La stabilité du Système solaire, des méthodes de perturbations aux intégrateurs symplectiques

Jacques Laskar

CNRS, Observatoire de Paris

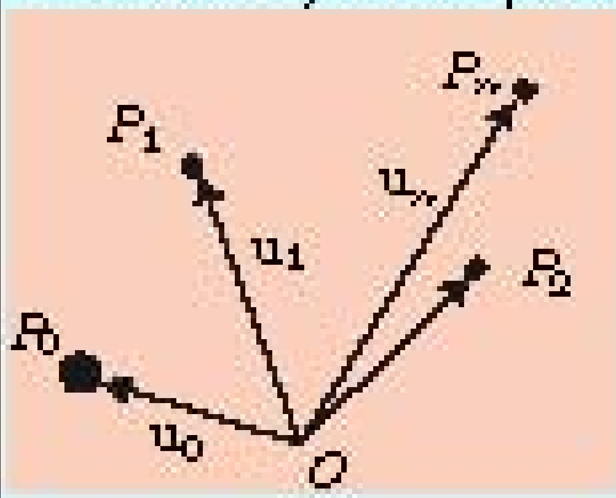
Paris, le 5 février 2010

Hamiltonien planétaire

$P_0, P_1, \dots, P_n$

$m_0, m_1, \dots, m_n$

Coord. Barycentriques

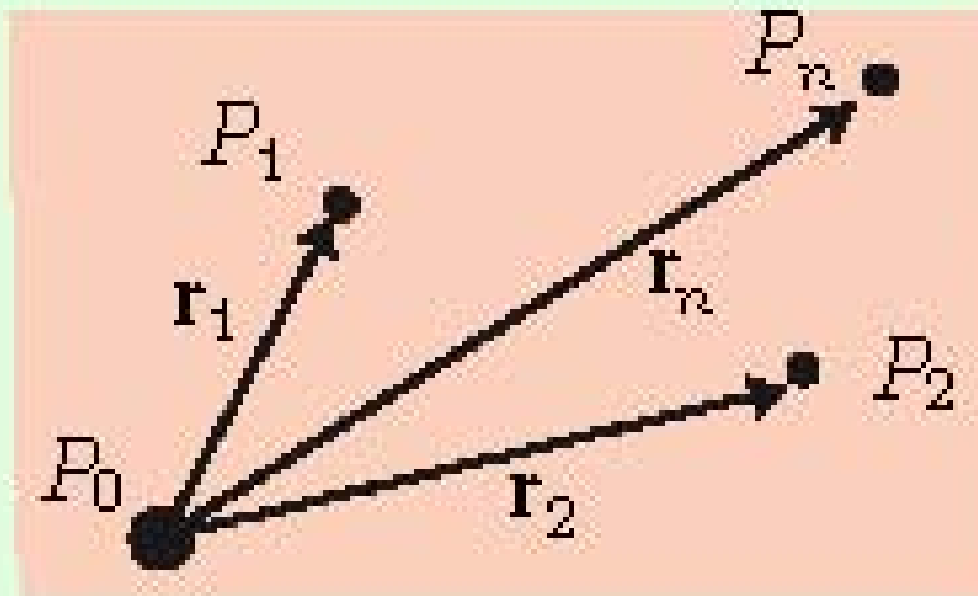


$\mathbf{u}_i = \overrightarrow{OP_i}$

$\tilde{\mathbf{u}}_i = m_i \dot{\mathbf{u}}_i$

$$H = \frac{1}{2}\sum_{i=0}^{n} \frac{\|\tilde{\mathbf{u}}_i\|^2}{m_i} - G \sum_{0 \le i < j} \frac{m_i m_j}{\Delta_{ij}}$$

Coord. Héliocentriques canoniques (Poincaré, 1896)



$\mathbf{r}_i = \mathbf{u}_i - \mathbf{u}_0 \quad i \neq 0$

$\tilde{\mathbf{r}}_i = \tilde{\mathbf{u}}_i$

$\mathbf{r}_0 = \mathbf{u}_0$

$\tilde{\mathbf{r}}_0 = \tilde{\mathbf{u}}_0 + \tilde{\mathbf{u}}_1 + \dots + \tilde{\mathbf{u}}_n = 0$

$$H = H_0 + H_1$$

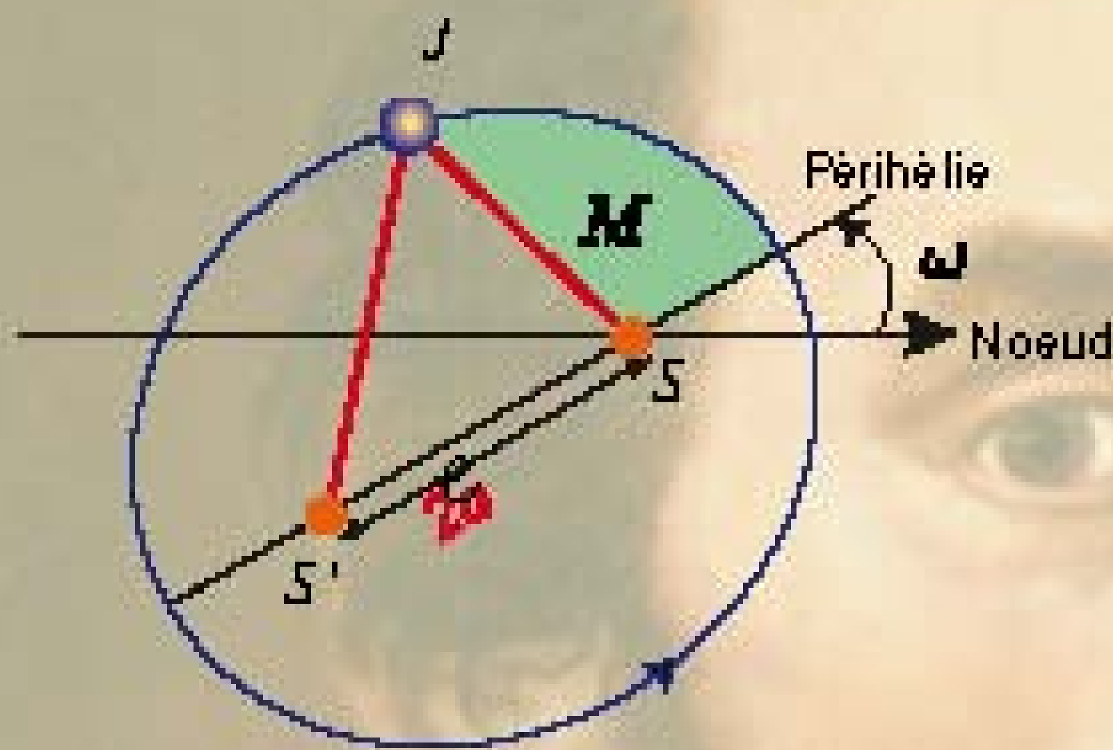
mouvements Képleriens

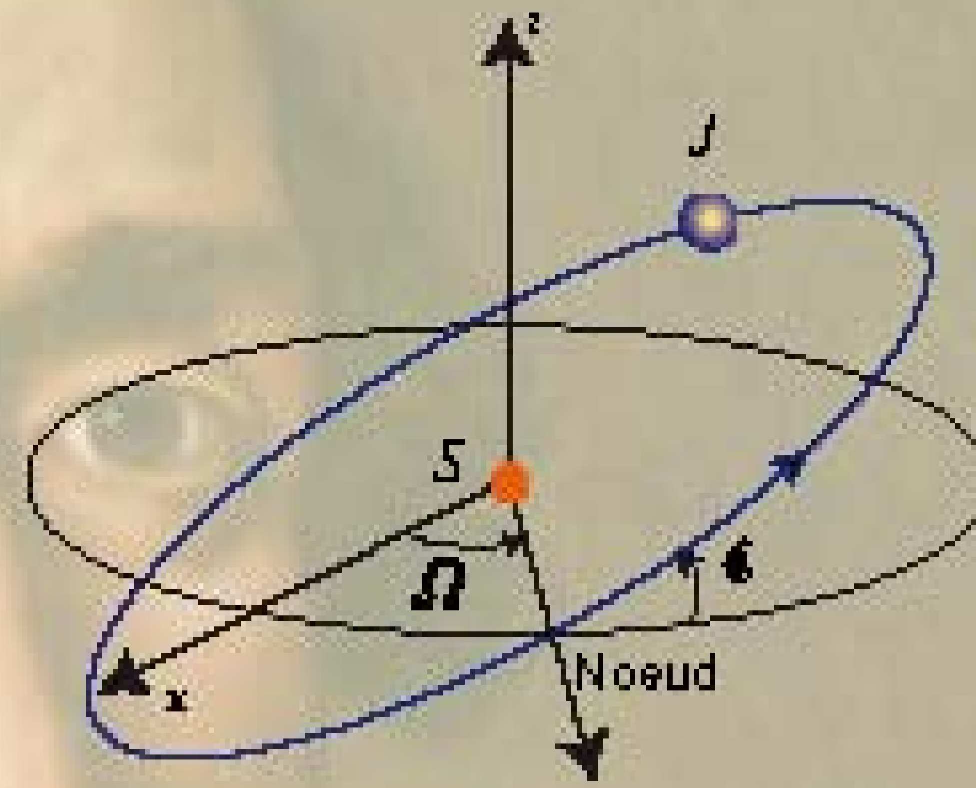
$$H_0 = \frac{1}{2}\sum_{i=1}^{n} \|\tilde{\mathbf{r}}_i\|^2 \left[\frac{1}{m_i} + \frac{1}{m_0}\right] - G \sum_{i=1}^{n} \frac{m_0 m_i}{r_i}$$

interactions planétaires

$$H_1 = \sum_{0 < i < j} \frac{\tilde{\mathbf{r}}_i \cdot \tilde{\mathbf{r}}_j}{m_0} - G \sum_{0 < i < j} \frac{m_i m_j}{\Delta_{ij}}$$

Kepler (1609)





ω : argument du périhélie

e : excentricité

M : anomalie moyenne

a : demi-grand axe

Ω : longitude du noeud

i : inclinaison

$$H_0(a) = -\frac{\mu m}{2a}$$

~ actions $(a, e, i,$

~ angles $M, \omega, \Omega)$

$$M = nt$$

Méthodes de Perturbations (moyennisation)

$$\boldsymbol{H = H_0(\Lambda) \ + H_1(\Lambda, \lambda, x, y)}$$

$$\Lambda = m\sqrt{\mu a}$$

$$x \sim \sqrt{\Lambda}\; e\; E^{i\varpi}$$

$$y \sim \sqrt{\Lambda}\; i\; E^{i\Omega}$$

$$T_W = \exp(\{W, .\})$$

$$W = W_1 + W_2 + \cdots$$

$$H'(\Lambda, x, y)$$

$$\{f, g\} = \sum_{j=1}^{n} \frac{\partial g}{\partial J_j}\frac{\partial f}{\partial \phi_j} - \frac{\partial g}{\partial \phi_j}\frac{\partial f}{\partial J_j}$$

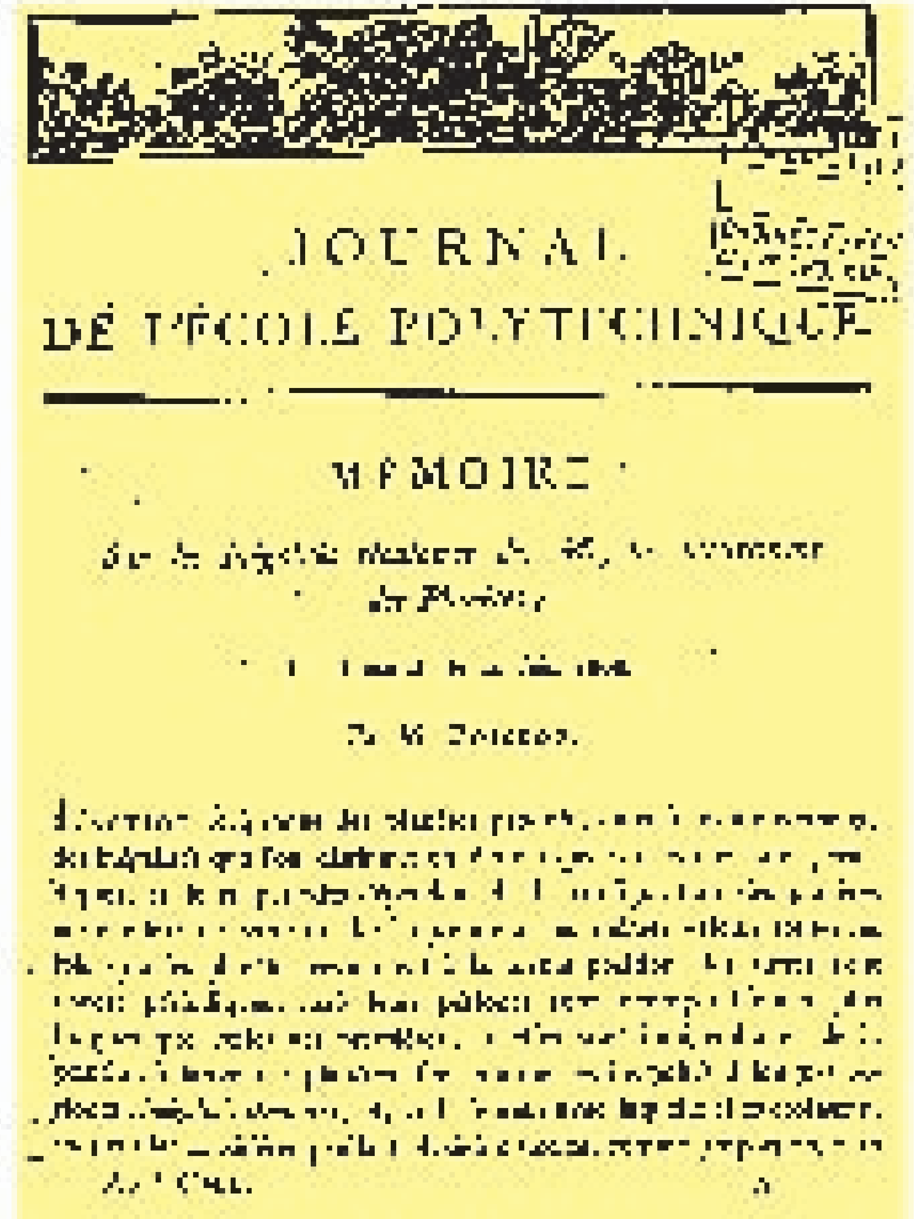
$$H'_0 = H_0$$

$$H'_1 = \{W_1, H_0\} + H_1$$

$$H'_2 = \{W_2, H_0\} + \frac{1}{2}\{W_1, \{W_1, H_0\}\} + \{W_1, H_1\}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Stabilité des grands axes


JOURNAL

DE L'ÉCOLE POLYTECHNIQUE

MÉMOIRE

(Laplace, 1772, Lagrange, 1776, 1809, Poisson, 1809,
Haretu, 1877, Poincaré, 1892,
Milani et al., 1987, Quinn et al. 1991 ...)

$$\frac{d\Lambda'_k}{dt} = \frac{\partial H'(\Lambda', x', y')}{\partial \lambda'_k} = 0$$

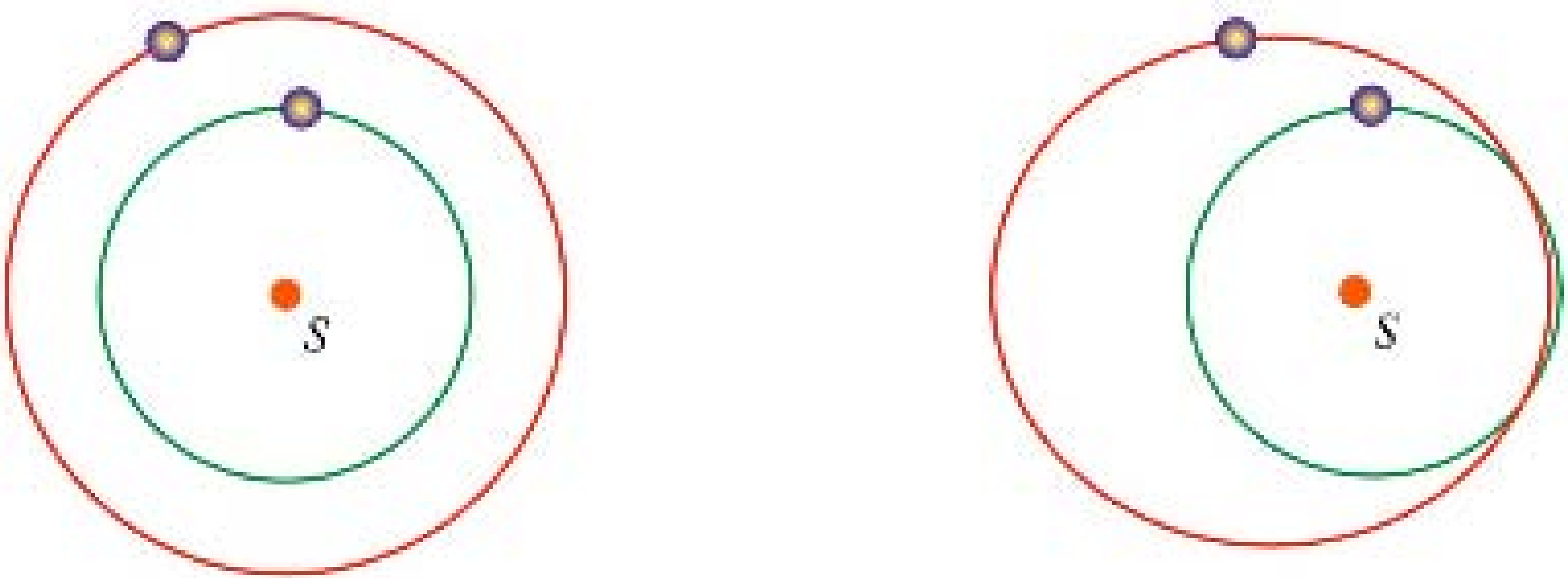
$$\Lambda_k = \Lambda'_k + \{W_1, \Lambda'_k\} + \{W_2, \Lambda'_k\} + \frac{1}{2}\{W_1, \{W_1, \Lambda'_k\}\} + \cdots$$

0 $\qquad$ 0 $\qquad$ $\neq 0$

The stability of the semi-major axes is not sufficient !

