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Trefftz-DG methods for the Helmholtz and the Maxwell equations

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Joint work with R. Hiptmair (ETH Zürich) & I. Perugia (Università di Pavia)

Time-harmonic PDEs

Helmholtz and (time-harmonic) Maxwell equations:

$$-\Delta u - \omega^2 u = 0 \qquad \nabla \times (\nabla \times \mathbf{E}) - \omega^2 \mathbf{E} = \mathbf{0} \qquad (\omega > 0)$$

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Why are they interesting?

1. Very general, related to any linear wave phenomena:

$$\left.\begin{array}{ll} \text{wave equation:} & \dfrac{\partial^2 U}{\partial t^2} - \Delta U = 0 \\ \text{time-harmonic regime:} & U(\mathbf{x}, t) = \Re\{u(\mathbf{x}) e^{-i\omega t}\} \end{array}\right\} \rightarrow \text{Helmholtz equation;}$$

2. plenty of applications;
3. easy to write...

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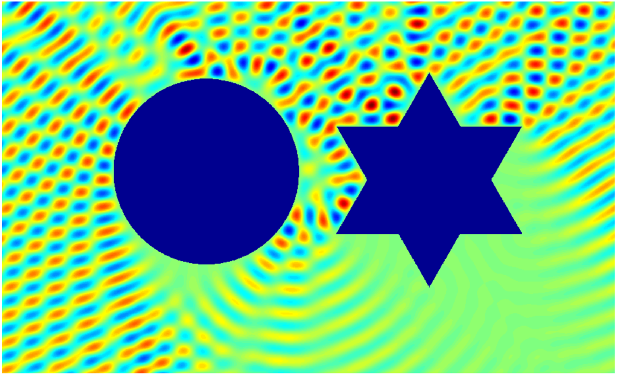
2. plenty of applications;
3. easy to write. . . but difficult to solve numerically ($\omega \gg 1$):
   - oscillating solutions $\rightarrow$ expensive to approximate;
   - numerical dispersion / pollution effect.

Difficulty #1: oscillations

Time-harmonic solutions are inherently oscillatory: a lot of DOFs needed for any polynomial discretisation!



(Helmholtz BVP, picture by T. Betcke)

Wavenumber $\omega = 2\pi/\lambda$ is the crucial parameter.

Difficulty #2: pollution effect

Big issue in FEM solution for high wavenumbers: pollution effect

$$\frac{\Big|\Big|\text{Galerkin error}\Big|\Big|}{\Big|\Big|\text{best approximation error}\Big|\Big|} \geq C\,\omega^a \qquad a > 0, \quad \omega \to \infty.$$

It affects every (low order) method in h: (BABUŠKA, SAUTER 2000).

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$$\Downarrow$$

Oscillating solutions + pollution effect
= standard FEM are too expensive at high frequencies!

Special schemes required, p-version preferred (hp even better).

ZIENKIEWICZ, 2000: *"Clearly, we can consider that this problem remains unsolved and a completely new method of approximation is needed to deal with the very short-wave solution."*

Trefftz methods

How to deal with these phenomena?

Trefftz methods are finite element schemes such that test and trial functions are solutions of Helmholtz/Maxwell equations in each element K of the mesh $\mathcal{T}_h$, e.g.:

$$V_h \subset T(\mathcal{T}_h) = \left\{ v \in L^2(\Omega) \;:\; -\Delta v - \omega^2 v = 0 \text{ in each } K \in \mathcal{T}_h \right\}.$$

Main idea: more accuracy for less DOFs.

Typical Trefftz basis functions for Helmholtz

1 plane waves, $\mathbf{x} \mapsto e^{i\omega \mathbf{x}\cdot\mathbf{d}}$ $\mathbf{d} \in \mathbb{S}^{N-1}$

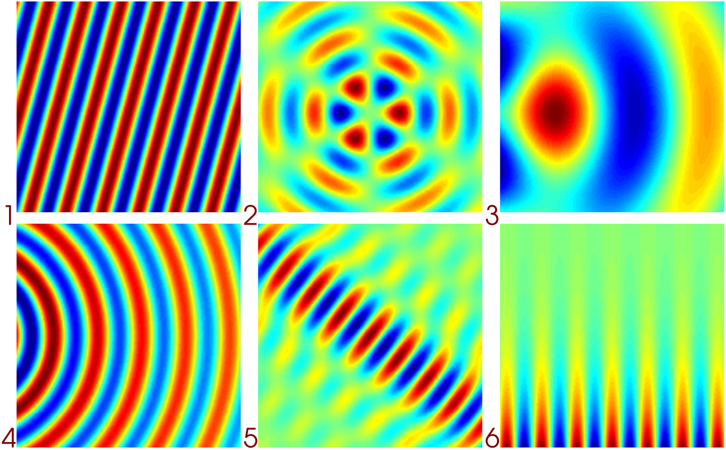
2 circular / spherical waves,

3 corner waves,

4 fundamental solutions/multipoles,

5 wavebands,

6 evanescent waves, . . .



(Plots of real parts.)

Wave-based methods

How to "match" traces across interelement boundaries?

Plenty of Trefftz schemes for Helmholtz/Maxwell available:

- Least squares: method of fundamental solutions (MFS), wave-based method (WBM);
- Lagrange multipliers: discontinuous enrichment (DEM);
- Partition of unity method (PUM/PUFEM), non-Trefftz;
- Variational theory of complex rays (VTCR);
- (Local) Discontinuous Galerkin (DG/LDG): Ultraweak variational formulation (UWVF).

