

# On the collision of two solitons for the generalized KdV equation in the nonintegrable case

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# Introduction

$$(gKdV) \quad \partial_t u + \partial_x(\partial_x^2 u + f(u)) = 0 \quad t, x \in \mathbb{R}$$

We call **soliton** a solution  $R(t, x) = Q_c(x - ct)$ ,  $c > 0$

General questions about the collision of two solitons

Let  $u(t)$  be a solution such that

$$u(t) \sim Q_{c_1}(x - c_1 t) + Q_{c_2}(x - c_2 t) \quad \text{as } t \rightarrow -\infty,$$

where  $Q_{c_1}(x - c_1 t)$ ,  $Q_{c_2}(x - c_2 t)$  are two solitons ( $0 < c_2 < c_1$ )

- ▶ What is the behavior of  $u(t)$  during and after the collision ?
- ▶ Do the two solitons survive the collision at the main orders ?
- ▶ If yes, are their speeds (size) and trajectories (shift) modified ?
- ▶ Is the collision elastic or inelastic ?

## Previous results concerning the collision of solitons

- ▶ Integrable case ( $f(u) = u^2$  or  $u^3$ )

There exist explicit multi-solitons describing the interaction of solitons and explicit formulas for the shifts on the solitons.

### **The collision is elastic**

[Fermi, Pasta and Ulam], [Zabusky and Kruskal] [Lax],  
[Hirota], [Miura et al.], etc.

- ▶ Numerical results for nonintegrable models and experiments

### **The collision seems inelastic but almost elastic**

(small dispersive trail)

[Eilbeck and McGuire], [Bona et al.] (BBM), [Shih] (gKdV),  
[Craig et al.] and references therein (Euler, KdV, experiments)  
etc.

No rigorous description of collision for nonintegrable models.

We report on a recent work describing collision for nonintegrable gKdV, in the case of two nonlinear objects of different scale:

$$0 < c_2 \ll c_1$$

Available results say that  $Q_{c_1}$  is globally stable and will survive the collision, up to a perturbation of order  $\|Q_{c_2}\|_{L^2}$ .

**However, in the nonintegrable case, without special algebraic structure, it is not clear whether  $Q_{c_2}$  survives the collision.**

## General setting

Assumption on  $f$ : for  $p = 2, 3, 4$

$$f(u) = u^p + f_1(u), \quad \lim_{u \rightarrow 0} \left| \frac{f_1(u)}{u^p} \right| = 0 \quad \text{subcriticality at 0}$$

Then, there exists  $c_* > 0$  s.t.  $\forall c \in (0, c_*)$ ,  $\exists Q_c \in H^1$  solution of

$$Q_c'' + f(Q_c) = cQ_c \quad \text{stable in } H^1$$

For  $c$  small:

$$Q_c(x) \sim Kc^{\frac{1}{p-1}} e^{-\sqrt{c}|x|}, \quad \|Q_c\|_{H^1} \sim \|Q_c\|_{L^2} \sim Kc^{\frac{1}{p-1} - \frac{1}{4}}$$

# Asymptotic results in the energy space

## Orbital stability by conservation laws

[Cazenave and Lions, 1982], [Weinstein, 1986]

$$\|u(0) - Q_c\|_{H^1} = \alpha_0 \text{ small} \Rightarrow \sup_t \|u(t) - Q_c(\cdot - \rho(t))\|_{H^1} \leq K\alpha_0$$

## Asymptotic stability [Martel and Merle, 2001-2007]

*Under the same assumptions, there exists  $c^+ \sim c$  such that*

$$u(t) - Q_{c^+}(\cdot - \rho(t)) \rightarrow 0 \quad \text{in } H^1(x > \frac{c}{10}t)$$

$$\rho'(t) \rightarrow c^+ \quad \text{as } t \rightarrow +\infty$$

## Existence of asymptotic multi-solitons [Martel, 2005]

There exists a unique solution in  $H^1$  such that

$$\lim_{t \rightarrow -\infty} \|U(t) - (Q_{c_1}(\cdot - c_1 t) + Q_{c_2}(\cdot - c_2 t))\|_{H^1} \rightarrow 0$$

The behavior of  $U(t)$  as  $t \rightarrow +\infty$  is not known, except that  $Q_{c_1}$  is stable up to a perturbation of order  $\|Q_{c_2}\|_{H^1} \sim K c_2^{\frac{1}{p-1} - \frac{1}{4}}$ .