Earth: $a \approx 1.$ AU

Mars : $a \approx 1.5$ AU



$e = 0$

$e = 0$

$e = 0.1$

$e = 0.3$

Excentricités et inclinaisons

$$\Lambda'_k = \text{Cte}$$

$$H' = \color{red}{\boldsymbol{H'_2(x, \bar{x}, y, \bar{y})}} + H'_4(x, \bar{x}, y, \bar{y}) + \cdots$$

$$x \sim \sqrt{\Lambda}\; e\; E^{i\varpi}$$
$$y \sim \sqrt{\Lambda}\; i\; E^{i\Omega}$$

$$\frac{dx}{dt} = i\frac{\partial H'}{\partial \bar{x}}$$

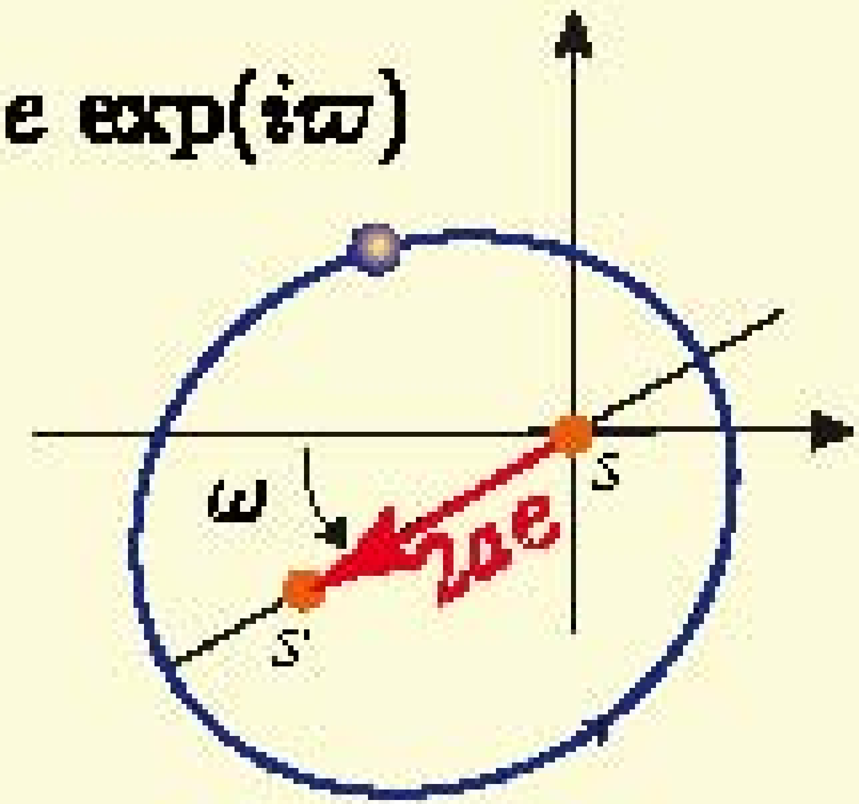
$$\frac{dy}{dt} = i\frac{\partial H'}{\partial \bar{y}}$$

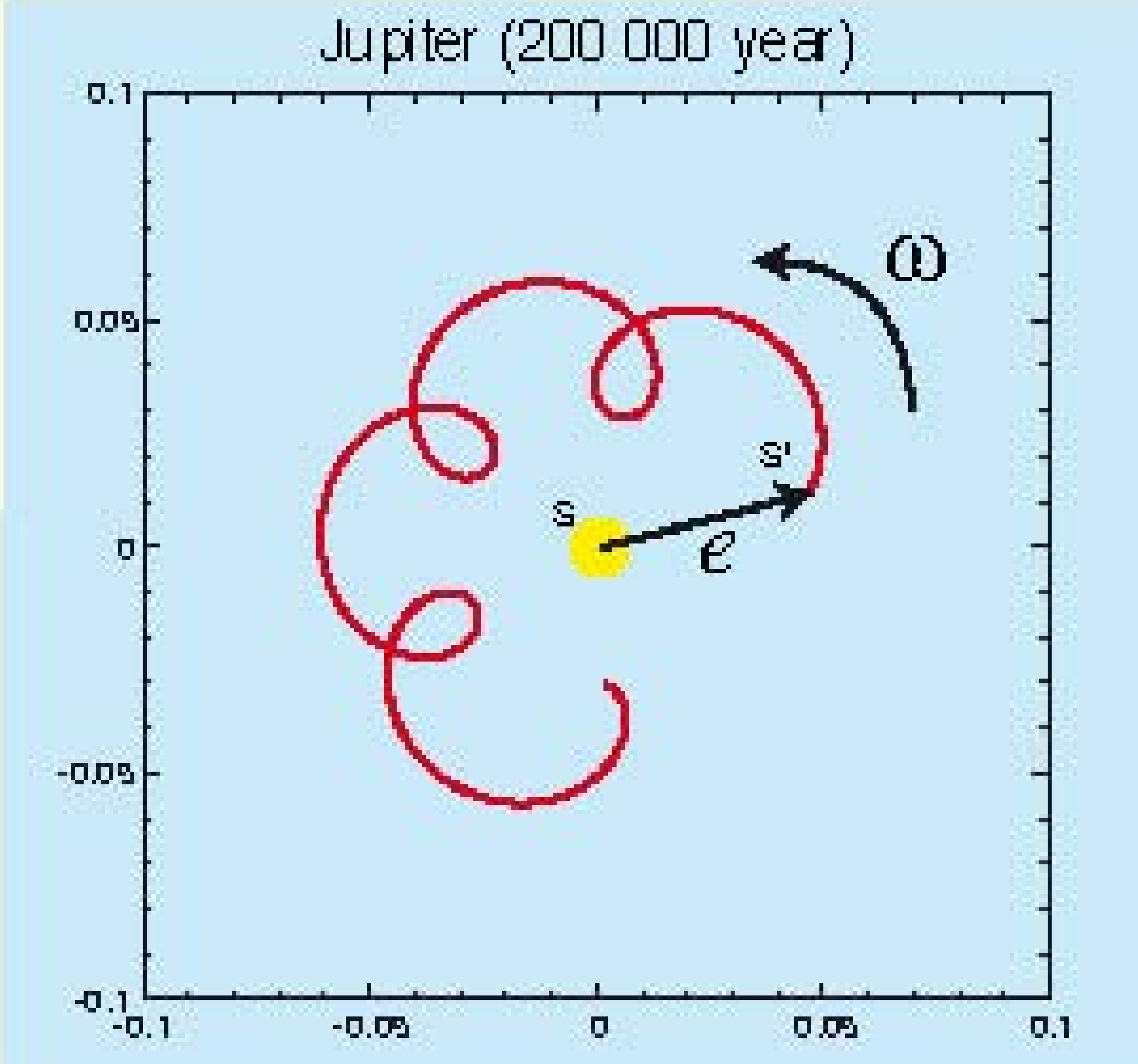
$$\frac{d}{dt}\begin{bmatrix} x_1 \\ \vdots \\ x_8 \\ y_1 \\ \vdots \\ y_8 \end{bmatrix} = \sqrt{-1}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_8 \\ y_1 \\ \vdots \\ y_8 \end{bmatrix}$$

Secular variations of the eccentricities and inclinations

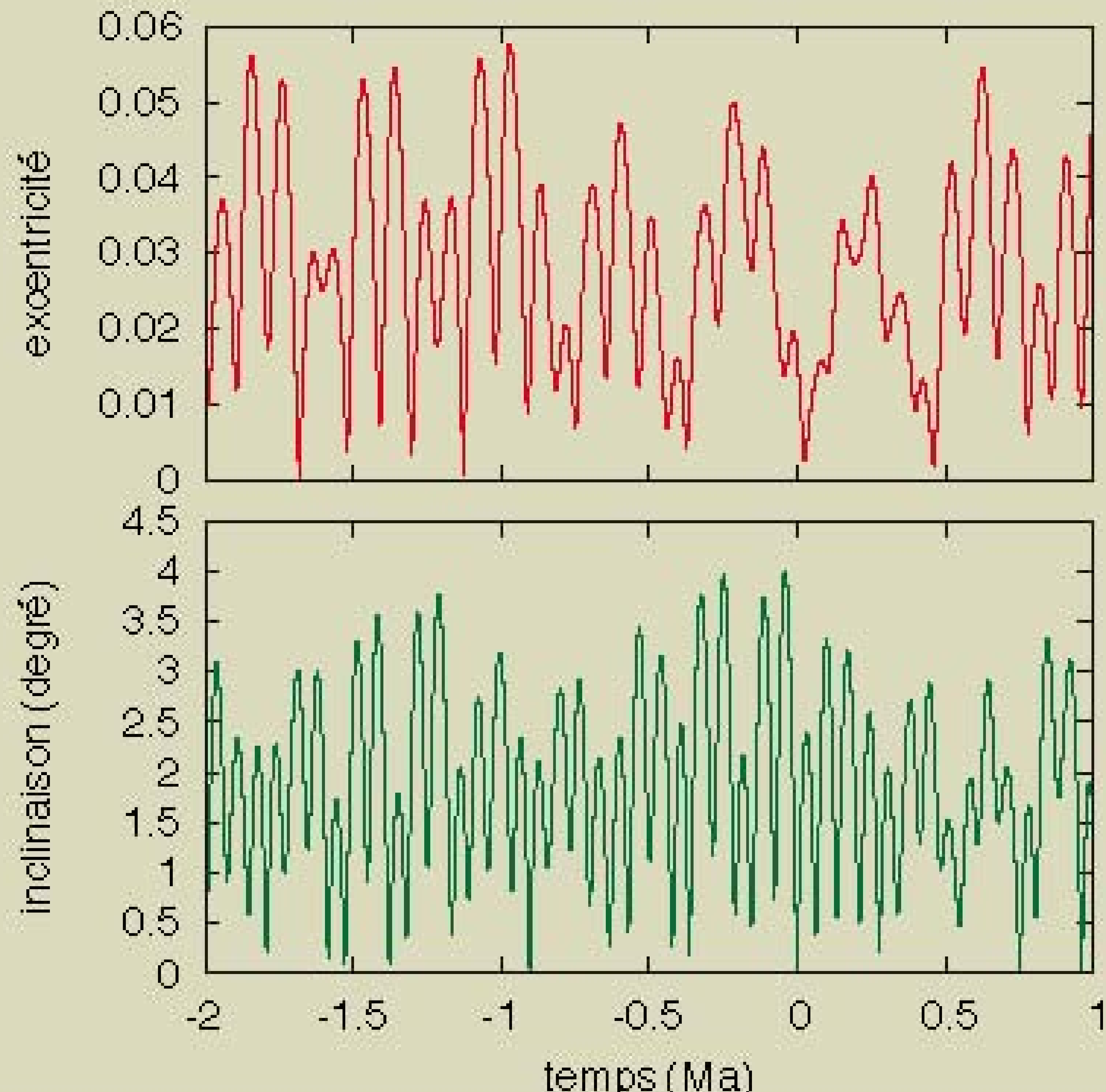
Lagrange 1774-78, Laplace 1774-75

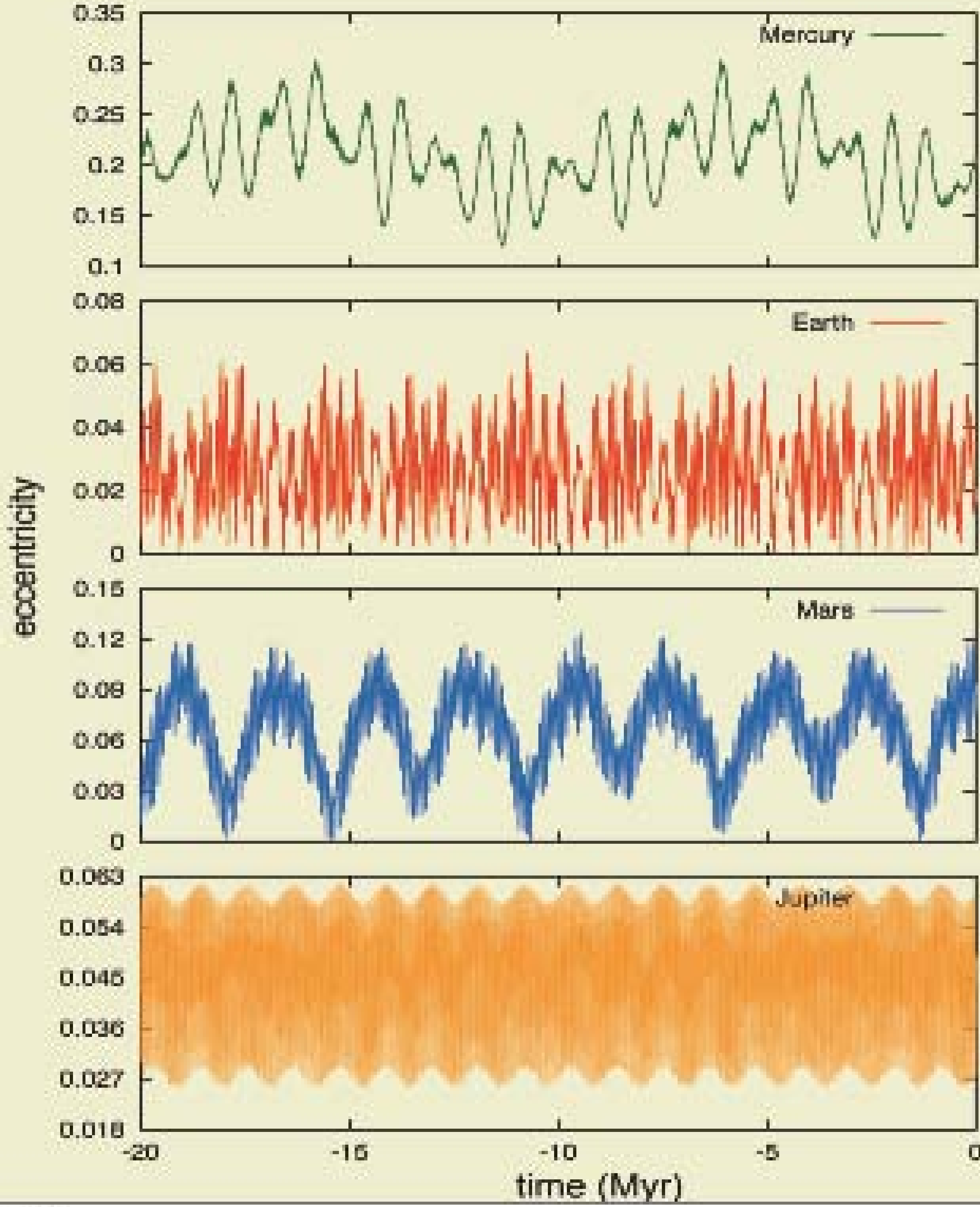
$$z = e\exp(i\varpi)$$





$$\frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_8 \end{pmatrix} = \sqrt{-1}\,A\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_8 \end{pmatrix}$$


excentricité
0.06
0.05
0.04
0.03
0.02
0.01
0
inclinaison (degré)
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
-2
-1.5
-1
-0.5
0
0.5
1
temps (Ma)



(Lagrange, 1774)