We are interested in a family of Trefftz-discontinuous Galerkin (TDG) methods that includes the UWVF of Cessenat–Després.

Focus: p-version.

Outline

- TDG method for Helmholtz
- TDG method for Maxwell
- Approximation theory for plane and spherical waves
- Exponential convergence of the hp-TDG
  —Work in progress—

Part I

TDG method for the Helmholtz equation

TDG: derivation — I

1. Consider Helmholtz equation with impedance (Robin) b.c.:

$$
\begin{aligned}
-\Delta u - \omega^2 u &= 0 && \text{in } \Omega \subset \mathbb{R}^N \text{ bdd., Lip., } N = 2, 3 \\
\nabla u \cdot \mathbf{n} + i\omega u &= g && \in L^2(\partial\Omega);
\end{aligned}
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2. introduce a mesh $\mathcal{T}_h$ on Ω;
3. multiply the Helmholtz equation with a test function v and integrate by parts on a single element $K \in \mathcal{T}_h$:

$$
\int_K \nabla u \nabla \overline{v} - \omega^2 u \overline{v} \, \mathrm{d}V - \int_{\partial K} (\mathbf{n} \cdot \nabla u) \overline{v} \, \mathrm{d}S = 0;
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$$\int_K \nabla u \nabla \overline{v} - \omega^2 u \overline{v} \, \mathrm{d}V - \int_{\partial K} (\mathbf{n} \cdot \nabla u) \overline{v} \, \mathrm{d}S = 0;$$

4. integrate by parts again: ultraweak step

$$\int_K -u \Delta \overline{v} - \omega^2 u \overline{v} \, \mathrm{d}V + \int_{\partial K} -(\mathbf{n} \cdot \nabla u) \overline{v} + u(\mathbf{n} \cdot \nabla \overline{v}) \, \mathrm{d}S = 0;$$

TDG: derivation — II

5. choose a discrete Trefftz space $V_p(K)$ and replace traces on ∂K with numerical fluxes $\widehat{u}_p$ and $\widehat{\boldsymbol{\sigma}}_p$:

$$u \to u_p \quad \text{(discrete solution)} \qquad \text{in } K\,,$$

$$u \to \widehat{u}_p\,, \qquad \frac{\nabla u}{i\omega} \to \widehat{\boldsymbol{\sigma}}_p \qquad \text{on } \partial K\,;$$

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\end{aligned}
$$

6 use the Trefftz property: $\forall\, v_p \in V_p(K)$

$$
\int_K u_p \underbrace{\overline{(-\Delta v_p - \omega^2 v_p)}}_{=0}\,\mathrm{d}V + \underbrace{\int_{\partial K} \widehat{u}_p\,\overline{\nabla v_p \cdot \mathbf{n}}\,\mathrm{dS} - \int_{\partial K} i\omega \widehat{\boldsymbol{\sigma}}_p \cdot \mathbf{n}\,\overline{v}_p\,\mathrm{dS} = 0}_{\text{TDG eq. on 1 element}}\,.
$$

Two things to set:
discrete space V_p and numerical fluxes $\widehat{u}_p$, $\widehat{\boldsymbol{\sigma}}_p$.

TDG: the space V_p

The abstract error analysis works for every discrete Trefftz space!

Possible choice: plane wave space $(\{\mathbf{d}_\ell\}_{\ell=1}^{p} \subset \mathbb{S}^{N-1})$

$$V_p(\mathcal{T}_h) = \Big\{ v \in L^2(\Omega) : \ v|_K(\mathbf{x}) = \sum_{\ell=1}^{p} \alpha_\ell e^{i\omega\, \mathbf{x}\cdot\mathbf{d}_\ell},\ \alpha_\ell \in \mathbb{C},\ \forall\, K \in \mathcal{T}_h \Big\}.$$

$p :=$ number of basis plane waves (DOFs) in each element.

Numerical fluxes

Choose the numerical fluxes as:

$$\begin{cases} \widehat{\boldsymbol{\sigma}}_p = \frac{1}{i\omega}\{\!\{\nabla_h u_p\}\!\} - \alpha\, [\![u_p]\!]_N \\ \widehat{u}_p = \{\!\{u_p\}\!\} - \beta\, \frac{1}{i\omega}[\![\nabla_h u_p]\!]_N \end{cases} \quad \text{on interior faces,}$$

$$\begin{cases} \widehat{\boldsymbol{\sigma}}_p = \frac{\nabla_h u_p}{i\omega} - (1-\delta)\frac{1}{i\omega}\left(\nabla_h u_p + i\omega u_p\,\mathbf{n} - g\,\mathbf{n}\right) \\ \widehat{u}_p = u_p - \delta\frac{1}{i\omega}\left(\nabla_h u_p\cdot\mathbf{n} + i\omega u_p - g\right) \end{cases} \quad \text{on } \partial\Omega.$$

$\{\!\{\cdot\}\!\}$ = averages, $[\![\cdot]\!]_N$ = normal jumps on the interfaces.

α, $\beta > 0$, $\delta \in (0, \frac{1}{2}]$ parameters at our disposal (in $L^\infty(\mathcal{F}_h)$).

- Here, p-version: α, β, δ independent of ω, h, p.
- UWVF: $\alpha = \beta = \delta = \frac{1}{2}$.
- hp-version, locally refined mesh: α, β, δ depend on local h, p.

