^2 u, Du) & \mathbb{R}^3 \times [0,\infty) \\ u(x,0) = \frac{1}{2} d(x,\mathcal{C})^2 \end{cases}$$

where

$$F(A,p) = \inf_{\nu \in \mathcal{N}(p)} \{\mathrm{trace}[AP^\nu]\}$$

$$\text{and } \mathcal{N}(p) = \{\nu \in S^2 : P^\nu p = 0\}, \text{ and } P^\nu = \nu \nu^T$$

Soner–Touzi formula

$$u(x,t) = \inf_{\nu \in \mathcal{U}} \{E\{u(y_\nu(x,t,t),0)\}\}$$

with $\mathcal{U} = \{\nu : [0,T] \to S^2 \subset \mathbb{R}^3 : \nu \cdot Du(x,t) = 0\}$

MCM in codimension 2

The generalized characteristics curves solve

$$\begin{cases} dy_\nu(x,t,s) = \sqrt{2}\nu(s)d\widehat{W}(s) \\ y_\nu(x,t,0) = x \end{cases}$$

L.Ambrosio, M.Soner, *Level Set Approach to Mean Curvature Flow in Arbitrary Codimension*, J.Differential Geometry, **43** (1996), 693-737.

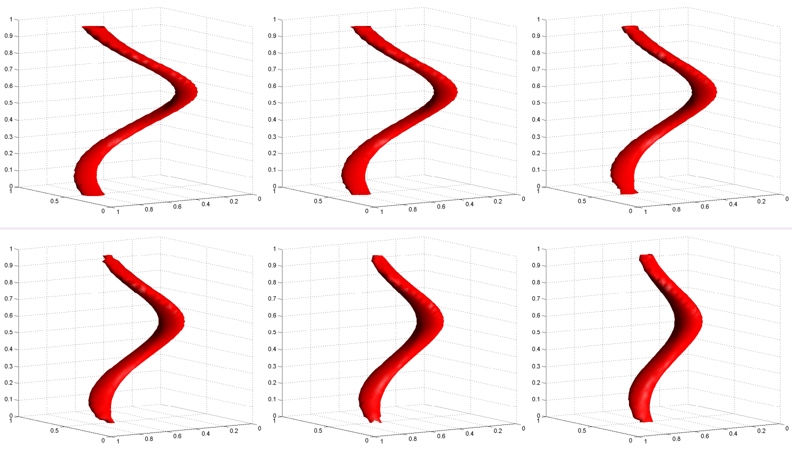
MCM cod 2: Numerical approximation

Time-discrete scheme

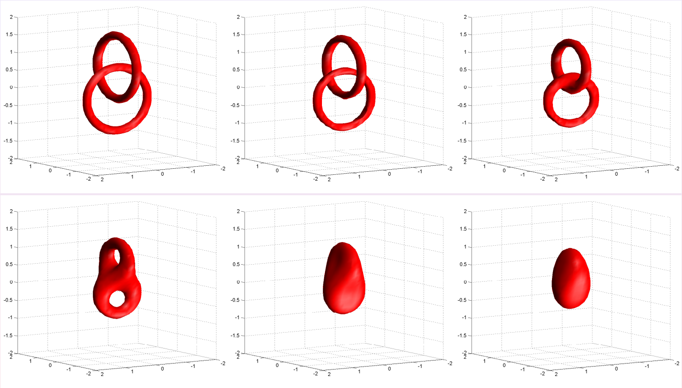
$$u_{\Delta t}(x, t_{n+1}) = \frac{1}{2} \inf_{\nu^n \in \widehat{\mathcal{U}}} \left\{ u_{\Delta t}(x + \sqrt{2\Delta t}\nu^n, t_n) + u_{\Delta t}(x - \sqrt{2\Delta t}\nu^n, t_n) \right\}$$

Fully-discrete scheme

$$u_j^n = \min_{\nu^n \in \mathbb{R}^3} \{ \tfrac{1}{2} I[u^n](x + \sqrt{2\Delta t}\nu^n) + \tfrac{1}{2} I[u^n](x - \sqrt{2\Delta t}\nu^n) + + \frac{(|D_j^n \nu^n|)^2}{\epsilon_1} + \frac{(|\nu^n| - 1)^2}{\epsilon_2} \}.$$



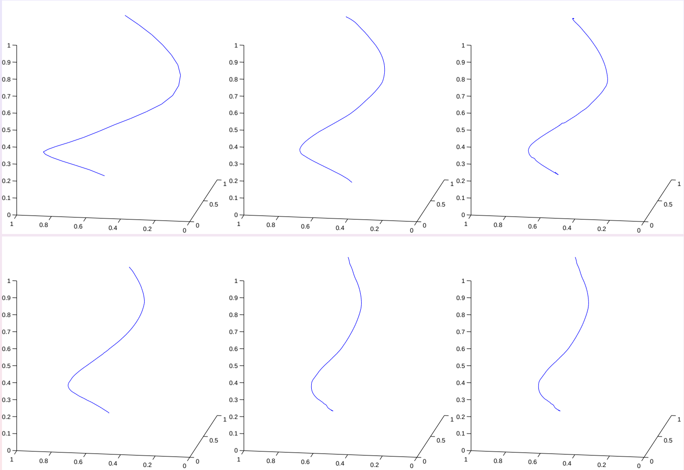
Evolution of ϵ helical surface, $\epsilon = 0.008$.



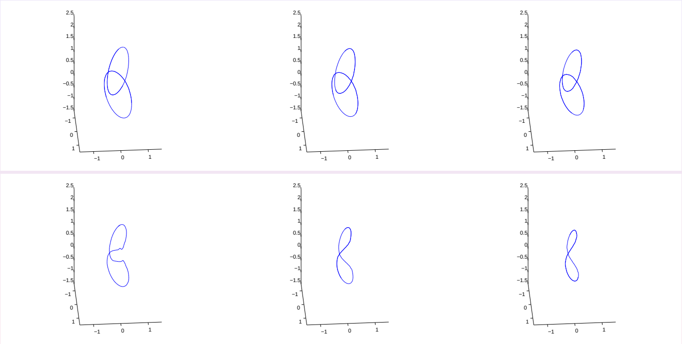
Evolution of two linked circles in $\mathbb{R}^3$ (ϵ-tube).

Optimal Trajectory Algorithm

- $s_0 := x_{jmin}$ point of the curve
- $\nu_j^{n*} = \underset{\nu_j^n \in \hat{\mathcal{U}}}{\operatorname{argmin}} \{u^n(s_j + \sqrt{2\Delta s}\nu_j^n)\} \quad j = 0, \ldots, \hat{j}$
- $s_{j+1} = s_j + \sqrt{2\Delta s}\nu_j^{n*} \quad j = 0, \ldots, \hat{j}$
- $|s_0 - s_{\hat{j}}| \leq \epsilon$



Evolution of a helix in $\mathbb{R}^3$.



Evolution of two linked circles in $\mathbb{R}^3$.