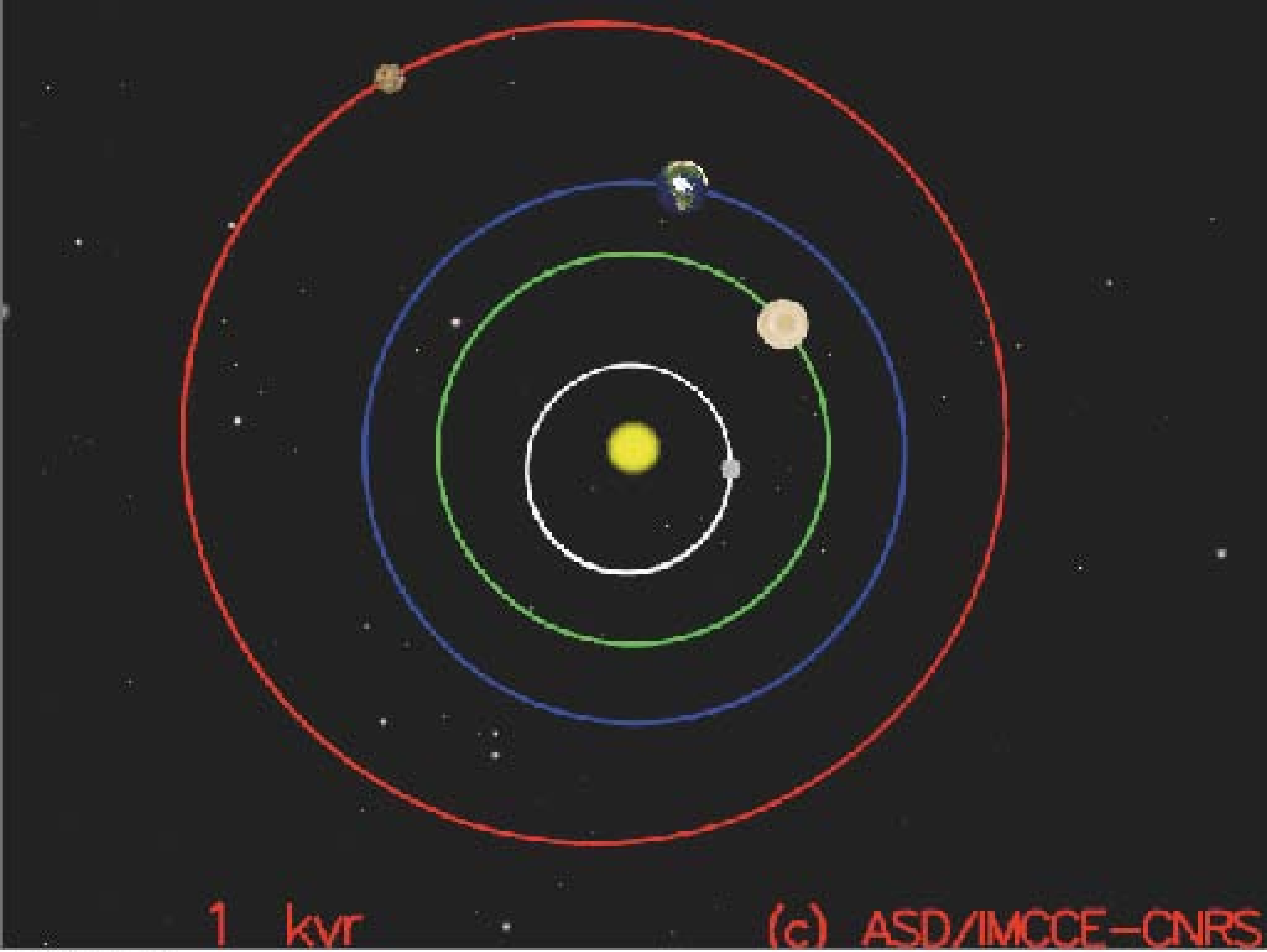
0.22208

0.07641

0.14726

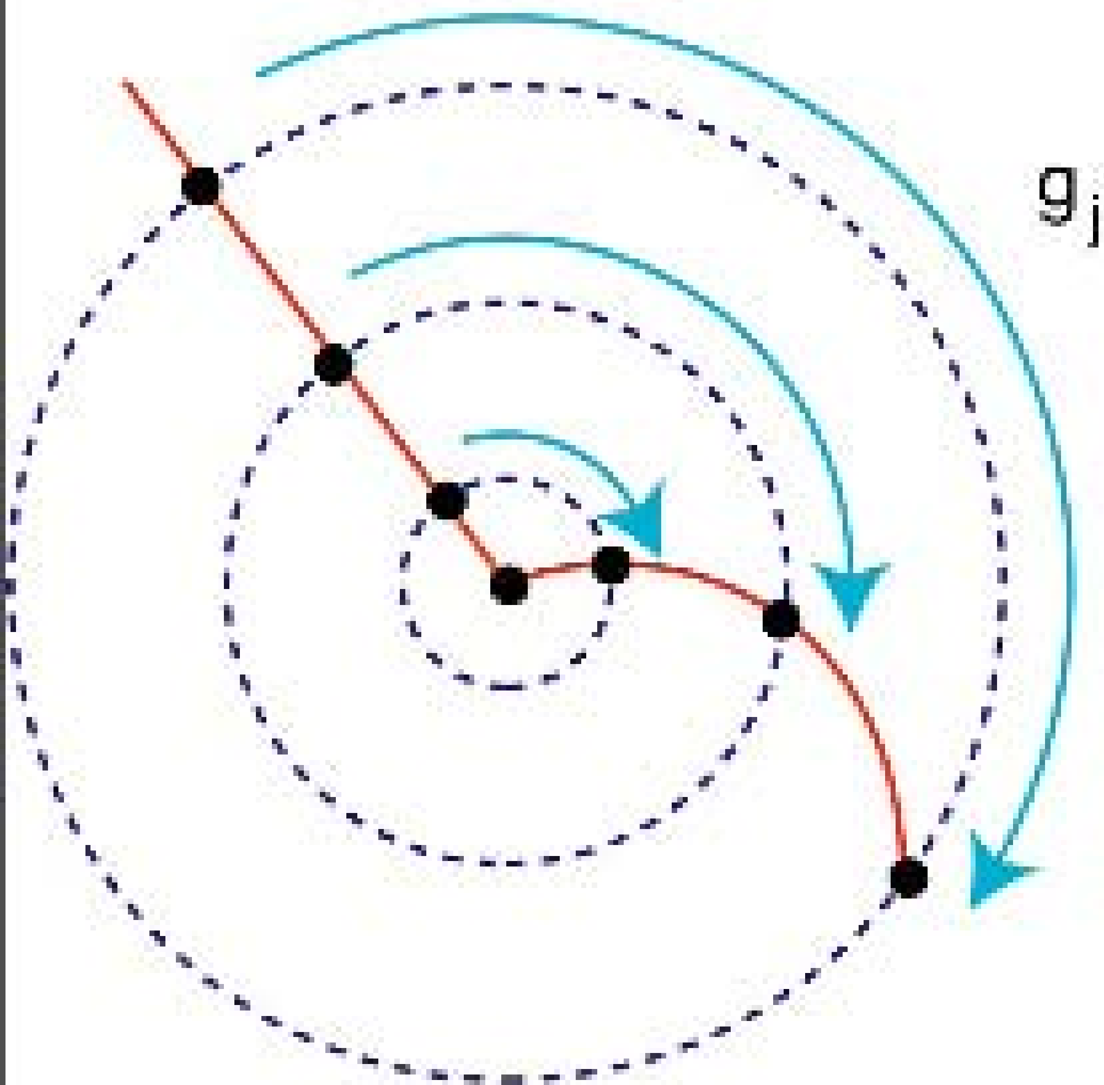
0.06036
0.02605

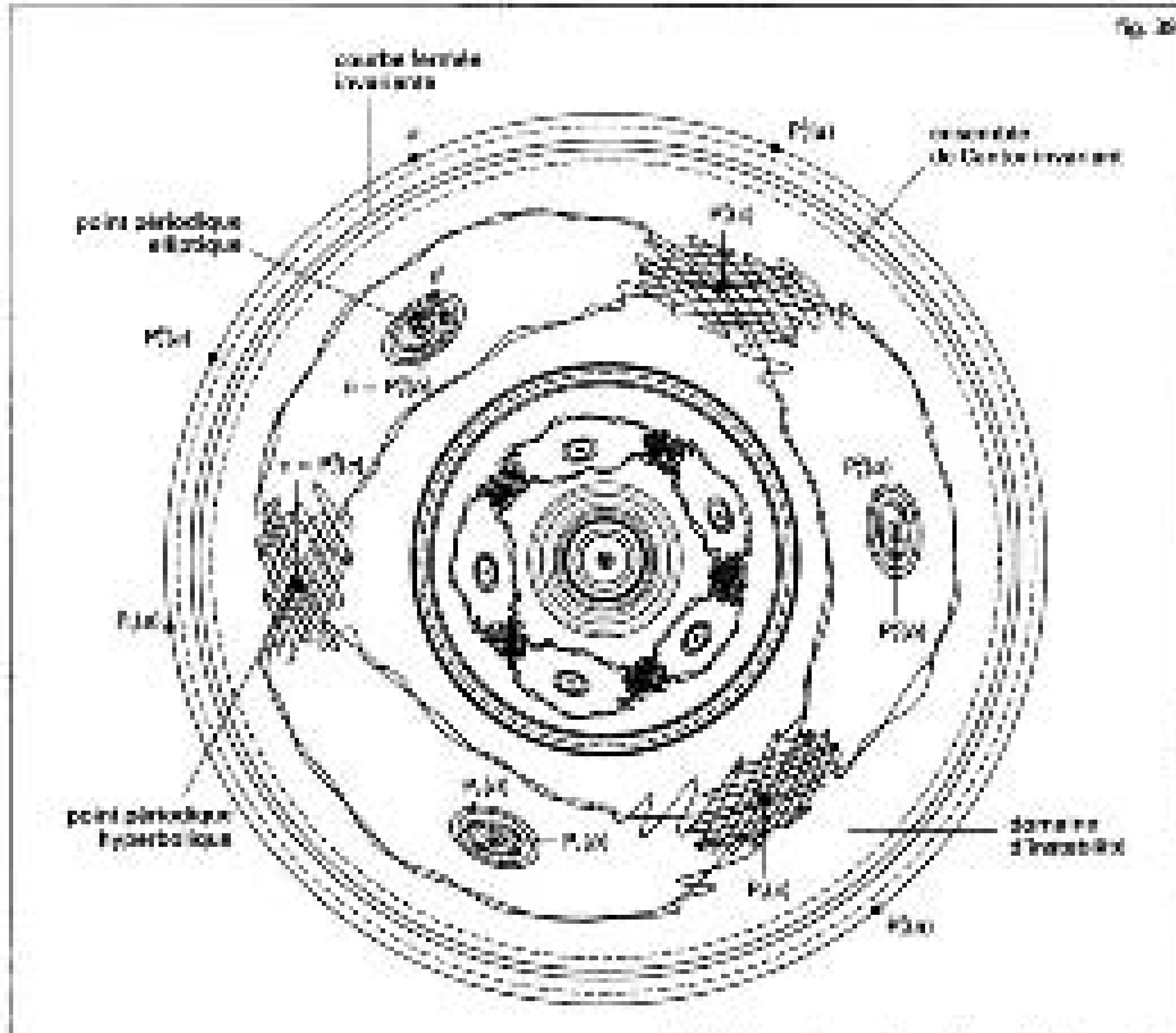
planets


1 kyr
(c) ASD/IMCCE-CNRS

Laplace, Lagrange, Le Verrier

Poincaré, KAM





(A. Chenciner: Encyclop. Universalis)

On sera frappé par la complexité de cette figure,
que je ne cherche même pas à tracer (Poincaré, 1899, MN, tII)

KAM Rigorous results

Arnold (1963) : 2-planets, planar, a/a' -> 0

Robutel (1995) : extension to 2 planets, to spatial case, larger values of a, a'

Herman, Fejoz (2004) : n planets, spatial case,

In all cases : very small values of the masses, eccentricities and inclinations

Orbital solutions for the Solar System planets

analytical

Lagrange, 1774	6 planets, analytical, deg 1, order 1
Le Verrier, 1856	7 planets, analytical, deg 1, order 1
Stockwell, 1873 ; Harzer, 1895	Le Verrier + Neptune
+Hill, 1897 Brouwer & Van Woerkom, 1950	Le Verrier + order 2 terms from Hill
Bretagnon, 1974	8 planets, analytical, deg 3, order 2
Laskar, 1988, 1990	analytical averaging deg 5, ord 2 numerical integ 200 Myr
NUMERICAL INTEGRATIONS	

Moyennisation d'ordre 2

$$H'_0 = H_0$$

$$H'_1 = \{W_1, H_0\} + H_1$$

$$H'_2 = \{W_2, H_0\} + \frac{1}{2}\{W_1, \{W_1, H_0\}\} + \{W_1, H_1\}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\{f, g\} = \sum_{j=1}^{n} \frac{\partial g}{\partial J_j}\frac{\partial f}{\partial \phi_j} - \frac{\partial g}{\partial \phi_j}\frac{\partial f}{\partial J_j}$$

Explosion du nombre de termes

8 planètes,

32 variables $x_i, \bar{x}_i, y_i, \bar{y}_i$ + 8 angles λ_i

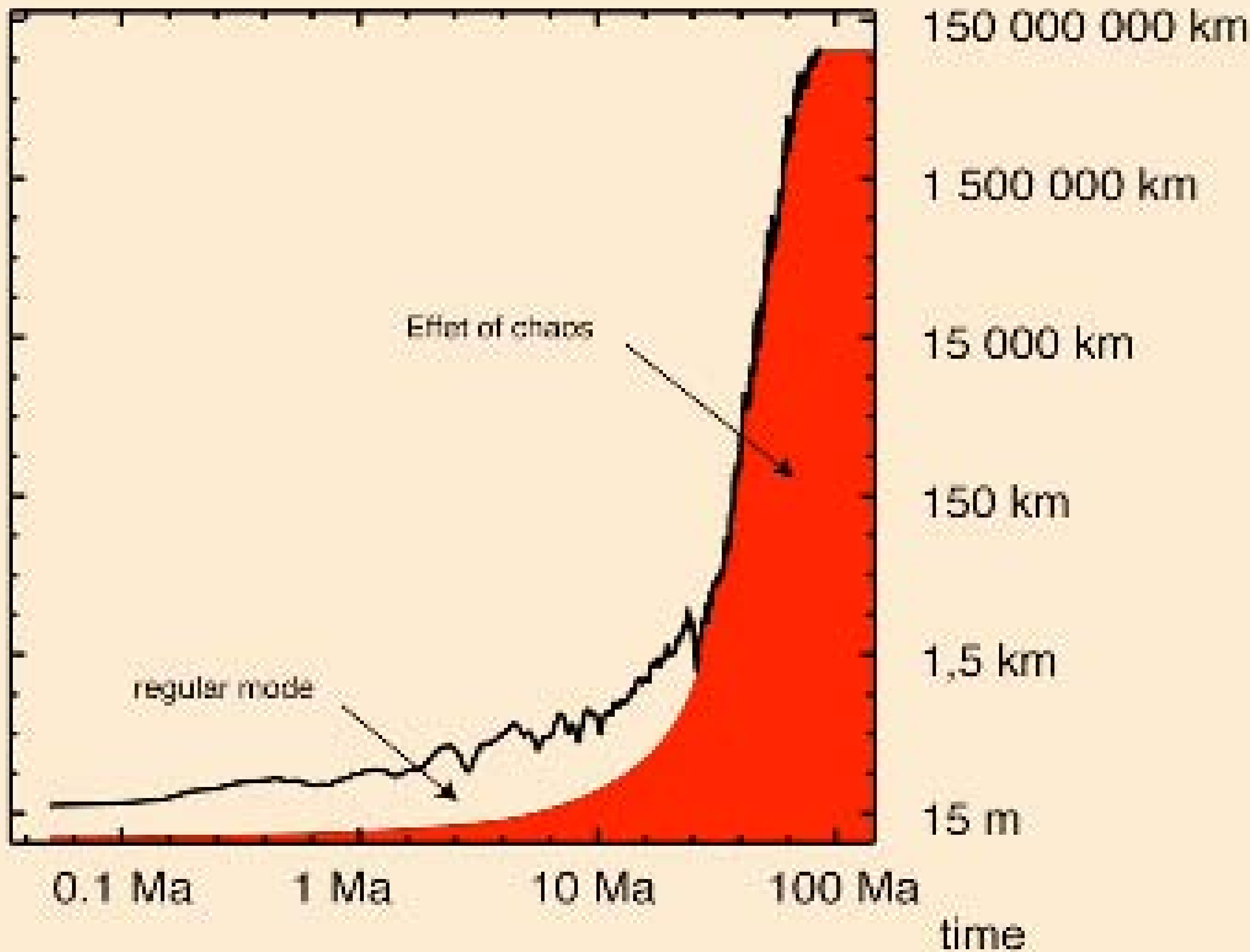
Système moyen d'ordre 2 au degré 5 (6 dans H)

153 824 termes $\alpha x^k \bar{x}^{k'} y^l \bar{y}^{l'}$ (Laskar, 1985-90)

Chaotic motion of the Solar System

Secular equations : 200 Ma : Laskar (1989,1990)

Direct integration : 100 Ma : Sussman and Wisdom (1992)



$$d(T) \approx d_0 10^{T/10}$$

Numerical Integrations

Newton equations
+ relativity
(model ~DE405)

INPOP (2006-8)
Adams integrator
h=0.055 days
7000 yr / 1 day

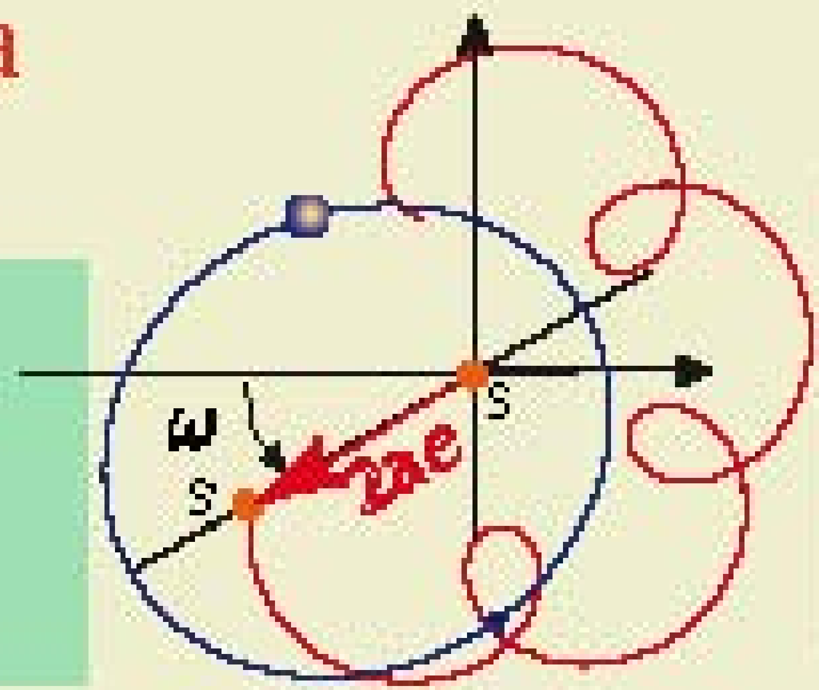
Newton equations
+ relativity
(simplified model)

La2004 (2004)
symplectic integrator
h=1.8625 days
5 M yr / 1 day

computer algebra

↓

secular equations



150 000 polynomial terms
Adams integrator (1989)
h= 250 yr
5 Gyr / 1 day

Short time integrations
T < 1Myr

References integations directly adjusted to observation (45 000 planetary obs)

JPL (NASA) integrations for 30 yrs : DE405
Russian solution (Pitjeva, 2001-5)

New solution (2008, 9) : INPOP06-8
(Fienga, Manche, Laskar, Gastineau et al ..)