Variational formulation of the TDG

With this fluxes, summing over the elements $K \in \mathcal{T}_h$, the TDG method reads: find $u_p \in V_p(\mathcal{T}_h)$ s.t.

$$\mathcal{A}_h(u_p, v_p) \;=\; i\omega^{-1} \int_{\partial\Omega} \delta\, g\, \overline{\nabla_h v_p \cdot \mathbf{n}}\, \mathrm{d}S + \int_{\partial\Omega} (1-\delta) g\, \overline{v_p}\, \mathrm{d}S,$$

$\forall\, v_p \in V_p(\mathcal{T}_h)$, where ($\mathcal{F}_h^I =$ interior skeleton)

$$\begin{aligned}
\mathcal{A}_h(u,v) := &\int_{\mathcal{F}_h^I} \{\!\{u\}\!\} [\![\overline{\nabla_h v}]\!]_N \, \mathrm{d}S && + i\,\omega^{-1} \int_{\mathcal{F}_h^I} \beta\, [\![\nabla_h u]\!]_N [\![\overline{\nabla_h v}]\!]_N \, \mathrm{d}S \\
&- \int_{\mathcal{F}_h^I} \{\!\{\nabla_h u\}\!\} \cdot [\![\overline{v}]\!]_N \, \mathrm{d}S && + i\,\omega \int_{\mathcal{F}_h^I} \alpha\, [\![u]\!]_N \cdot [\![\overline{v}]\!]_N \, \mathrm{d}S \\
&+ \int_{\partial\Omega} (1-\delta)\, u\, \overline{\nabla_h v \cdot \mathbf{n}}\, \mathrm{d}S && + i\,\omega^{-1} \int_{\partial\Omega} \delta\, \nabla_h u \cdot \mathbf{n}\, \overline{\nabla_h v \cdot \mathbf{n}}\, \mathrm{d}S \\
&- \int_{\partial\Omega} \delta\, \nabla_h u \cdot \mathbf{n}\, \overline{v}\, \mathrm{d}S && + i\,\omega \int_{\partial\Omega} (1-\delta) u\, \overline{v}\, \mathrm{d}S\,.
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\end{aligned}$$

$u_p \mapsto (\mathrm{Im}\, \mathcal{A}_h(u_p, u_p))^{\frac{1}{2}}$ is a norm on the Trefftz space $\Rightarrow$ $\exists\,!\, u_p$.

Unconditional quasi-optimality

On the Trefftz space

$$T(\mathcal{T}_h) := \left\{ v \in L^2(\Omega) : v_{|K} \in H^2(K),\ -\Delta v - \omega^2 v = 0 \text{ in each } K \in \mathcal{T}_h \right\},$$

$$\left.\begin{array}{l} \forall\, v, w \in T(\mathcal{T}_h) : \\ \operatorname{Im} \mathcal{A}_h(v,v) = |||v|||^2_{\mathcal{F}_h} \\ |\mathcal{A}_h(w,v)| \le 2\, |||w|||_{\mathcal{F}_h^+}\ |||v|||_{\mathcal{F}_h} \end{array}\right\} \Rightarrow \begin{array}{c} \text{quasi-optimality:} \\ |||u - u_p|||_{\mathcal{F}_h} \le 3 |||u - v_p|||_{\mathcal{F}_h^+} \\ \forall v_p \in T(\mathcal{T}_h). \end{array}$$

Using norms

$$|||v|||^2_{\mathcal{F}_h} := \omega^{-1} \left\| \beta^{1/2} [\![\nabla_h v]\!]_N \right\|^2_{0,\mathcal{F}_h^I} + \omega \left\| \alpha^{1/2} [\![v]\!]_N \right\|^2_{0,\mathcal{F}_h^I} + \omega^{-1} \left\| \delta^{1/2} \nabla_h v \cdot \mathbf{n} \right\|^2_{0,\partial\Omega} + \omega \left\| (1-\delta)^{1/2} v \right\|^2_{0,\partial\Omega},$$

$$|||v|||^2_{\mathcal{F}_h^+} := |||v|||^2_{\mathcal{F}_h} + \omega \left\| \beta^{-1/2} \{\!\{ v \}\!\} \right\|^2_{0,\mathcal{F}_h^I} + \omega^{-1} \left\| \alpha^{-1/2} \{\!\{ \nabla_h v \}\!\} \right\|^2_{0,\mathcal{F}_h^I} + \omega \left\| \delta^{-1/2} v \right\|^2_{0,\partial\Omega}.$$

TDG p-convergence

Monk–Wang duality technique
$\rightarrow$ quasi-optimality in $L^2(\Omega)$-norm.

Assume for now: best approximation estimates for plane or circular waves (shown later in this talk).

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We obtain (h- and) p-estimates for plane/circular waves (2D):

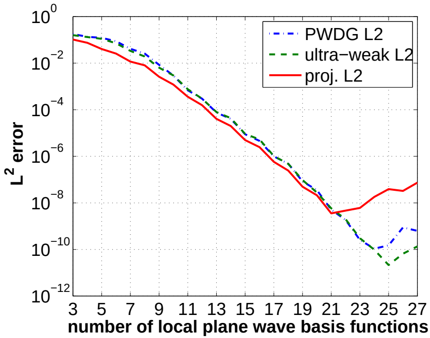
$$|||u - u_p|||_{\mathcal{F}_h} \leq C(\omega h)\, \omega^{-\frac{1}{2}}\, h^{k-\frac{1}{2}} \left(\frac{\log(p)}{p}\right)^{k-\frac{1}{2}} \|u\|_{k+1,\omega,\Omega} \;,$$

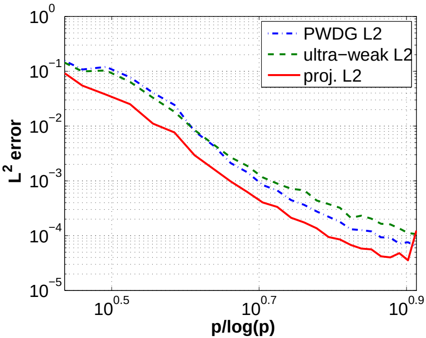
$$\omega\; \|u - u_p\|_{L^2(\Omega)} \leq C(\omega h)\; \mathrm{diam}(\Omega)\, h^{k-1} \left(\frac{\log(p)}{p}\right)^{k-\frac{1}{2}} \|u\|_{k+1,\omega,\Omega} \;.$$

Slightly different orders of convergence in p in 3D.