## Stability of multi-solitons in $H^1$ [Martel, Merle and Tsai, 2002]

For  $\alpha_0$  small,  $T$  large:

$$\|u(T) - (Q_{c_1}(\cdot - c_1 T) + Q_{c_2}(\cdot - c_2 T))\|_{H^1} \leq \alpha_0 \quad \Rightarrow$$
$$\sup_{t \geq T} \|u(t) - (Q_{c_1}(\cdot - \rho_1(t)) + Q_{c_2}(\cdot - \rho_2(t)))\|_{H^1} \leq K(\alpha_0 + e^{-\gamma T})$$

# PLAN

- ▶ Stability of two soliton collision for the gKdV equation
- ▶ Detailed description for the quartic gKdV equation  $f(u) = u^4$
- ▶ Existence of symmetric 2-soliton-like solutions
- ▶ Classification of the nonlinearities
- ▶ Some elements of proof



# Stability of two soliton collision (general nonlinearity)

**THM 1.** [Martel and Merle, 2007]

Assume  $0 < c_2 < c_0(c_1) \ll c_1$ . Let  $U(t)$  be s.t.

$$\lim_{t \rightarrow -\infty} \|U(t) - (Q_{c_1}(\cdot - c_1 t) + Q_{c_2}(\cdot - c_2 t))\|_{H^1(\mathbb{R})} \rightarrow 0$$

There exist  $c_1^+ \underset{c_2 \sim 0}{\sim} c_1$ ,  $c_2^+ \underset{c_2 \sim 0}{\sim} c_2$ , s.t.

$$c_1^+ \geq c_1, \quad c_2^+ \leq c_2$$

$$w^+(t, x) = U(t, x) - (Q_{c_1^+}(x - \rho_1(t)) + Q_{c_2^+}(x - \rho_2(t)))$$

$$\lim_{t \rightarrow +\infty} \|w^+(t)\|_{H^1(x \geq \frac{c_2}{10}t)} = 0, \quad \sup_t \|w^+(t)\|_{H^1} \leq Kc_2^{\frac{1}{p-1}}$$

# Comments on THM 1

- ▶ **The two solitons are preserved through the collision**

$$\sup_t \|w^+(t)\|_{H^1} \leq Kc_2^{\frac{1}{p-1}} \quad \text{and} \quad \|Q_{c_2}\|_{H^1} \sim Kc_2^{\frac{1}{p-1} - \frac{1}{4}}$$

- ▶ **Speed change is related to a dispersive residue**

$$\|w^+(t)\|_{H^1} \not\rightarrow 0 \text{ as } t \rightarrow +\infty \text{ iff } c_1^+ > c_1 \text{ and } c_2^+ < c_2$$

- ▶ Center of mass :  $\lim_{+\infty} |\rho'_j(t) - c_j^+| = 0$
- ▶ Stability in  $H^1$  of the behavior of  $U(t)$  for all time

## Detailed description for the quartic KdV equation

$$\partial_t u + \partial_x(\partial_x^2 u + u^4) = 0 \quad t, x \in \mathbb{R}$$

Assume  $0 < c \ll 1$ . Let  $U(t)$  be s.t.  $(Q = Q_1)$

$$\lim_{t \rightarrow -\infty} \|U(t) - Q(\cdot - t) - Q_c(\cdot - ct)\|_{H^1(\mathbb{R})} \rightarrow 0$$

**THM 2.** [Martel and Merle, 2007]

$$c_1^+ - 1 \geq K c^{\frac{17}{6}}, \quad 1 - \frac{c_2^+}{c} \geq K c^{\frac{8}{3}}$$

$$0 < K c^{\frac{17}{12}} \leq \|w_x^+(t)\|_{L^2} + c^{\frac{1}{2}} \|w^+(t)\|_{L^2} \leq K' c^{\frac{11}{12}}, \quad t \text{ large}$$