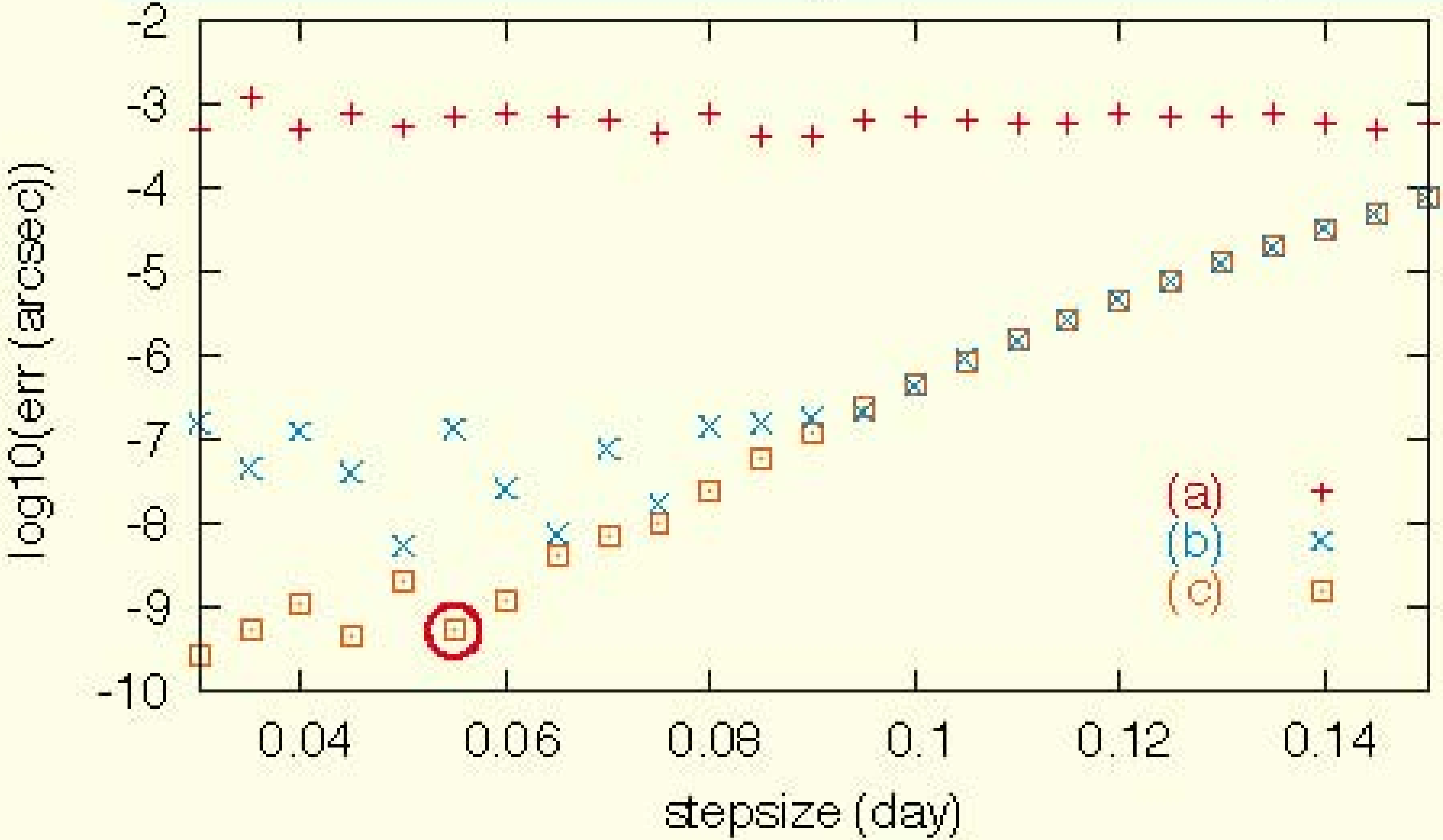
Interactions in La2004, LaX, DE405, INPOP

- Newtonian (planets ↔ planets, asteroïds ↔ asteroïds (5), planets ↔ asteroïds (300))
- Relativistic corrections (planets, asteroïds)
- Non-spherical body ↔ point mass
- Sun (J_2) ↔ Planets
- Earth (J2, J3, J4)↔ (Moon, Sun, Venus, Jupiter)
- Moon (J2, J3, J4, Cnm, Snm, n:2,4; m:1,n) ↔ (Earth, Sun, Venus, Jupiter)
- Deformation of extended bodies (tides) ↔ point mass
- Earth (Sun, Moon) ↔ (Moon, Sun, Venus, Jupiter)
- Moon (Spin,Earth, Sun) ↔ (Earth, Sun, Venus, Jupiter)
- Earth Shape ↔ Moon shape (torque exerted by the Moon)

The numerical integrator

Adams PECE integrator h = 0.055 day
80 bits arithmetic on itanium II
Quad prec for the corrector step (1 addition)

INPOP : Adams PECE order 12
Error in the Moon longitude after 100 years



(a) + double precision 64 bits
(b) × extended precision 80 bits
(c) ⊡ extended precision 80 bits + 1 quad. prec. addition

INPOP06

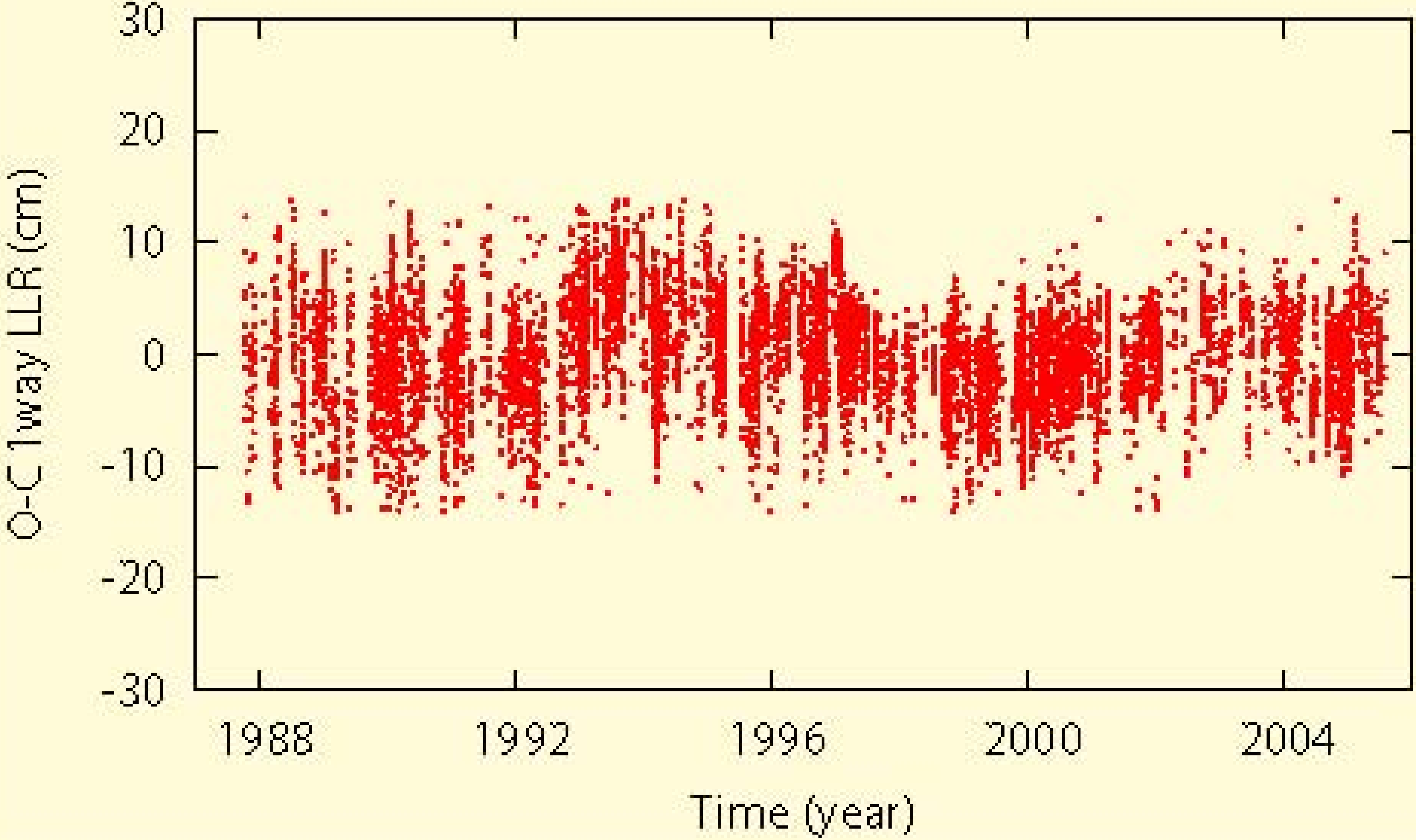
Error in position after 100 or 10000 years

	100 yr (micro m)	10000 yr (mm)
Mercury	93.3	41.29
Venus	7.5	4.90
EMB	14.0	5.34
Mars	3.4	0.46
Jupiter	0.6	0.04
Saturn	0.2	0.04
Uranus	5.5	0.02
Neptune	3.1	0.03
Pluto	2.2	0.04
Moon	1.0	2.51

INPOP07 LLR residuals Grasse

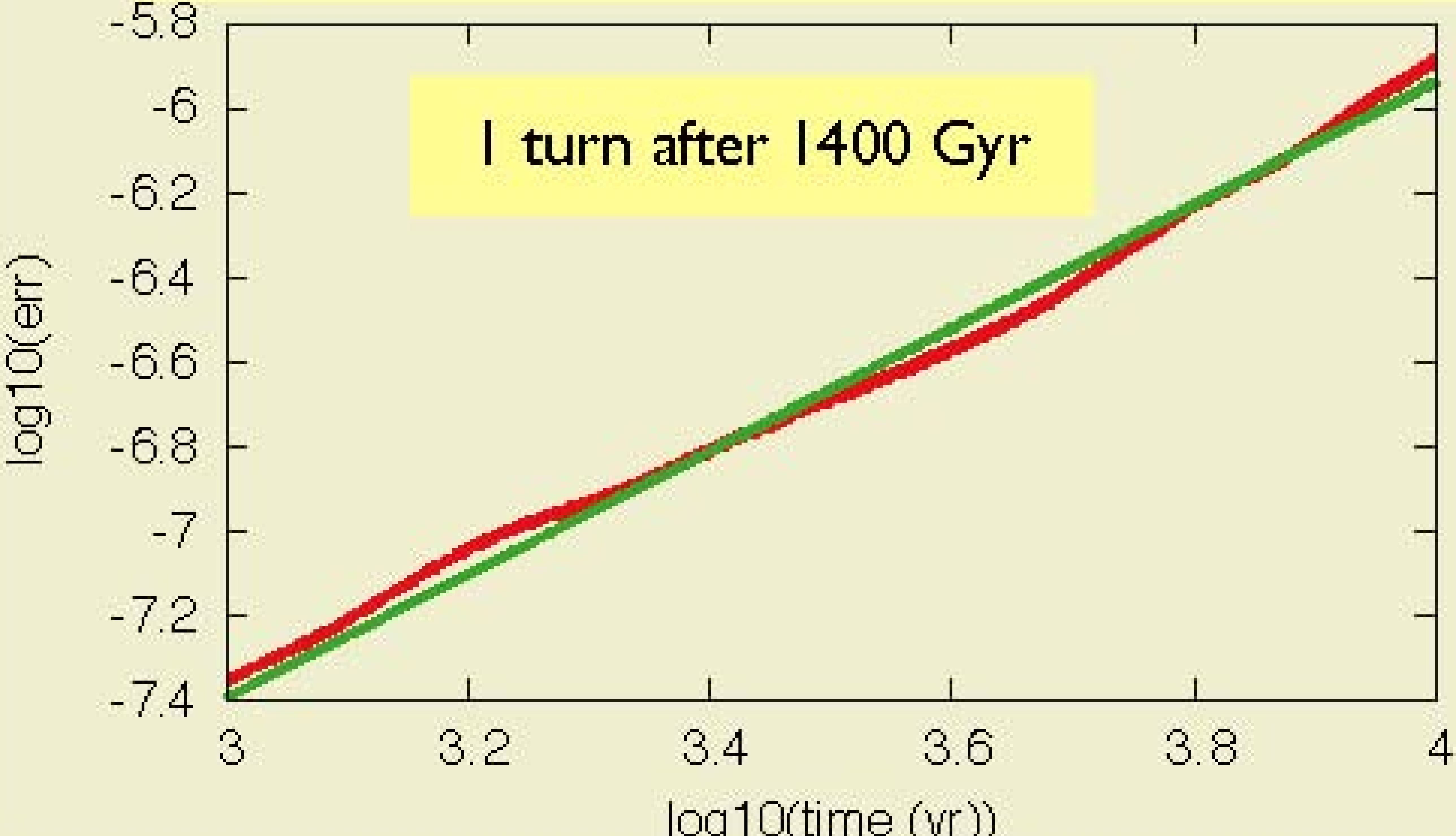
(H. Manche, IMCCE, S. Bouquillon, SYRTE)

8262 pts moy: 0.011 cm sig: 4.64 cm (4.4 cm)



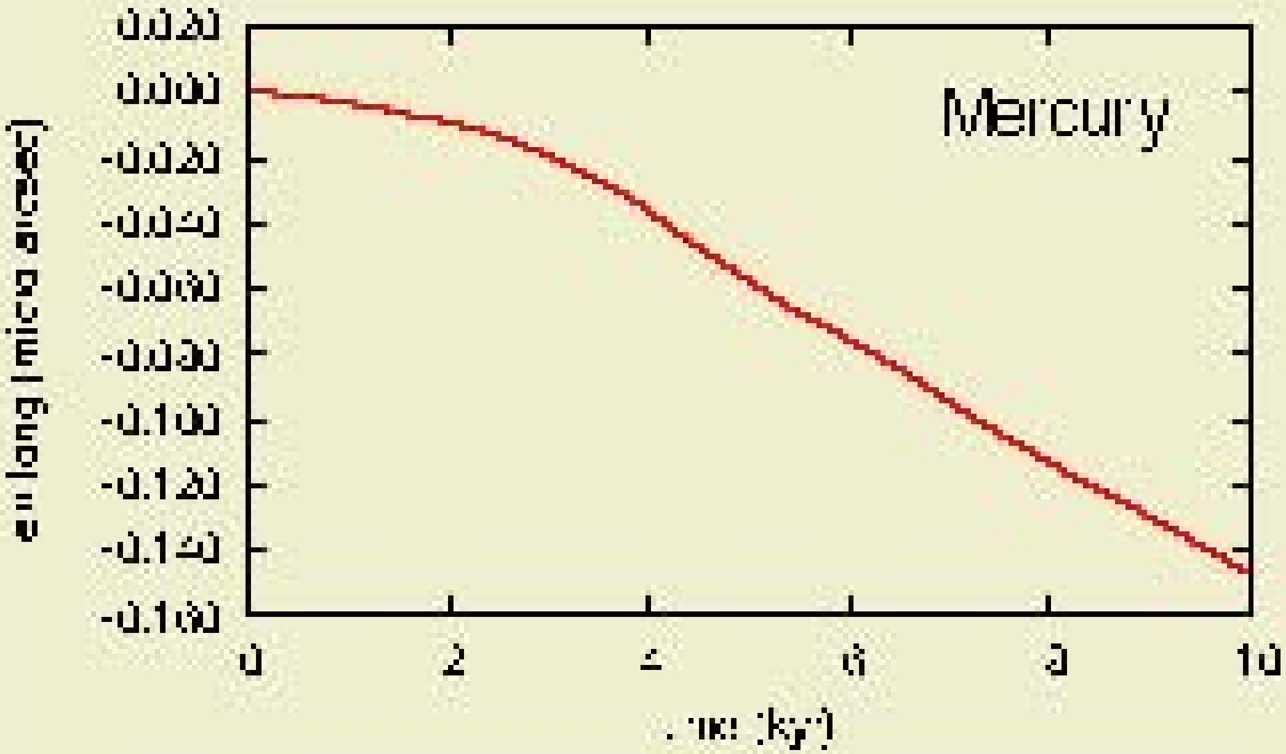
Longitude of the Moon

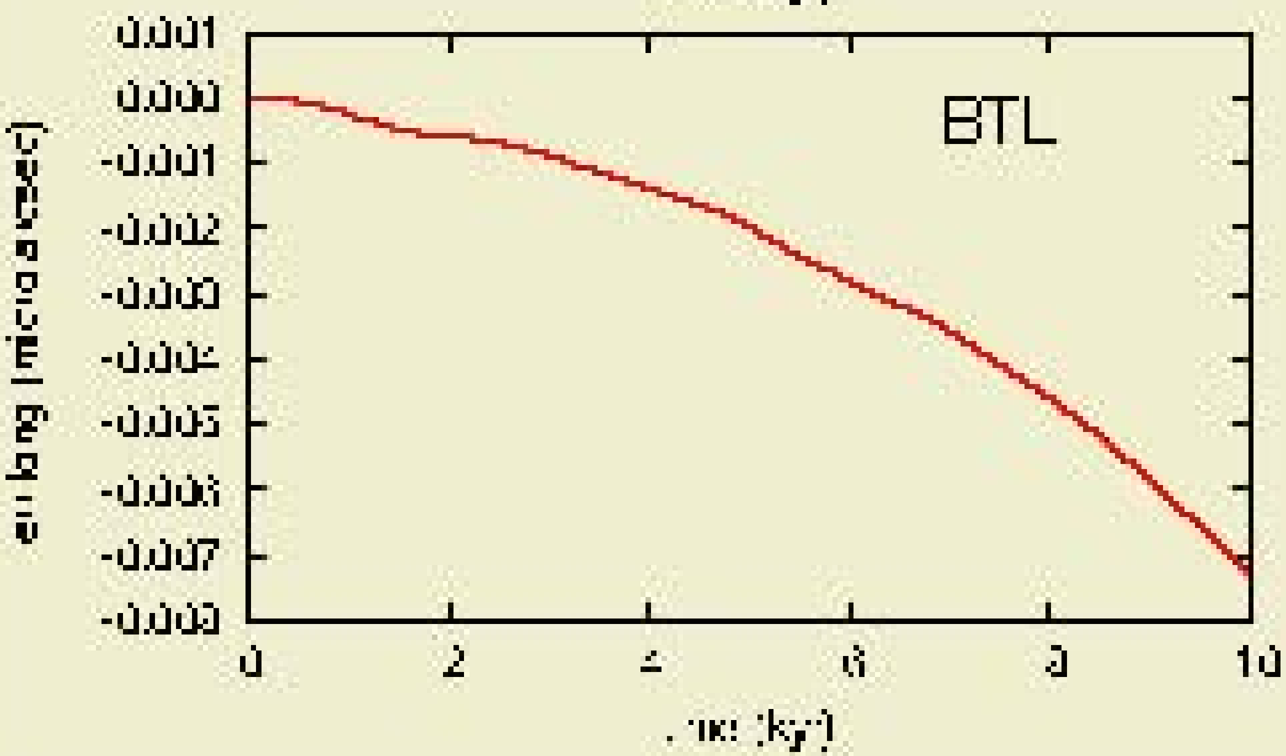
$$\delta\lambda('') = 2.3 \times 10^{-12} \times T^{1.46}; \quad T(yr)$$