Numerical tests

Plane wave spaces, $\omega = 10$, $h = 1/\sqrt{2}$, L^2-norm of errors:





Smooth solution in $C^\infty(\mathbb{R}^2)$

$$u = J_1(\omega|x|)\cos\theta$$

exponential convergence.

Singular solution in $H^{\frac{5}{2}-\epsilon}(\Omega)$

$$u = J_{\frac{3}{2}}(\omega|x|)\cos(\tfrac{3}{2}\theta)$$

algebraic convergence.

Disclaimer: ill-conditioning

TDG has:

- unconditional quasi-optimality,
- good approximation properties,

Great!

but with high frequency problems no free lunch is expected!

Where is the cheat?

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Where is the cheat?
All wave-based methods (including TDG / UWVF) are
strongly ill-conditioned.
(And no great preconditioner is available yet.)

Consequence of Trefftz basis; intuitively, think at (equispaced) plane waves:

$$V_h(K) = \mathrm{span}\{e^{i\omega\mathbf{x}\cdot\mathbf{d}_1}, \dots, e^{i\omega\mathbf{x}\cdot\mathbf{d}_p}\} \quad \text{“} \xrightarrow{\omega h_K \to 0} \text{”} \quad \mathrm{span}\{1\},$$

$$\left\| e^{i\omega\mathbf{x}\cdot\mathbf{d}_{\ell+1}} - e^{i\omega\mathbf{x}\cdot\mathbf{d}_\ell} \right\| \quad \xrightarrow{p\to\infty} \quad 0.$$

Ideas: precise balance h vs p, adaptivity on $\mathbf{d}_\ell$'s, new basis...

The road map

	Helmholtz	Maxwell
Formulation of TDG	$\checkmark$	
TDG $\vert\vert\vert \cdot \vert\vert\vert_{\mathcal{F}_h}$-quasi optimality	$\checkmark$	
TDG duality argument	$L^2(\Omega)$	
Approximation by GHPs		
Approximation by PWs		

Part II

TDG method for Maxwell's equations

The TDG for time-harmonic Maxwell's equations

Homogeneous Maxwell equations with impedance b.c.:

$$\begin{cases} \nabla \times (\mu^{-1} \nabla \times \mathbf{E}) - \omega^2 \epsilon\, \mathbf{E} = \mathbf{0} & \text{in } \Omega, \\ \mu^{-1}(\nabla \times \mathbf{E}) \times \mathbf{n} - i\omega\vartheta\, (\mathbf{n} \times \mathbf{E}) \times \mathbf{n} = \mathbf{g} & \in L^2_T(\partial\Omega). \end{cases}$$

($\epsilon, \mu > 0$ (piecewise) constant, assume $\equiv 1$ in this presentation.)

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($\epsilon, \mu > 0$ (piecewise) constant, assume $\equiv 1$ in this presentation.)

Derivation of the TDG method similar to the Helmholtz case:

- $\exists \, ! \, \mathbf{E}_p$ discrete solution,
- quasi optimality in mesh- and flux-dependent norm, containing only tangential jumps and traces:

  $\rightarrow$ no direct control on the divergence.

We obtain error estimates in $|||\cdot|||_{\mathcal{F}_h}$, we want them in a mesh-independent norm (e.g., $L^2(\Omega)$).

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Monk–Wang duality does not apply directly, we need:

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Monk–Wang duality does not apply directly, we need:

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- new wavenumber-explicit stability bounds for the dual BVP:

$$\begin{cases} \nabla\times(\nabla\times\mathbf{\Phi})-\omega^2\mathbf{\Phi}=\mathbf{w}_0 \quad \in H(\mathrm{div}^0;\Omega) & \text{in } \Omega,\\ (\nabla\times\mathbf{\Phi})\times\mathbf{n}+i\omega\vartheta(\mathbf{n}\times\mathbf{\Phi})\times\mathbf{n}=\mathbf{0} & \text{on } \partial\Omega, \end{cases}$$

$$\Rightarrow \text{S} \qquad \|\nabla\times\mathbf{\Phi}\|_{0,\Omega}+\omega\,\|\mathbf{\Phi}\|_{0,\Omega}\le C\ \|\mathbf{w}_0\|_{0,\Omega}\,, \qquad C\ne C(\omega),$$

(using novel Rellich identities for Maxwell, star-shaped Ω);

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$$\Rightarrow \text{S} \qquad \|\nabla \times \mathbf{\Phi}\|_{0,\Omega} + \omega \|\mathbf{\Phi}\|_{0,\Omega} \le C \ \|\mathbf{w}_0\|_{0,\Omega}, \qquad C \neq C(\omega),$$

  (using novel Rellich identities for Maxwell, star-shaped Ω);

- new regularity result for polyhedral domains ($0 < s < 1/2$):

$$\text{R} \qquad \|\nabla \times \mathbf{\Phi}\|_{1/2+s,\Omega} + \omega \|\mathbf{\Phi}\|_{1/2+s,\Omega} \le C\,(1+\omega) \ \|\mathbf{w}_0\|_{0,\Omega}.$$

We control the error in a mesh-independent norm slightly weaker than $\mathbf{L}^2(\Omega)$.