## Comments on THM 2

- ▶ **Nonexistence of a pure 2-soliton solution in this regime**
- ▶ THM 2 is the first rigorous result describing an inelastic collision between two nonlinear objects
- ▶ The collision is almost elastic

$$\|w^+(t)\|_{L^2} \leq K \|Q_c\|_{L^2}^7$$

# Generalized 2-soliton and explicit shifts (quartic)

**THM 3.** [Martel and Merle, 2007]

Assume  $0 < c \ll 1$ . There exists a solution  $\varphi(t)$  s.t.

$$\varphi(-t, -x) = \varphi(t, x)$$

$$w^-(t, x) = \varphi(t, x) - Q(x - t + \frac{1}{2}\Delta) - Q_c(x - ct + \frac{1}{2}\Delta_c)$$

$$w^+(t, x) = \varphi(t, x) - Q(x - t - \frac{1}{2}\Delta) - Q_c(x - ct - \frac{1}{2}\Delta_c)$$

$$\lim_{-\infty} \|w^-(t)\|_{H^1(x \leq \frac{1}{10}ct)} = 0, \quad \lim_{+\infty} \|w^+(t)\|_{H^1(x \geq \frac{1}{10}ct)} = 0$$

$$K c^{\frac{17}{12}} \leq \|w_x^\pm(t)\|_{L^2} + c^{\frac{1}{2}} \|w^\pm(t)\|_{L^2} \leq K' c^{\frac{17}{12}}, \quad \pm t \text{ large}$$

$$\Delta \underset{c \sim 0}{\sim} -\frac{K_1}{c^{1/6}} < 0, \quad \Delta_c \underset{c \sim 0}{\sim} -K_2 < 0$$

## Comments on THM 3

- ▶ The solution  $\varphi(t)$  is a generalization of multi-soliton in the nonintegrable situation.  
We can obtain  $\varphi(t)$  **at any order of  $c$** .
- ▶ Speeds at  $t = \pm\infty$  are identical.  
The shift  $\Delta$  on  $Q$  becomes negative infinite as  $c \rightarrow 0$ .  
The shift  $\Delta_c$  on  $Q_c$  is negative and of size 1
- ▶  $\varphi(t)$  is not unique but lower bound on the defect is universal.
- ▶ From critical Cauchy theory [Tao, 2006], we conjecture that  $w^+$  is pure dispersion.
- ▶ THM 3 extends to general nonlinearity  $f(u)$ , except the lower bound on  $w^+(t)$  ( $\Delta \sim K_1 \int Q_c$ ).

## Classification of the nonlinearities

We go back to the general framework: for  $p = 2, 3$ ,  $m \geq p + 1$ ,

$$f(u) = u^p + f_1(u), \quad \lim_{u \rightarrow 0} \left| \frac{f_1(u)}{u^p} \right| = 0, \quad f_1^{(m)}(0) \neq 0.$$

Considering small solutions, after scaling, we reduce to

$$f(u) = u^p + \varepsilon u^m + f_2(u), \quad \lim_{u \rightarrow 0} \left| \frac{f_2(u)}{u^m} \right| = 0.$$

**THM 4.** [Muñoz, 2009]

Assume  $0 < \varepsilon \ll 1$ ,  $0 < c \ll 1$ .

$$0 < \mathbf{K} \varepsilon \mathbf{c}^{\frac{2}{p-1} + \frac{3}{4}} \leq \|w_x^+(t)\|_{L^2} + c^{\frac{1}{2}} \|w^+(t)\|_{L^2}, \quad t \text{ large}$$

**Non existence of a pure 2-soliton solution in this regime.**

## Sketch of the method (quartic case)

Proofs are based on both algebraic computations (during the interaction) and asymptotic analysis. Define  $T_c = c^{-\frac{1}{2} - \frac{1}{100}}$ .