numerical errors

$$\delta\lambda = \alpha \times T^2$$

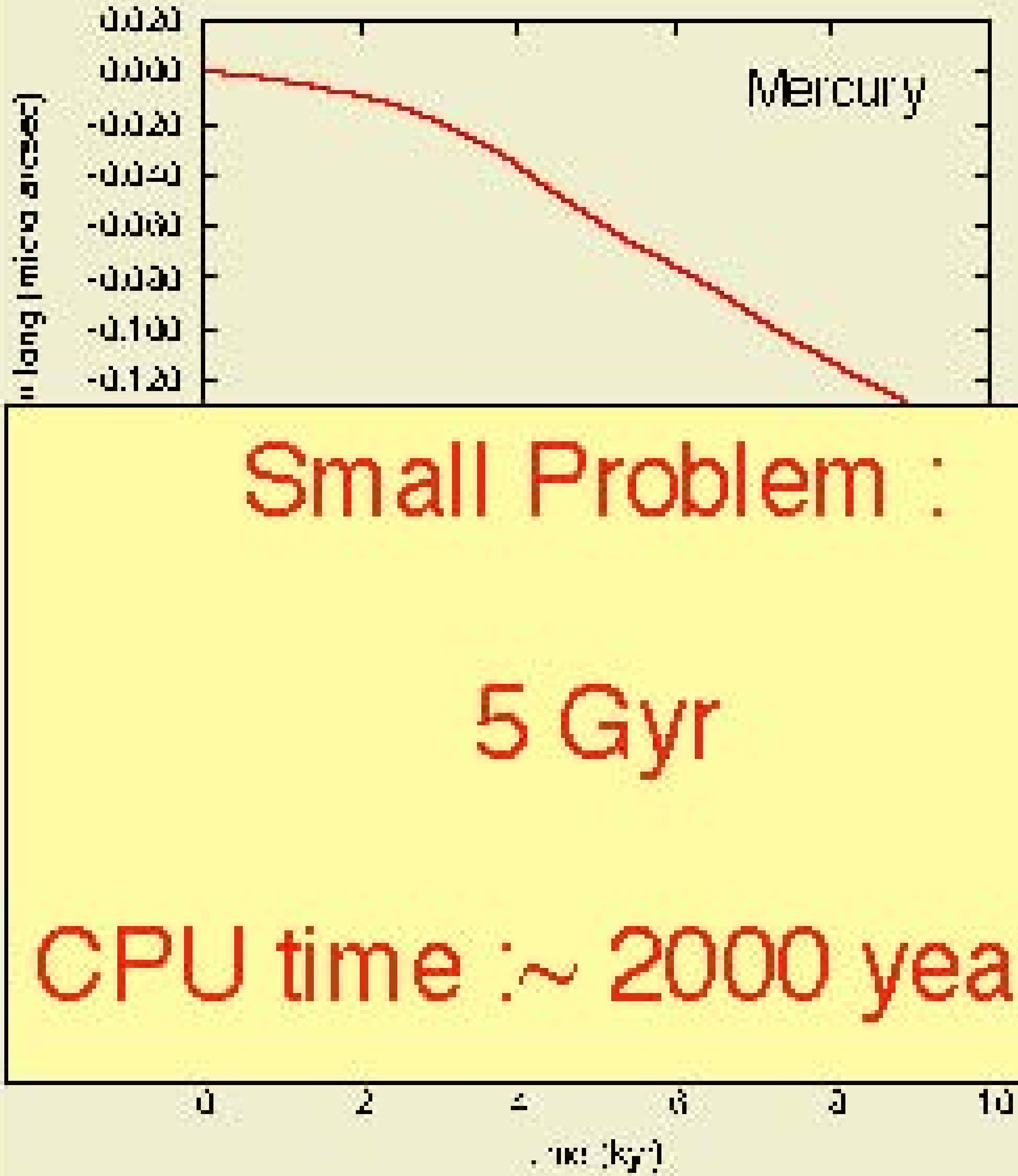




	(Gyr)
Mercury	30
Venus	120
EMB	130
Mars	560
Jupiter	3800
Saturn	5000
Uranus	9000
Neptune	10000
Pluton	10000
Moon	10

numerical errors

$$\delta\lambda = \alpha \times T^2$$



Small Problem :

5 Gyr

CPU time :~ 2000 years !

	(Gyr)
Mercury	30
Venus	120
EMB	130
Mars	560
Jupiter	3800
Saturn	5000
Uranus	9000
Neptune	10000
Pluton	10000
Moon	10

Long time integrations

1 Myr < T < 250 Myr

Numerical Integration

Complete Solar System (Moon + relativity)

1983	1991	1992	2003	2004
Newhall et al. (DE102)	Quinn, Tremaine, Duncan	Sussman, Wisdom	Varadi et al. (R7)	Laskar et al.
0.004 Myr	3 Myr	100 Myr	100 Myr	250 Myr
Moon Relativity	Av Moon Relativity (S)	Av Moon Av Relativity	Av Moon Relativity (S)	Moon Relativity (S, E)
Adams-Cowell < 14	Symmetric (13)	Symplectic (2)	Stormer (14)	Symplectic (4+8)
	0.75 d	7.2 d	0.3125 d	1.8625 d
	~ 65 d	~ 40 d	~ 100 d	~ 50 d

Integrateur symplectique

$$H = A(p) + B(q)$$

But $e^{\tau L_A} e^{\tau L_B} \neq e^{\tau(L_A + L_B)}$

Campbell-Baker-Hausdorf (CBH) formulas

$e^{\tau L_A} e^{\tau L_B} = e^{\tau(L_K)}$ with

$$K = A + B + \frac{\tau}{2}\{A, B\} + \frac{\tau^2}{12}\{A, \{A, B\}\} + \frac{\tau^2}{12}\{\{A, B\}, B\}$$
$$+ \frac{\tau^3}{24}\{A, \{\{A, B\}, B\}\}$$
$$- \frac{\tau^4}{720}\{A, \{A, \{A, \{A, B\}\}\}\} + \frac{\tau^4}{180}\{A, \{A, \{\{A, B\}, B\}\}\}$$
$$+ \frac{\tau^4}{360}\{\{A, \{A, B\}\}, \{A, B\}\} + \frac{\tau^4}{180}\{A, \{\{\{A, B\}, B\}, B\}\}$$
$$+ \frac{\tau^4}{120}\{\{A, B\}, \{\{A, B\}, B\}\} - \frac{\tau^4}{720}\{\{\{\{A, B\}, B\}, B\}, B\} + O(\tau^5)$$

Higher orders

$$S_n(\tau) = e^{c_1 \tau L_A} e^{d_1 \tau L_B} \cdots e^{c_n \tau L_A} e^{d_n \tau L_B} = e^{\tau L_{K(\tau)}}$$

$$\begin{aligned} K(\tau) = & \; k_{1,1} A + k_{1,2} B + \tau k_{2,1} \{A, B\} \\ & + \tau^2 k_{3,1} \{A, \{A, B\}\} + \tau^2 k_{3,2} \{\{A, B\}, B\} \\ & + \tau^3 k_{4,1} \{A, \{A, \{A, B\}\}\} + \tau^3 k_{4,2} \{A, \{\{A, B\}, B\}\} \\ & + \tau^3 k_{4,3} \{\{\{A, B\}, B\}, B\} + O(\tau^4) \end{aligned}$$

order p : $K(\tau) = A + B + O(\tau^p)$.

Set of algebraic equations

$k_{1,1} = c_1 + c_2 + \cdots c_n = 1$, $k_{1,2} = d_1 + d_2 + \cdots d_n = 1$, for $p \geq 1$.

$$k_{i,j} = 0 \qquad \text{for} \qquad (2 \leq i \leq p) \; .$$

Mouvements planétaires

$$H = A(p,q) + \varepsilon B(q)$$

$$H = A(a) + \varepsilon B(a, \lambda, e, \varpi, i, \Omega)$$

Keplerian motion

planetary interactions

$$(p,q) \longrightarrow (a, \lambda, e, \varpi, i, \Omega)$$

λ : fast angle

(Wisdom & Holman, 1991
Kinoshita, Yoshida, Nakai, 1991)

Perturbed Hamiltonian $H = A + \varepsilon B$

$$S_n(\tau) = e^{c_1 \tau L_A} e^{d_1 \tau \varepsilon L_B} \dots e^{c_n \tau L_A} e^{d_n \tau \varepsilon L_B} = e^{\tau L_{K(\tau)}}$$

$$K(\tau) = A + \varepsilon B + \tau^2 \varepsilon k_{3,1} \{A, \{A, B\}\} + \tau^2 \varepsilon^2 k_{3,2} \{\{A, B\}, B\} + O(\tau^4 \varepsilon)$$

Kill only terms in $\tau^n \varepsilon$!

$$K(\tau) = A + \varepsilon B + O(\tau^n \varepsilon) + O(\tau^2 \varepsilon^2)$$

symplectic integrator SABA(C)4 (Laskar & Robutel, 2001)
(McLachlan, 1995)

$$S_0(\tau) = \mathrm{e}^{\tau c_1 L_A} \mathrm{e}^{\tau d_1 L_B} \mathrm{e}^{\tau c_2 L_A} \mathrm{e}^{\tau d_2 L_B} \mathrm{e}^{\tau c_3 L_A} \times \mathrm{e}^{\tau d_2 L_B} \mathrm{e}^{\tau c_2 L_A} \mathrm{e}^{\tau d_1 L_B} \mathrm{e}^{\tau c_1 L_A}$$

$$\begin{aligned} c_1 &= 1/2 - \sqrt{525 + 70\sqrt{30}}/70 \\ c_2 &= \left(\sqrt{525 + 70\sqrt{30}} - \sqrt{525 - 70\sqrt{30}}\right)/70 \\ c_3 &= \left(\sqrt{525 - 70\sqrt{30}}\right)/35 \\ d_1 &= 1/4 - \sqrt{30}/72 \\ d_2 &= 1/4 + \sqrt{30}/72. \end{aligned}$$

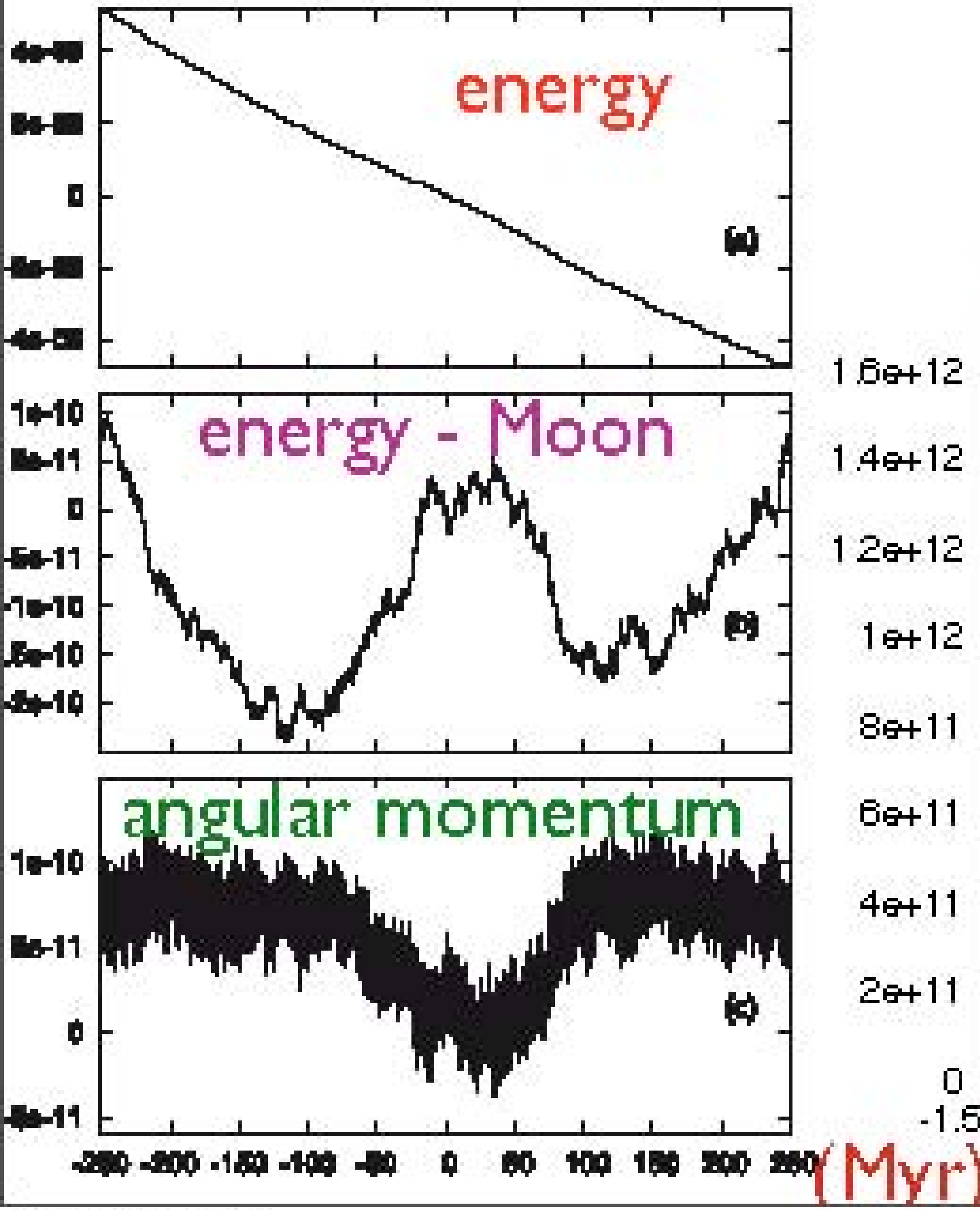
$$H = A + \varepsilon B$$

$$S_1(\tau) = \mathrm{e}^{-\tau^3 \varepsilon^2 b/2 L_C} S_0(\tau) \mathrm{e}^{-\tau^3 \varepsilon^2 b/2 L_C} \qquad (3)$$

where $b = 0.003396775048208601331$ and $C = \{\{A, B\}, B\}$.