TDG error bounds for Maxwell

Conclusion: quasi-optimality of TDG in two norms

$$|||\mathbf{E}-\mathbf{E}_p|||_{\mathcal{F}_h} \le \ 3 \inf_{\boldsymbol{\xi}_p\in\mathbf{V}_p^E(\mathcal{T}_h)} |||\mathbf{E}-\boldsymbol{\xi}_p|||_{\mathcal{F}_h^+} \ ,$$

$$\begin{aligned}\left\|\mathbf{E}-\mathbf{E}_p\right\|_{H(\mathrm{div},\Omega)'} &:= \sup_{\mathbf{v}\in H(\mathrm{div};\Omega)} \frac{\int_\Omega(\mathbf{E}-\mathbf{E}_p)\cdot\overline{\mathbf{v}}\,\mathrm{d}V}{\left\|\mathbf{v}\right\|_{H(\mathrm{div};\Omega)}}\\ &\le C\left(\frac{\omega^{-\frac12}+\omega^{-\frac32}}{h^{\frac12}}+h^s(\omega^{\frac12}+\omega^{-\frac32})\right)|||\mathbf{E}-\mathbf{E}_p|||_{\mathcal{F}_h}.\end{aligned}$$

(First one from coercivity, second one from duality.)

Assumptions: constant ϵ and μ, polyhedral star-shaped Ω,
shape-regular and quasi-uniform $\mathcal{T}_h$,
$\mathbf{E}\in H^{1/2+s}(\mathrm{curl};\Omega)$ only ($\rightarrow$ no spurious solutions).

The road map

	Helmholtz	Maxwell
Formulation of TDG	$\checkmark$	$\sim$ Helm.
TDG $\|\|\|\cdot\|\|\|_{\mathcal{F}_h}$-quasi optimality	$\checkmark$	$\sim$ Helm.
TDG duality argument	$L^2(\Omega)$	$\boldsymbol{H}(\mathbf{div},\Omega)'$
Approximation by GHPs		
Approximation by PWs		

Part III

Approximation in Trefftz spaces

The best approximation estimates

The analysis of any plane wave Trefftz method requires best approximation estimates:

$$-\Delta u - \omega^2 u = 0 \qquad \text{in } D \in \mathcal{T}_h, \qquad u \in H^{k+1}(D),$$

$$\mathrm{diam}(D) = h, \qquad p \in \mathbb{N}, \qquad \mathbf{d}_1, \dots, \mathbf{d}_p \in \mathbb{S}^{N-1},$$

$$\inf_{\vec{\alpha} \in \mathbb{C}^p} \left\| u - \sum_{\ell=1}^{p} \alpha_\ell e^{i\omega\, \mathbf{d}_\ell \cdot \mathbf{x}} \right\|_{H^j(D)} \leq C\, \epsilon(h,p) \, \|u\|_{H^{k+1}(D)},$$

with *explicit* $\epsilon(h,p) \xrightarrow[p\to\infty]{h\to 0} 0$.

The best approximation estimates

The analysis of any plane wave Trefftz method requires best approximation estimates:

$$-\Delta u - \omega^2 u = 0 \qquad \text{in } D \in \mathcal{T}_h, \qquad u \in H^{k+1}(D),$$

$$\operatorname{diam}(D) = h, \qquad p \in \mathbb{N}, \qquad \mathbf{d}_1, \dots, \mathbf{d}_p \in \mathbb{S}^{N-1},$$

$$\inf_{\vec{\alpha} \in \mathbb{C}^p} \left\| u - \sum_{\ell=1}^{p} \alpha_\ell e^{i\omega\, \mathbf{d}_\ell \cdot \mathbf{x}} \right\|_{H^j(D)} \le C\, \epsilon(h,p)\ \|u\|_{H^{k+1}(D)},$$

with *explicit* $\epsilon(h,p) \xrightarrow[p\to\infty]{h\to 0} 0$.

Goal: precise estimates on $\epsilon(h,p)$

- for plane and circular/spherical waves;
- both in h and p (simultaneously);
- in 2 and 3 dimensions;
- with explicit bounds in the wavenumber ω.

The Vekua theory in N dimensions

We need an old (1940s) tool from PDE analysis: Vekua theory.

$D \subset \mathbb{R}^N$ star-shaped wrt. $\mathbf{0}$, $\omega > 0$.
Define two continuous functions:

$$M_1, M_2 : D \times [0, 1] \to \mathbb{R}$$

$$M_1(\mathbf{x}, t) = -\frac{\omega|\mathbf{x}|}{2} \frac{\sqrt{t}^{N-2}}{\sqrt{1-t}} J_1\left(\omega|\mathbf{x}|\sqrt{1-t}\right),$$

$$M_2(\mathbf{x}, t) = -\frac{i\omega|\mathbf{x}|}{2} \frac{\sqrt{t}^{N-3}}{\sqrt{1-t}} J_1\left(i\omega|\mathbf{x}|\sqrt{t(1-t)}\right).$$

The Vekua operators

$$V_1, V_2 : C(D) \to C(D),$$

$$V_j[\phi](\mathbf{x}) := \phi(\mathbf{x}) + \int_0^1 M_j(\mathbf{x}, t)\phi(t\mathbf{x})\,\mathrm{d}t, \qquad \forall\, \mathbf{x} \in D,\ j = 1, 2.$$