- ▶ Asymptotic arguments for  $|t| > T_c$

For  $|t| > T_c$ , we expect the solitons to be decoupled

We apply refinements of previous stability and asymptotic stability arguments ([Martel and Merle, 2001]).

Monotonicity of localized  $L^2$  quantities, Viriel type identities.

- ▶ Construction of an explicit approximate solution

Algebraic computations relevant in the collision region  $|t| < T_c$

- ▶ Justification of the algebra on  $[-T_c, T_c]$

Stability arguments using a modified Hamiltonian structure (refinement of [Weinstein, 1986])



## Approximate solution at all order for $|t| < T_c$

Let

$$y_c = x + (1 - c)t, \quad y = x - \alpha(y_c)$$

$$v(t, x) = Q(y) + Q_c(y_c) + W(t, x)$$

$$\alpha'(s) = \sum_{\substack{1 \leq k \leq k_0 \\ 0 \leq \ell \leq \ell_0}} a_{k,\ell} c^\ell Q_c^k(s)$$

$$W(t, x) = \sum_{\substack{1 \leq k \leq k_0 \\ 0 \leq \ell \leq \ell_0}} c^\ell \left( Q_c^k(y_c) A_{k,\ell}(y) + (Q_c^k)'(y_c) B_{k,\ell}(y) \right)$$

where  $(a_{k,\ell}, A_{k,\ell}, B_{k,\ell})$  are to be determined so that

$$\|\partial_t v + \partial_x(\partial_x^2 v - v - v^4)\|_{L^2} \leq Kc^{N(k_0, \ell_0)}$$

$N(k_0, \ell_0) \rightarrow +\infty$  as  $k_0, \ell_0 \rightarrow +\infty$

(Introduction of parameters  $(a_{k,\ell})$  is related to the shift of  $Q$ )

# Model system

For each  $(k, \ell)$ , we obtain the system

$$(\Omega_{k,\ell}) \quad \begin{cases} (\mathcal{L}A_{k,\ell})' + a_{k,\ell}(3Q - 2Q^4)' = F_{k,\ell} \\ (\mathcal{L}B_{k,\ell})' + a_{k,\ell}(3Q'') - 3A''_{k,\ell} - 4Q^3A_{k,\ell} = G_{k,\ell} \end{cases}$$

where  $F_{k,\ell}$  and  $G_{k,\ell}$  are given in terms of  $(a_{k',\ell'}, A_{k',\ell'}, B_{k',\ell'})$ , for  $k' \leq k$ ,  $\ell' \leq \ell$ , with either  $k' < k$  or  $\ell' < \ell$  and where  $\mathcal{L}A = -A'' + A - 4Q^3A$ .

System  $(\Omega_{k,\ell})$  can be solved when  $F_{k,\ell}$  and  $G_{k,\ell}$  have certain parity properties (no uniqueness: two free parameters).

We obtain  $A_{k,\ell}$ ,  $B_{k,\ell}$  which are localized functions plus polynomial.

## Recomposition at $t = \pm T_c$ and identification of a defect

We find  $A_{1,0}, A_{2,0} \in L^2$  and

$$B_{1,0}(x) = -b_{1,0} \frac{Q'(x)}{Q(x)} + \tilde{B}_{1,0}(x), \quad \tilde{B}_{1,0} \in L^2, \quad b_{1,0} \neq 0$$

$$B_{2,0}(x) = -b_{2,0} \frac{Q'(x)}{Q(x)} + \tilde{B}_{2,0}(x), \quad \tilde{B}_{2,0} \in L^2, \quad b_{2,0} \neq 0$$

For  $t = +T_c$ , we have  $y_c \ll y$ , the two solitons are decoupled and

$$\begin{aligned} v(T_c, x) &\sim Q(y) + Q_c(y_c) - b_{1,0} Q'_c(y_c) - b_{2,0} (Q_c^2)'(y_c) + \dots \\ &\sim Q(y) + Q_c(y_c - b_{1,0}) - b_{2,0} (Q_c^2)'(y_c - b_{1,0}) + \dots \end{aligned}$$