$$O(\tau^8 \varepsilon) + O(\tau^{2\,(4)} \varepsilon^2)$$

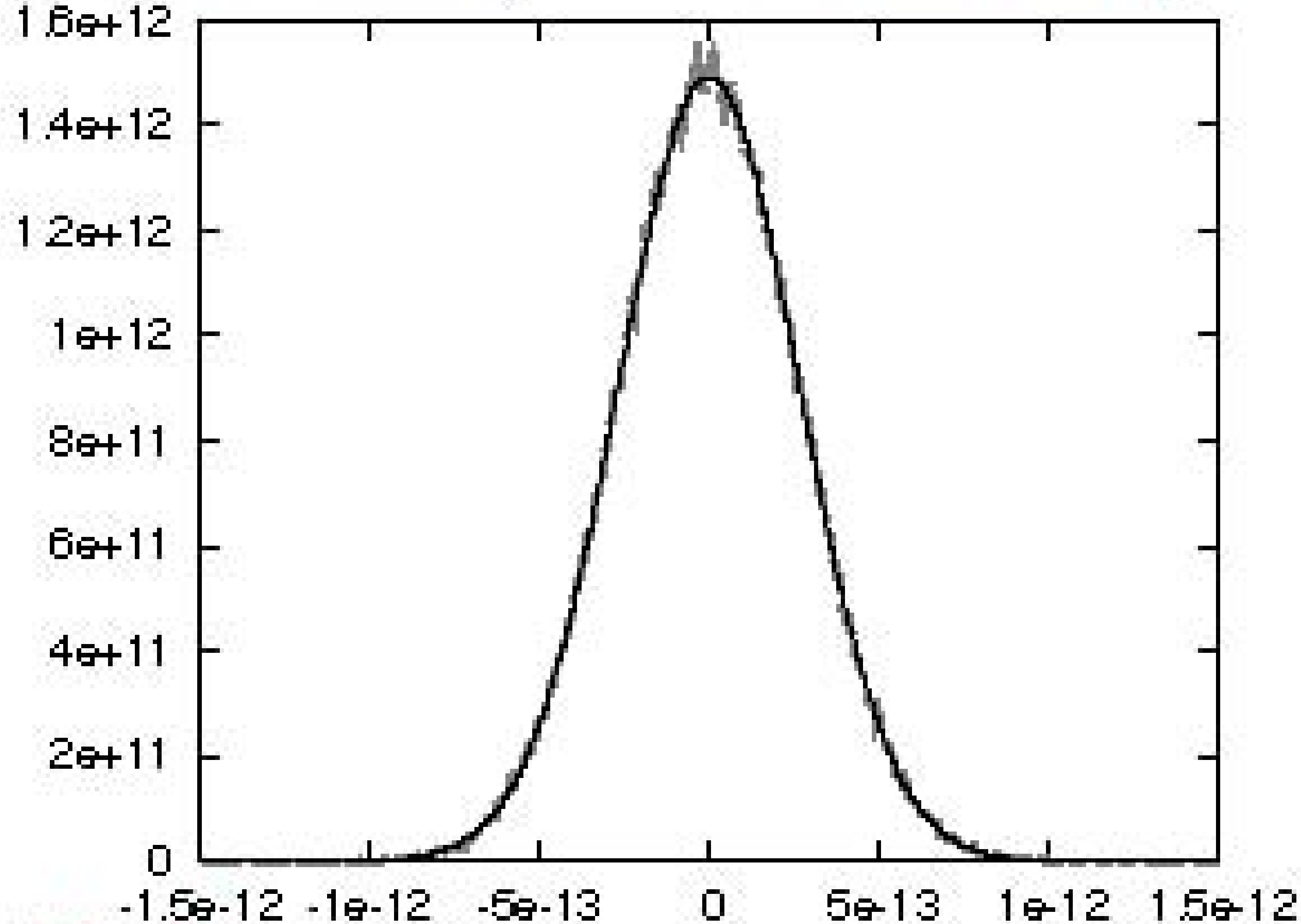
Symplectic integrator SABA(C)4
(McLachlan, 1995, Laskar & Robutel, 2001)



$$O(\tau^8\varepsilon) + O(\tau^{2\,(4)}\varepsilon^2)$$

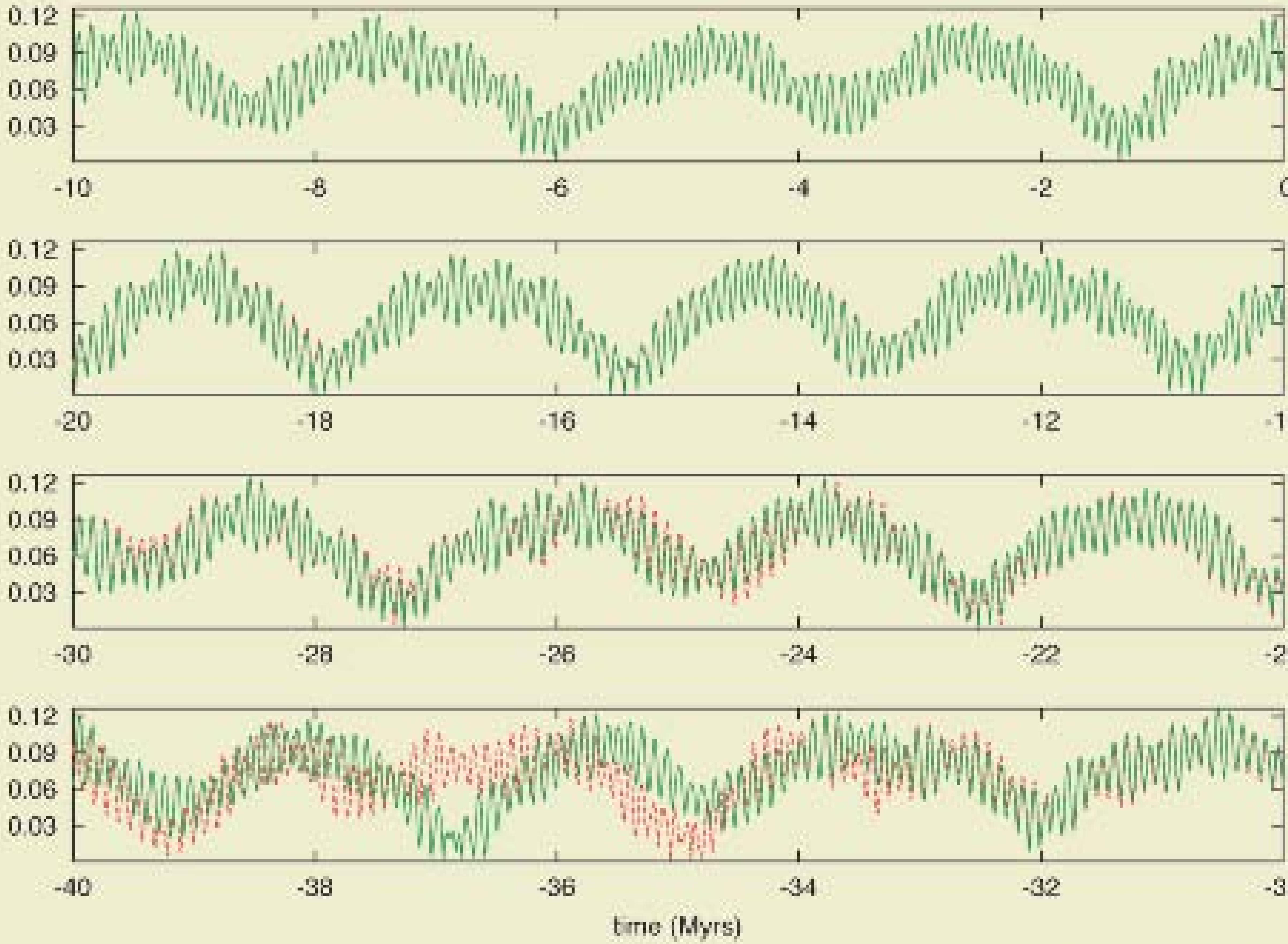
$$2.7\varepsilon_m/step$$

$$0.005\ yr \qquad 87\ elem\ op$$

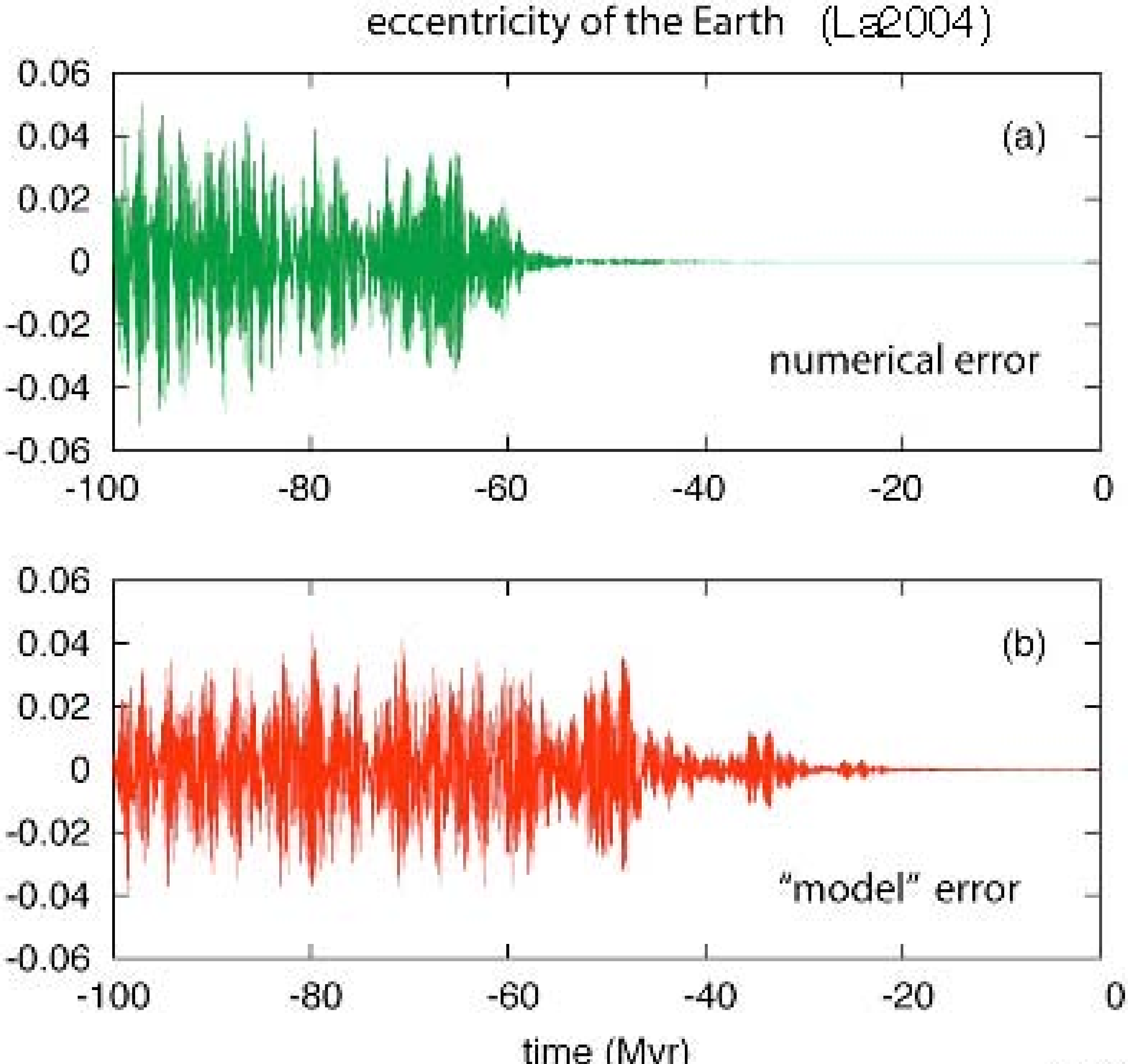


(Laskar et al, 2004)

Mars eccentricity : num2004-sec2004



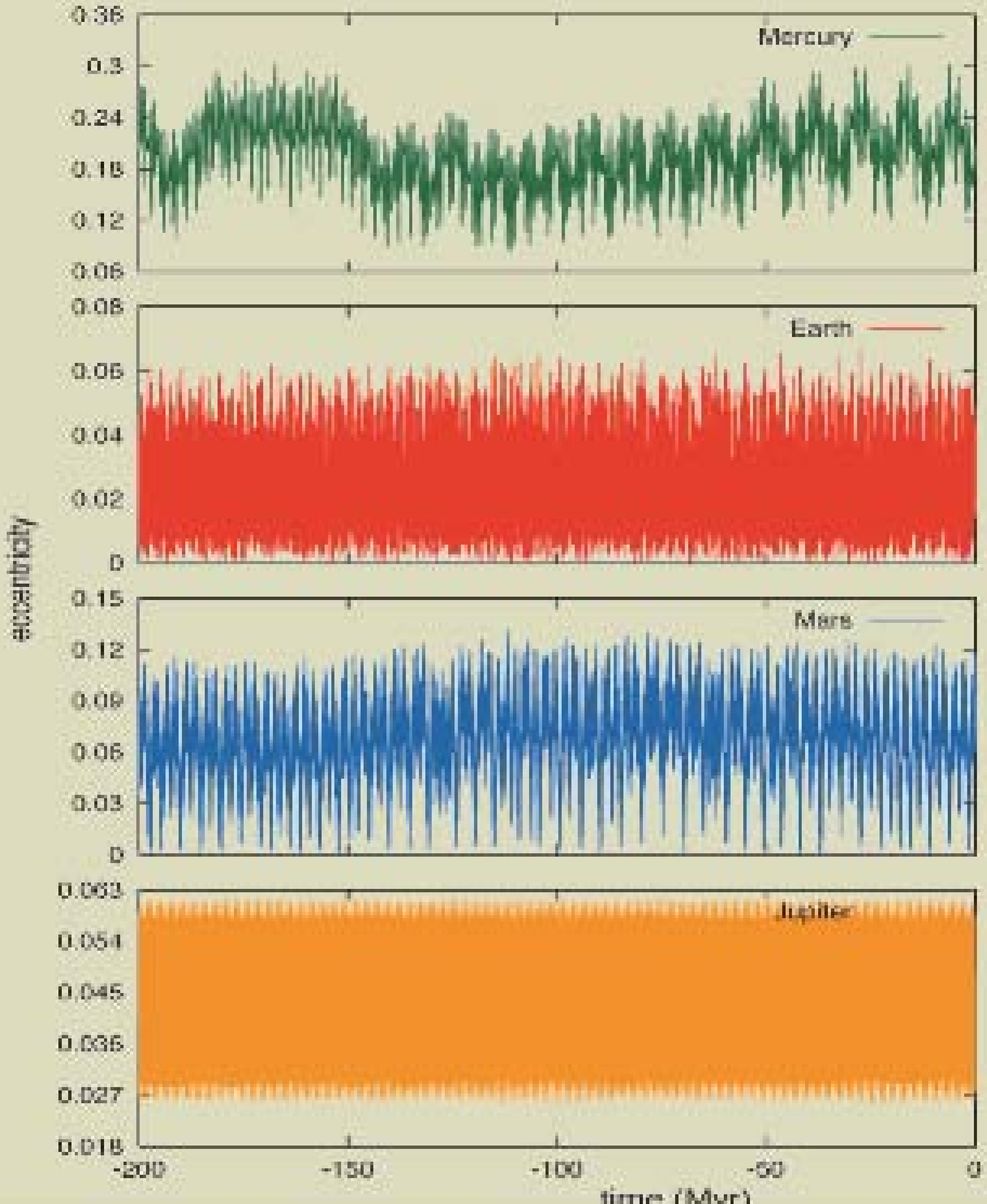
(Laskar et al, 2004)

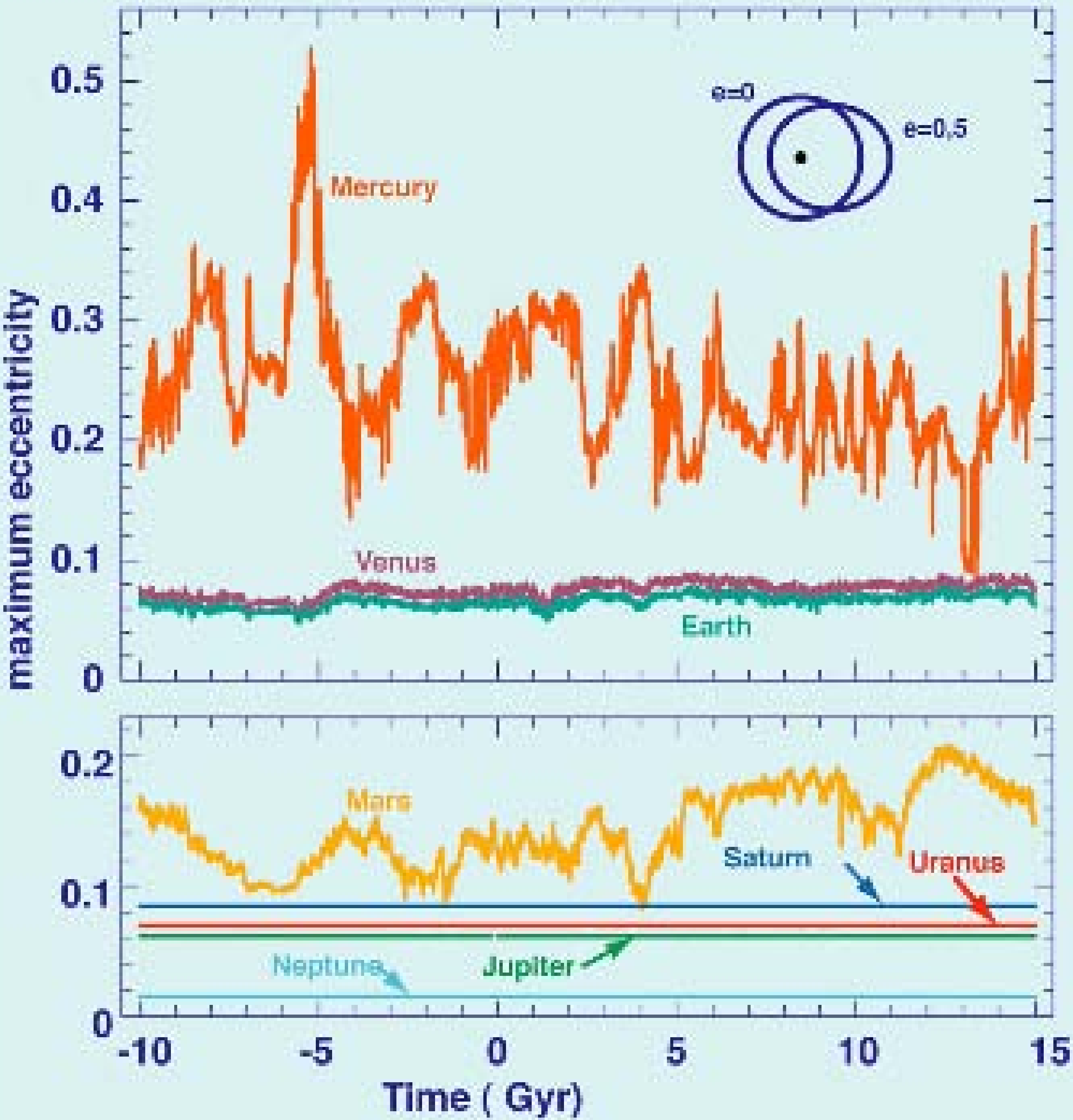


(Laskar et al, 2004)