4 properties of Vekua operators

1 $$V_2 = (V_1)^{-1}$$

2 $$\Delta\phi = 0 \qquad \Longleftrightarrow \qquad (-\Delta - \omega^2)\; V_1[\phi] = 0$$

Main idea of Vekua theory:

$$\text{Harmonic functions} \; \underset{V_1}{\overset{V_2}{\rightleftarrows}} \; \text{Helmholtz solutions}$$

3 Continuity in (ω-weighted) Sobolev norms, explicit in ω

($H^j(D)$, $W^{j,\infty}(D)$, $j \in \mathbb{N}$)

4 $P =$ Harmonic polynomial $\iff$ $V_1[P] =$ circular/spherical wave

$$\Big[\underbrace{e^{il\psi}\, J_l(\omega r)}_{2D}, \quad \underbrace{Y_l^m(\tfrac{\mathbf{x}}{|\mathbf{x}|})\, j_l(\omega|\mathbf{x}|)}_{3D}\Big]$$

Vekua operators & approximation by GHPs

$$-\Delta u - \omega^2 u = 0, \qquad\qquad u \in H^{k+1}(D),$$

$$\downarrow V_2$$

$V_2[u]$ is harmonic $\Longrightarrow$ can be approximated by harmonic polynomials

(harmonic Bramble–Hilbert in h,
$\mathbb{C}$omplex analysis in p-2D (Melenk), new result in p-3D),

$$\downarrow V_1$$

u can be approximated by GHPs:

generalized harmonic polynomials $:= V_1\left[\begin{array}{c}\text{harmonic}\\ \text{polynomials}\end{array}\right]$ = circular/spherical waves.

(Obtained bound applicable to GHP-based Trefftz schemes!)

The approximation of GHPs by plane waves

Link between plane waves and circular/spherical waves:
Jacobi–Anger expansion

2D $$e^{iz\cos\theta} = \sum_{l\in\mathbb{Z}} i^l J_l(z)\, e^{il\theta} \qquad z\in\mathbb{C},\ \theta\in\mathbb{R},$$

3D $$\underbrace{e^{ir\boldsymbol{\xi}\cdot\boldsymbol{\eta}}}_{\text{plane wave}} = 4\pi \sum_{l\geq 0}\sum_{m=-l}^{l} i^l\ \underbrace{j_l(r)\ Y_{l,m}(\boldsymbol{\xi})}_{\text{GHP}}\ \overline{Y_{l,m}(\boldsymbol{\eta})} \qquad \boldsymbol{\xi},\ \boldsymbol{\eta}\in\mathbb{S}^2,\ r\geq 0.$$

We need the other way round:

GHP $\approx$ linear combination of plane waves

- truncation of J–A expansion,
- careful choice of directions (in 3D),
- solution of a linear system,
- residual estimates,

$\rightarrow$ explicit error bound.

The final approximation by plane waves

$-\Delta u - \omega^2 u = 0$

$\downarrow$

GHPs

$\downarrow$

Plane waves

Vekua theory,	
harmonic appr.:	algebraic in h & p,
(Jacobi–Anger)$^{-1}$:	algebraic in h,
	$>$ exponential in p.

Final estimate

$$\inf_{\alpha\in\mathbb{C}^p} \left\| u - \sum_{\ell=1}^{p} \alpha_\ell e^{i\omega\, \mathbf{x}\cdot\mathbf{d}_\ell} \right\|_{j,\omega,D} \leq C(\omega h)\; h^{k+1-j} q^{-\lambda(k+1-j)}\; \|u\|_{k+1,\omega,D}$$

In 2D:	$p = 2q+1$,	$\lambda(D)$	explicit,	$\forall\, \mathbf{d}_\ell$.
In 3D:	$p = (q+1)^2$,	$\lambda(D)$	unknown,	special $\mathbf{d}_\ell$.

If u extends outside D: exponential order in q. (Same for GHPs.)

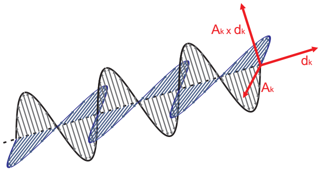
Approximation by Maxwell plane waves

Basis of Maxwell plane waves:

$$\left\{ \mathbf{a}_\ell e^{i\omega \mathbf{x}\cdot\mathbf{d}_\ell}, \quad \mathbf{a}_\ell \times \mathbf{d}_\ell e^{i\omega \mathbf{x}\cdot\mathbf{d}_\ell} \right\}_{\ell=1,\dots,(q+1)^2}$$

$$|\mathbf{a}_\ell| = |\mathbf{d}_\ell| = 1, \ \mathbf{d}_\ell \cdot \mathbf{a}_\ell = 0.$$

Spherical waves defined via
vector spherical harmonics.



Easy proof of approximation bounds by applying Helmholtz results to potentials.
Suboptimal orders, can be partially improved using Vekua.

Same technique (+ special potential representation) used for elastic wave equation and Kirchhoff–Love plates (Chardon).

The road map

	Helmholtz	Maxwell
Formulation of TDG	$\checkmark$	$\sim$ Helm.
TDG $\|\|\|\cdot\|\|\|_{\mathcal{F}_h}$-quasi optimality	$\checkmark$	$\sim$ Helm.
TDG duality argument	$L^2(\Omega)$	$H(\mathbf{div},\Omega)'$
Approximation by GHPs	$\checkmark$	$\checkmark$ (p non sharp)
Approximation by PWs	$\checkmark$	$\checkmark$ (non sharp)

Part IV

What about hp-TDG?