**But the term  $-b_{2,0} (Q_c^2)'$  is a defect of size  $\|(Q_c^2)'\|_{L^2} = Kc^{\frac{11}{12}}$ .**  
We cannot recompose  $v(T_c, x)$  as the sum of two solitons at this order ( $Q_c''$  is related to  $Q_c^4$  and  $Q_c^{(3)}$  is related to  $(Q_c^4)'$ )

## Nonexistence of a pure 2-soliton (quartic case)

- By contradiction, assume that there exists a pure 2-soliton  $U(t)$ :

$$\|U(t) - Q(\cdot - t - x_{1,\pm}) - Q_c(\cdot - ct - x_{2,\pm})\|_{H^1} \rightarrow 0 \quad \text{as } t \rightarrow \pm\infty$$

By stability, after time and space translations,  $\exists T_+, \delta_+$

$$\|U(-T_c, \cdot - T_c) - Q(\cdot + \frac{1}{2}\Delta) - Q_c(\cdot - (1-c)T_c + \frac{1}{2}\Delta_c)\|_{H^1} \leq Kc$$

$$\|U(T_+, \cdot - \delta_+) - Q(\cdot - \frac{1}{2}\Delta) - Q_c(\cdot + (1-c)T_c - \frac{1}{2}\Delta_c)\|_{H^1} \leq Kc$$

A priori no relation between  $T_+$  and  $T_c$ .

- From the algebra, there exists a nonsymmetric approx. solution

$$\|\tilde{v}(-T_c) - Q(\cdot + \frac{1}{2}\Delta) - Q_c(\cdot - (1-c)T_c + \frac{1}{2}\Delta_c)\|_{H^1} \leq Kc$$

$$\|\tilde{v}(T_c) - Q(\cdot - \frac{1}{2}\Delta) - Q_c(\cdot + (1-c)T_c - \frac{1}{2}\Delta_c) + \mathbf{2b}_{2,0}(\mathbf{Q}_c^2)'\|_{H^1} \leq Kc$$

By stability analysis on  $[-T_c, T_c]$  applied on  $\tilde{v}$ ,  $\exists \delta$  s.t.

$$\|U(T_c) - \tilde{v}(T_c, \cdot - \delta)\|_{H^1} \leq Kc$$

- $T_+ \sim T_c$  by stability of 2 soliton structure

$$\|U(T_c) - Q(\cdot - \rho_1) - Q_c(\cdot - \rho_2)\|_{H^1} \leq Kc^{\frac{11}{12}}, \quad \rho_1 - \rho_2 \sim T_c,$$

$$\Rightarrow \forall t > T_c, \|U(t) - Q(\cdot - \rho_1(t)) - Q_c(\cdot - \rho_2(t))\|_{H^1} \leq Kc^{\frac{5}{12}}$$

with  $\rho_1(t) - \rho_2(t) \sim (1 - c)t$

- Contradiction follows from

$$\|U(T_+) - Q(\cdot - \tilde{\rho}_1) - Q_c(\cdot - \tilde{\rho}_2)\|_{H^1} \leq Kc$$

$$\|U(T_+) - Q(\cdot - \rho_1) - Q_c(\cdot - \rho_2) + \mathbf{2b}_{2,0}(\mathbf{Q}_c^2)'(\cdot - \rho_2)\|_{H^1} \leq Kc$$

## Case of the BBM equation [Martel, Merle and Mizumachi]

For the BBM equation, we study the collision of a soliton of speed  $c_0 > 1$  with a small soliton of speed  $c > 1$  close to 1.

After renormalization, it is equivalent to study the collision of  $Q$  by a small soliton  $R_\sigma \sim Q_\sigma$ ,  $\sigma = c - 1 > 0$  small, for the equation

$$(1 - \lambda \partial_x^2) \partial_t u + \partial_x (\partial_x^2 u - u + u^2) = 0, \quad \lambda = \frac{c_0 - 1}{c_0} \in (0, 1)$$

Similar analysis can be done, and shows the existence of a nonzero defect.