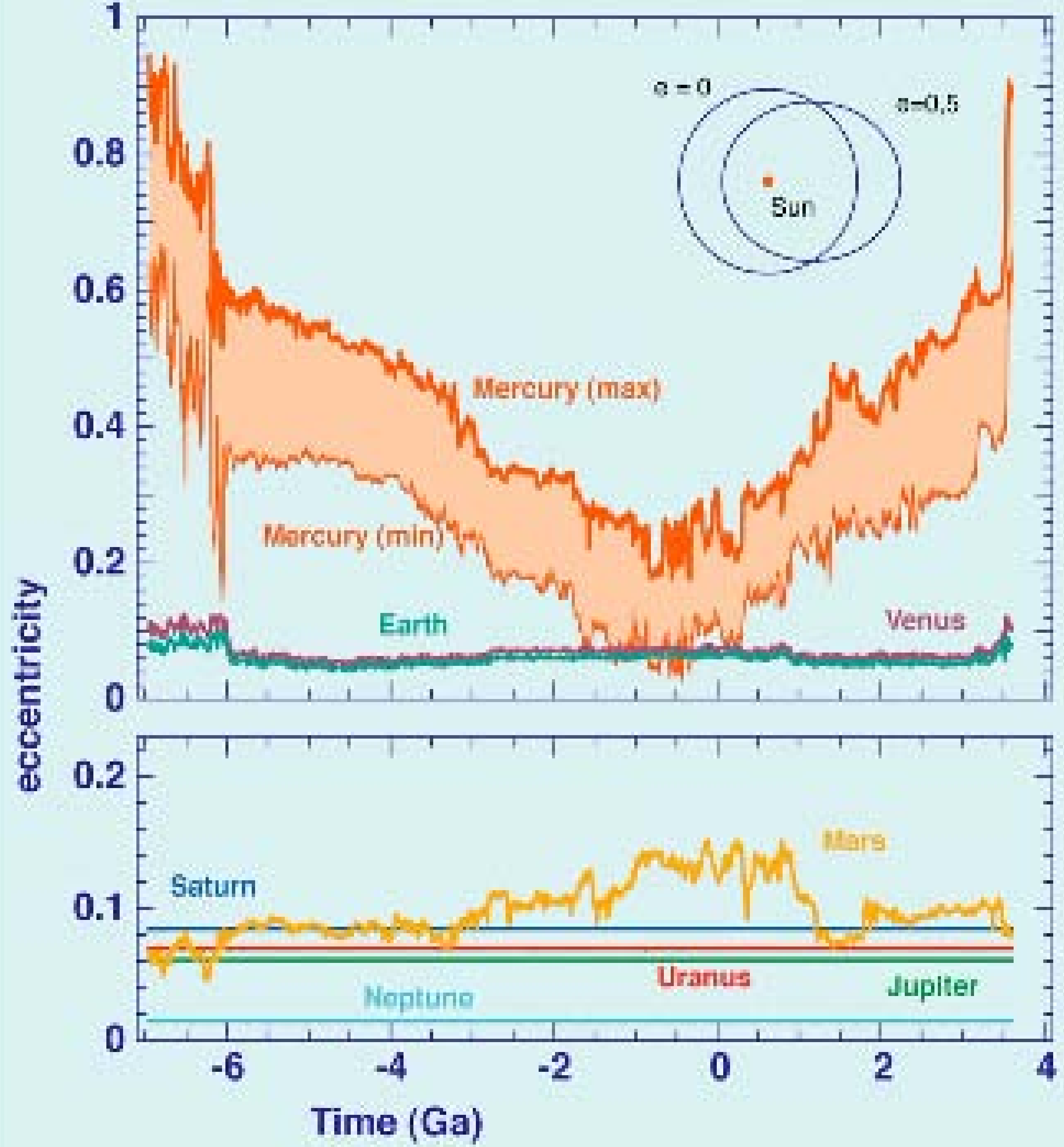
Very long times

T > 250 Myr


Mercury
Earth
Mars
Jupiter
eccentricity
time (Myr)
-200
-150
-100
-50
0


maximum eccentricity
Mercury
e=0
e=0.5
Venus
Earth
Mars
Saturn
Uranus
Neptune
Jupiter
Time (Gyr)
0
0.1
0.2
0.3
0.4
0.5
-10
-5
5
10
15

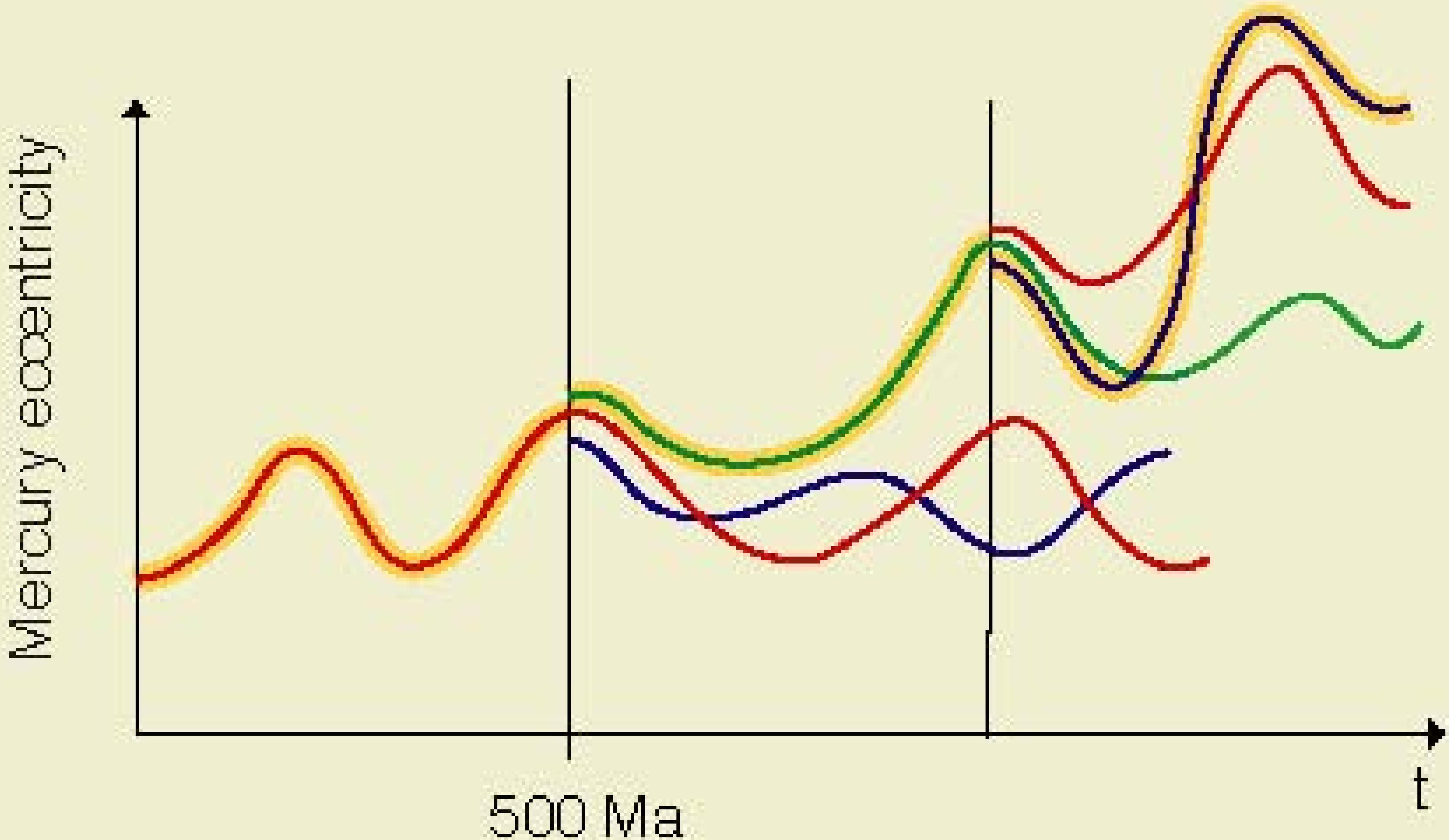
(Laskar, 1994)



e = 0
e=0,5
Sun
Mercury (max)
Mercury (min)
Earth
Venus
Mars
Saturn
Uranus
Jupiter
Neptune
eccentricity
Time (Ga)
1
0.8
0.6
0.4
0.2
0
0.2
0.1
0
-6
-4
-2
0
2
4

Laskar (1994)

shadow orbit for the collision
of Mercury with Venus



$$d_0 = 15m \times 10^{-50}$$

Statistical view

(Laskar, Icarus, 2008)

Numerical experiment

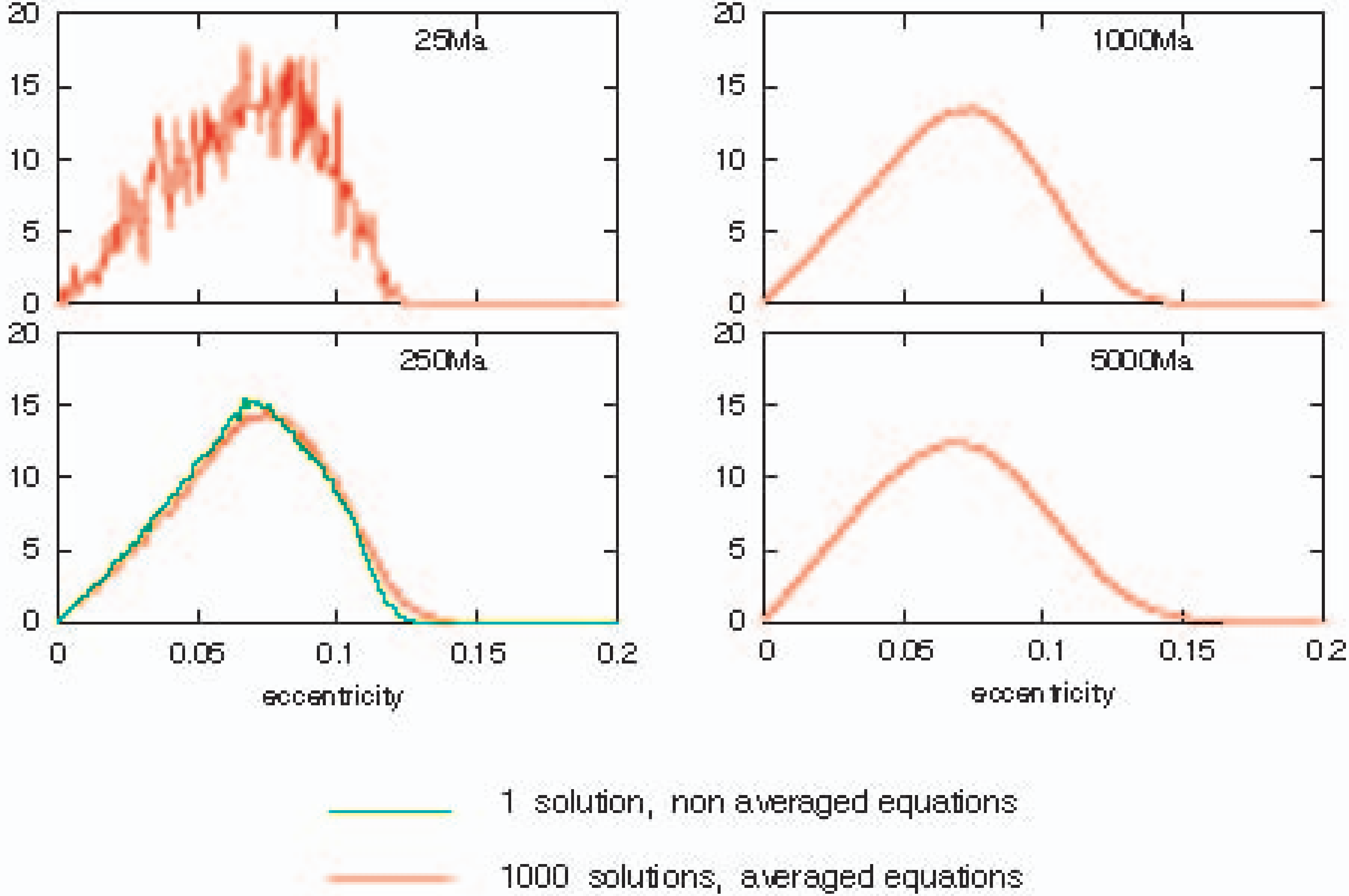
1000 orbits from 0 to -5 Gyr with close initial conditions :

$$-500 \times 10^{-10} \longrightarrow +500 \times 10^{-10}$$

Secular equations (Laskar, 1989,2004)