What else is needed?

So far we have proved:

- unconditional well-posedness and quasi-optimality,
- approximation bounds in h and p simultaneously.

What else do we need to obtain exponential convergence of hp-version of TDG?

(Mental picture: 2D, piecewise analytic domain/data, geometrically graded mesh, expected error $\sim e^{-b\sqrt{\#DOFs}}$.)

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Two annoying subtleties:

(i) one related to approximation $\rightarrow$ solved!

(ii) one related to TDG flux parameters (α, β, δ) and $\mathcal{F}_h$-norm $\rightarrow$ still causing headache...

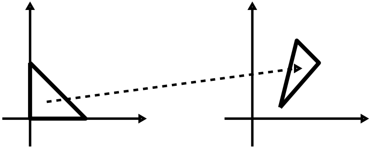
Moreover: what about analytic extension of Helmholtz solutions across impedance boundaries?

(This is joint work with Ch. Schwab (ETH Zürich), RH, IP.)

Fully-explicit approximation — issue (i)

Polynomial FEM: best approximation bounds on $K \in \mathcal{T}_h$ obtained by scaling to reference element $\hat{K}$.

Consider Trefftz methods for Laplace eq.: local basis made of harmonic polynomials is not preserved by affine scaling.



$$\mathbb{P}^q(\hat{K}) \longrightarrow \mathbb{P}^q(K)$$
$$\mathbb{H}^q(\hat{K}) \longrightarrow ???$$

Every element K has "its own" approximation bound.
The bounding constants depend on the shape of K: in unstructured graded meshes they are not uniformly bounded.

We want "universal bounds" independent of the geometry, but. . .

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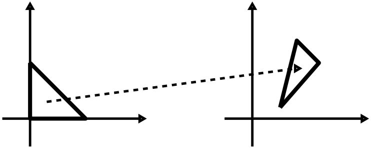

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We want "universal bounds" independent of the geometry, but... we get more: fully explicit bounds for curvilinear non-convex elements.

Assumption and tools

Assumption: (Very weak!)

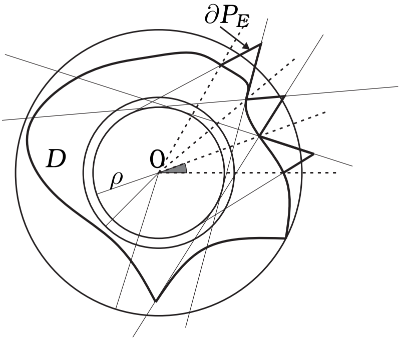
- $D \subset \mathbb{R}^2$ s.t. $\mathrm{diam}(D) = 1$, star-shaped wrt B_ρ, $0 < \rho < 1/2$.

Define:

- $D_\delta := \{z \in \mathbb{R}^2, d(z, D) < \delta\}$, $\quad \xi := \begin{cases} 1 & D \text{ convex}, \\ \frac{2}{\pi} \arcsin \frac{\rho}{1-\rho} < 1. \end{cases}$

Use:

- M. Melenk's ideas;
- complex variable, identification $\mathbb{R}^2 \leftrightarrow \mathbb{C}$, harmonic $\leftrightarrow$ holomorphic;
- conformal map level sets, Schwarz–Christoffel;
- Hermite interpolant q_n;
- lot of "basic" geometry and trigonometry, nested polygons, plenty of pictures. . .



Explicit approximation estimate

Approximation result

Let $n \in \mathbb{N}$, f holomorphic in D_δ, $0 < \delta \le 1/2$,
$h := \min\left\{(\xi\delta/27)^{1/\xi}/3,\ \rho/4\right\}$, $\Rightarrow$ $\exists q_n$ of degree $\le n$ s.t.

$$\|f - q_n\|_{L^\infty(D)} \le 7\rho^{-2}\ h^{-\frac{72}{\rho^4}}(1+h)^{-n}\,\|f\|_{L^\infty(D_\delta)}\,.$$

- Fully explicit bound;
- exponential in degree n;
- $h \ge$ "conformal distance" $(D, \partial D_\delta)$, related to physical dist. δ;
- in convex case $h = \min\{\delta/27,\ \rho/4\}$;
- extends to harmonic f/q_n and derivatives ($W^{j,\infty}$-norm);
- easily extended to GHPs and Helmholtz solutions;
- $\Rightarrow$ $\|u - u_h\|_{H^1(\Omega)} \lesssim e^{-b\sqrt{\#DOFs}}$ for Trefftz *hp* IP-DG (Laplace), by analytic extension of Laplace solutions (Babuška–Guo).

Summary and open problems

What we have done:

- TDG formulation, well-posedness,
- h- and p-convergence, duality in $L^2(\Omega)/H(\mathbf{div};\Omega)'$ norms,
- $h\&p$ approximation estimates for spherical/plane waves,
- new Rellich-type identity and stability estimate for Maxwell,
- ideas towards exponential convergence of hp-TDG.

A lot of possible research directions:

- non-constant coefficients $\omega(\mathbf{x})$, $\epsilon(\mathbf{x})$, $\mu(\mathbf{x})$,
- adaptivity on PW directions,
- reaction-diffusion type equations ($\omega \mapsto i\omega$), time-harmonic elasticity and other PDEs,
- more general domains and Rellich-type identities,
- improved approximation bounds, new bases,
- defeat ill-conditioning, . . .

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Thank you!

Trick: Rellich-type identity for Maxwell

$\forall\, \mathbf{E}, \mathbf{H} \in C^1(B_r(\mathbf{x}) \to \mathbb{C}^3)$:

$$
\begin{aligned}
2\,\mathrm{Re}&\Big\{ \big(\nabla \times \mathbf{E} - i\omega \mathbf{H}\big) \cdot (\overline{\mathbf{E}} \times \mathbf{x}) + \big(\nabla \times \mathbf{H} + i\omega \mathbf{E}\big) \cdot (\overline{\mathbf{H}} \times \mathbf{x}) \Big\} \\
&= 2\,\mathrm{Re}\Big\{ \nabla\cdot \Big[(\mathbf{E}\cdot\mathbf{x})\overline{\mathbf{E}} + (\mathbf{H}\cdot\mathbf{x})\overline{\mathbf{H}} \Big] - (\mathbf{E}\cdot\mathbf{x})(\nabla\cdot\overline{\mathbf{E}}) - (\mathbf{H}\cdot\mathbf{x})(\nabla\cdot\overline{\mathbf{H}}) \Big\} \\
&\qquad - \nabla\cdot \Big[|\mathbf{E}|^2 \mathbf{x} + |\mathbf{H}|^2 \mathbf{x} \Big] + |\mathbf{E}|^2 + |\mathbf{H}|^2 \,.
\end{aligned}
$$

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$\forall\, \mathbf{E}, \mathbf{H} \in C^1(B_r(\mathbf{x}) \to \mathbb{C}^3)$:

$$2\,\mathrm{Re}\left\{ (\nabla \times \mathbf{E} - i\omega\mathbf{H}) \cdot (\overline{\mathbf{E}} \times \mathbf{x}) + (\nabla \times \mathbf{H} + i\omega\mathbf{E}) \cdot (\overline{\mathbf{H}} \times \mathbf{x}) \right\}$$
$$= 2\,\mathrm{Re}\left\{ \nabla\cdot \left[(\mathbf{E}\cdot\mathbf{x})\overline{\mathbf{E}} + (\mathbf{H}\cdot\mathbf{x})\overline{\mathbf{H}} \right] - (\mathbf{E}\cdot\mathbf{x})(\nabla\cdot\overline{\mathbf{E}}) - (\mathbf{H}\cdot\mathbf{x})(\nabla\cdot\overline{\mathbf{H}}) \right\}$$
$$- \nabla\cdot \left[|\mathbf{E}|^2\mathbf{x} + |\mathbf{H}|^2\mathbf{x} \right] + |\mathbf{E}|^2 + |\mathbf{H}|^2 \,.$$

Ω bounded polyhedron, star-shaped wrt. $B_\gamma(\mathbf{0})$ (i.e., $\mathbf{x}\cdot\mathbf{n} \geq \gamma$),
$\mathbf{E}, \mathbf{H} \in H(\mathrm{curl};\Omega) \cap H(\mathrm{div};\Omega)$, $\quad \mathbf{E}\times\mathbf{n},\ \mathbf{H}\times\mathbf{n} \in L^2_T(\partial\Omega)$:

$$\|\mathbf{E}\|^2_{0,\Omega} + \|\mathbf{H}\|^2_{0,\Omega} \leq \frac{(\mathrm{diam}(\Omega))^2}{\gamma}\left(\|\mathbf{E}_T\|^2_{0,\partial\Omega} + \|\mathbf{H}_T\|^2_{0,\partial\Omega} \right)$$
$$+ 2\left| \int_\Omega (\mathbf{E}\cdot\mathbf{x})(\underbrace{\nabla\cdot\overline{\mathbf{E}}}_{=0}) + (\mathbf{H}\cdot\mathbf{x})(\underbrace{\nabla\cdot\overline{\mathbf{H}}}_{=0})\,\mathrm{d}V \right|$$
$$+ 2\left| \int_\Omega (\underbrace{\nabla \times \mathbf{E} - i\omega\mathbf{H}}_{=0}) \cdot (\overline{\mathbf{E}} \times \mathbf{x}) + (\underbrace{\nabla \times \mathbf{H} + i\omega\mathbf{E}}_{=\text{Maxw. source term}}) \cdot (\overline{\mathbf{H}} \times \mathbf{x})\,\mathrm{d}V \right| \,.$$

Maxwell plane wave approximation

1 $\mathbf{E}$ Maxwell $\Rightarrow$ $\nabla\times\mathbf{E}$ Maxwell $\Rightarrow$ $(\nabla\times\mathbf{E})_{1,2,3}$ Helmholtz

$$\left\|\nabla\times\mathbf{E}-\begin{matrix}\text{Helmholtz}\\ \text{vector p.w.}\end{matrix}\right\|_{j,\omega,D}\le C(hq^{-\lambda})^{k+1-j}\,\|\nabla\times\mathbf{E}\|_{k+1,\omega,D}\ .$$

2 With $j\ge 1$, apply $\nabla\times$ and reduce j (bad!):

$$\left\|\nabla\times\nabla\times\mathbf{E}-\nabla\times\begin{bmatrix}\text{Helmholtz}\\ \text{vector p.w.}\end{bmatrix}\right\|_{j-1,\omega,D}\le C(hq^{-\lambda})^{k+1-j}\,\|\nabla\times\mathbf{E}\|_{k+1,\omega,D}\,.$$

$$\Downarrow$$

3 $$\left\|\omega^2\mathbf{E}-\text{Maxwell p.w.}\right\|_{j-1,\omega,D}\le C(hq^{-\lambda})^{k+1-j}\,\|\nabla\times\mathbf{E}\|_{k+1,\omega,D}\ .$$

Mismatch between Sobolev indices and convergence order:
not sharp!