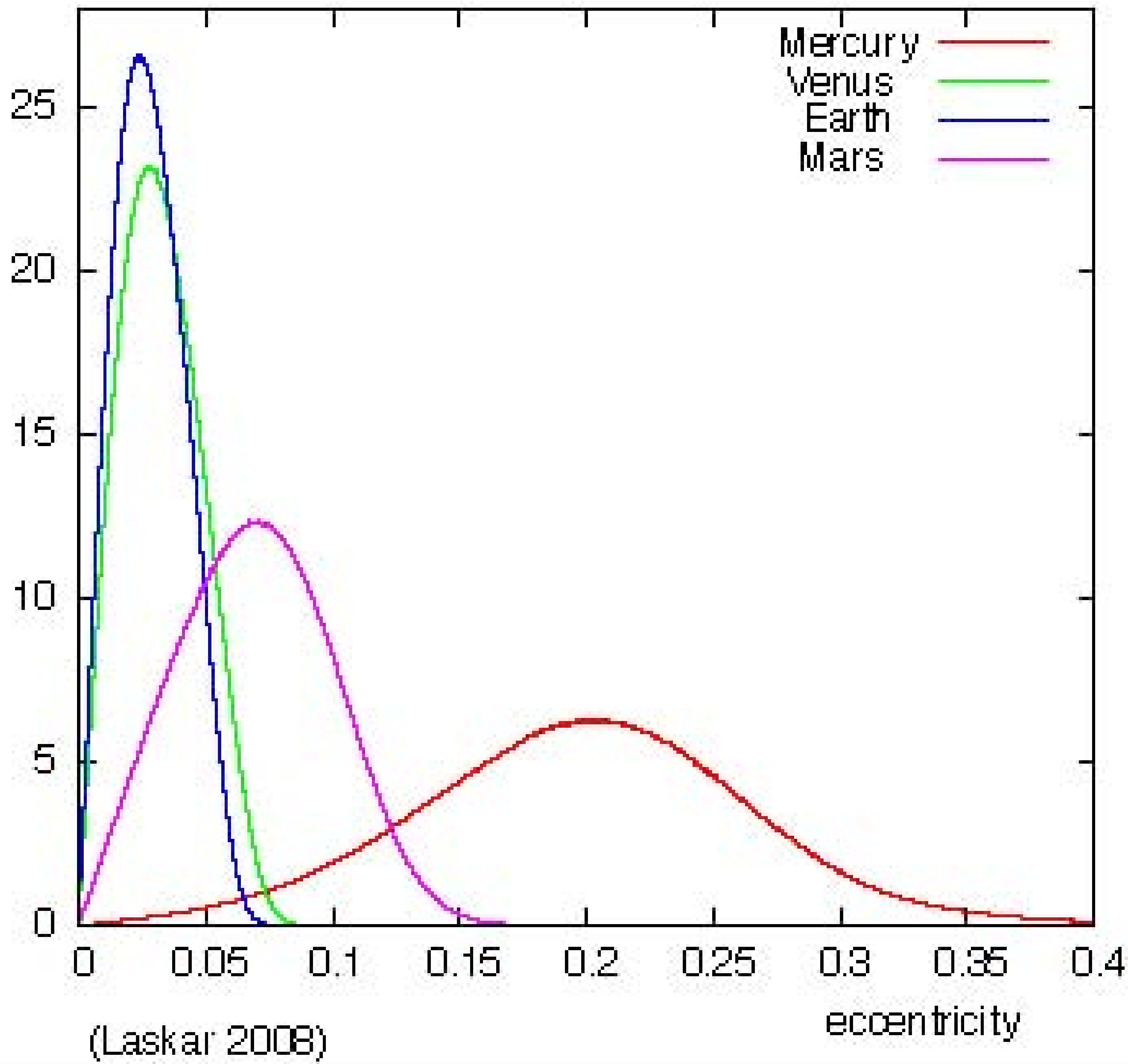
Probability Density Functions

Diffusion of Mars eccentricity

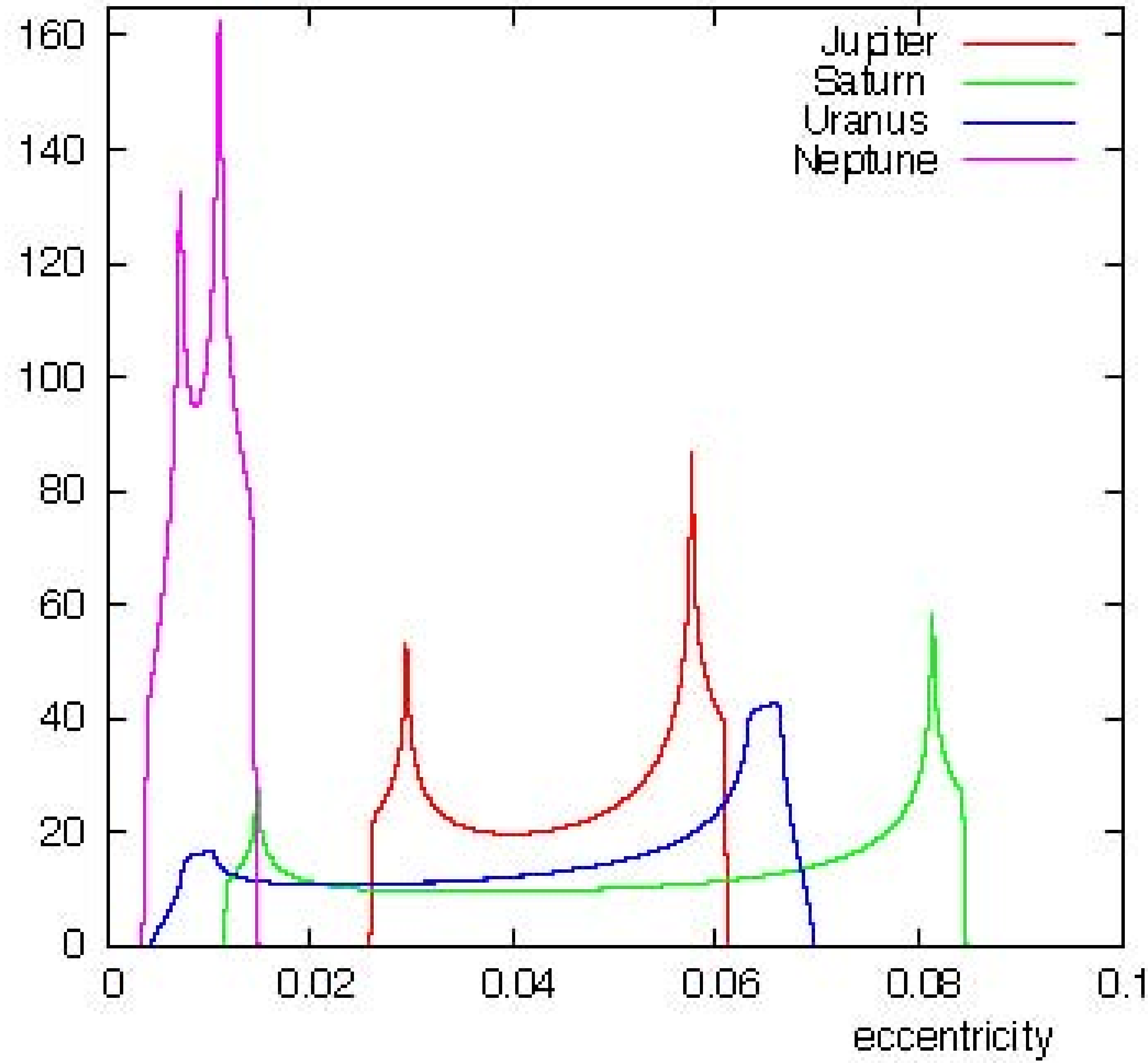


(Laskar et al, 2004)

density function

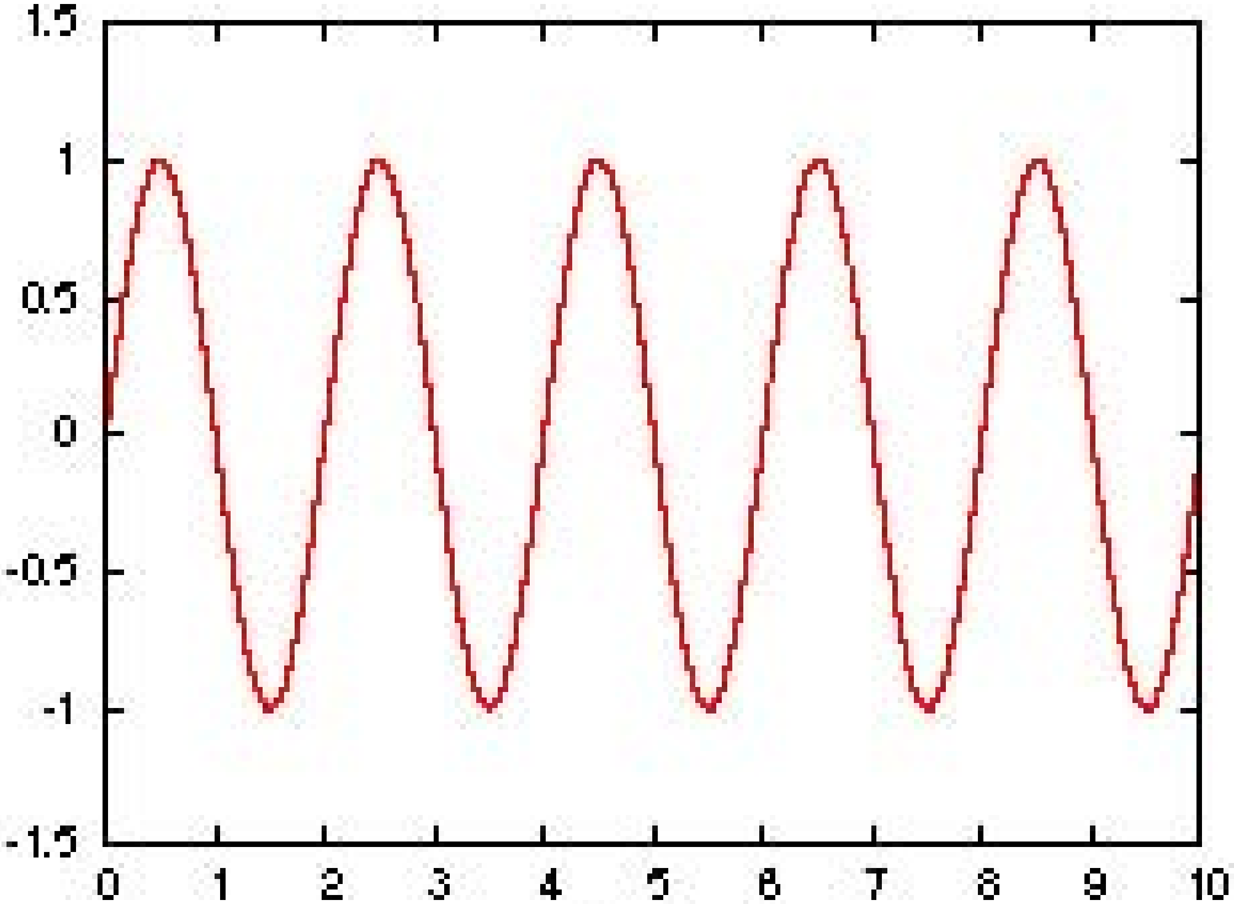


density function



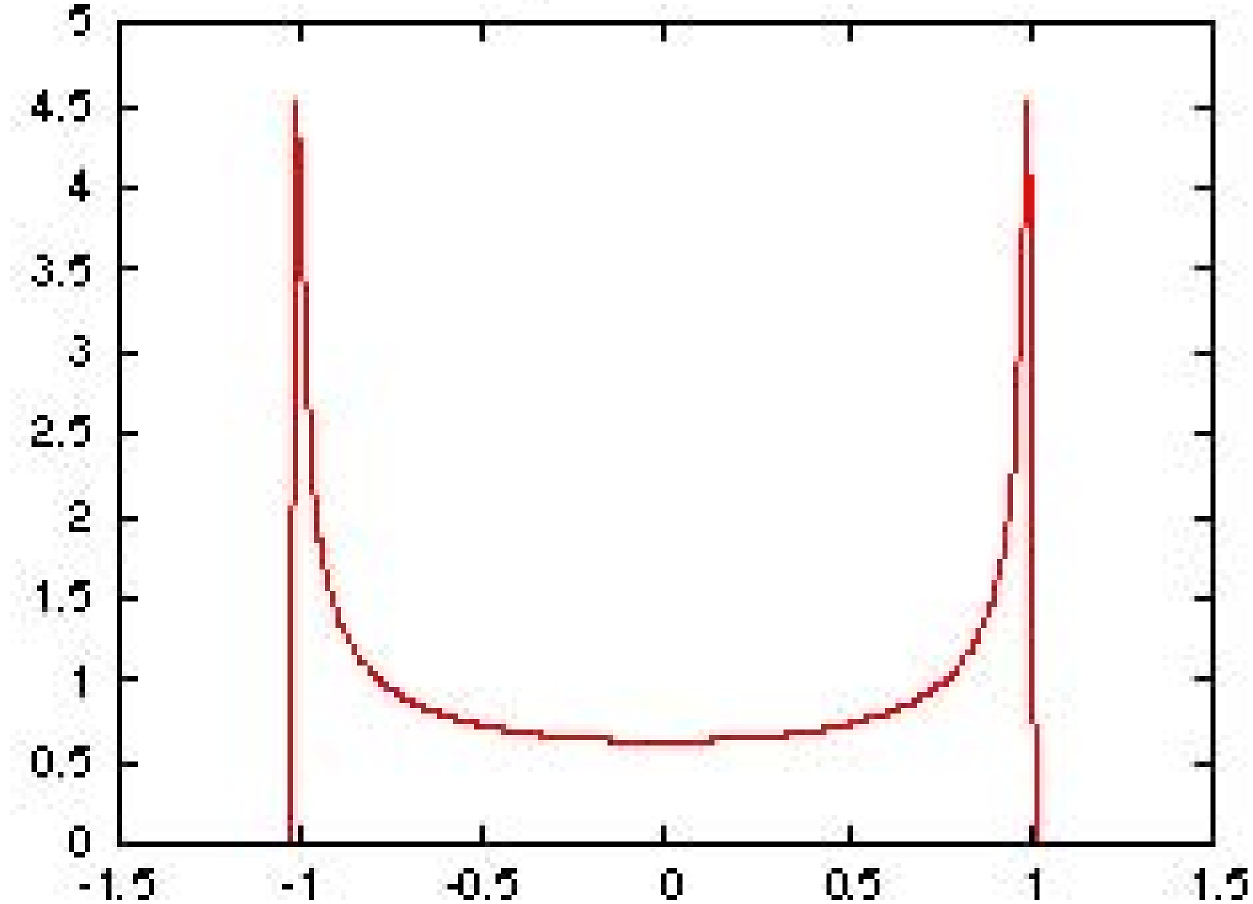
Density function of sin(x)

$$\sin(x)$$



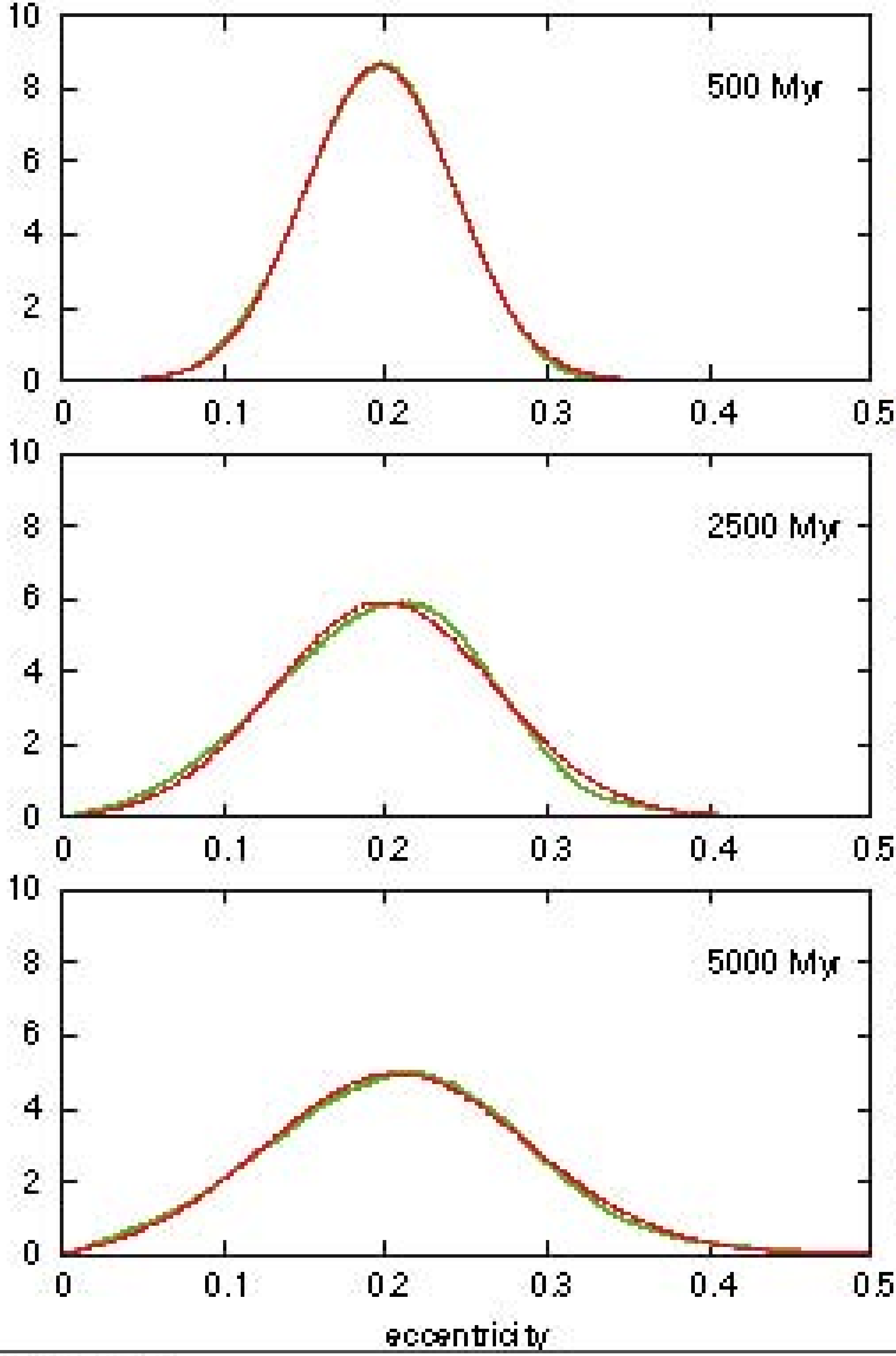
$$\sin(x)$$

$$\frac{1}{\pi\sqrt{1-x^2}}$$



Chaotic diffusion

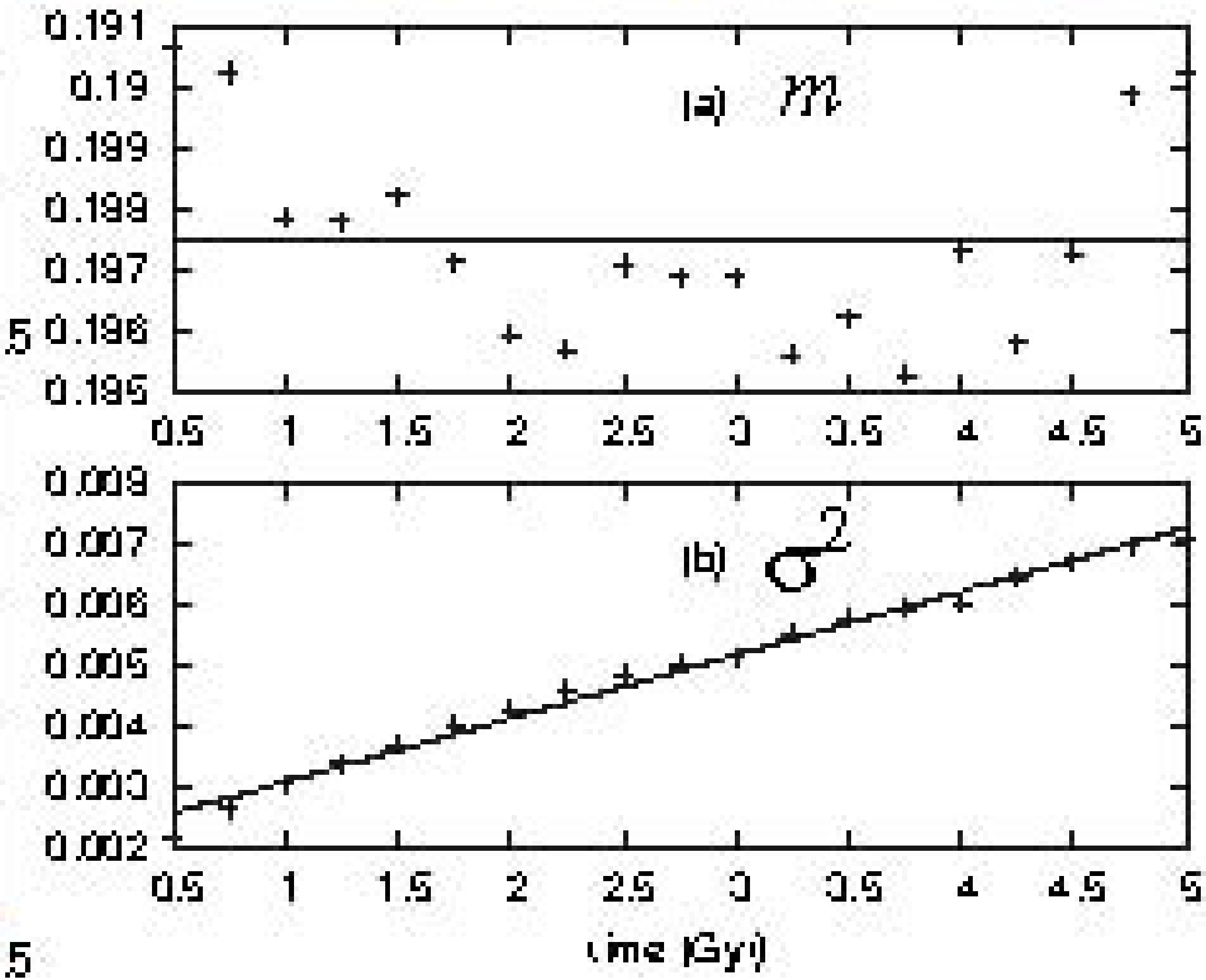
Mercury

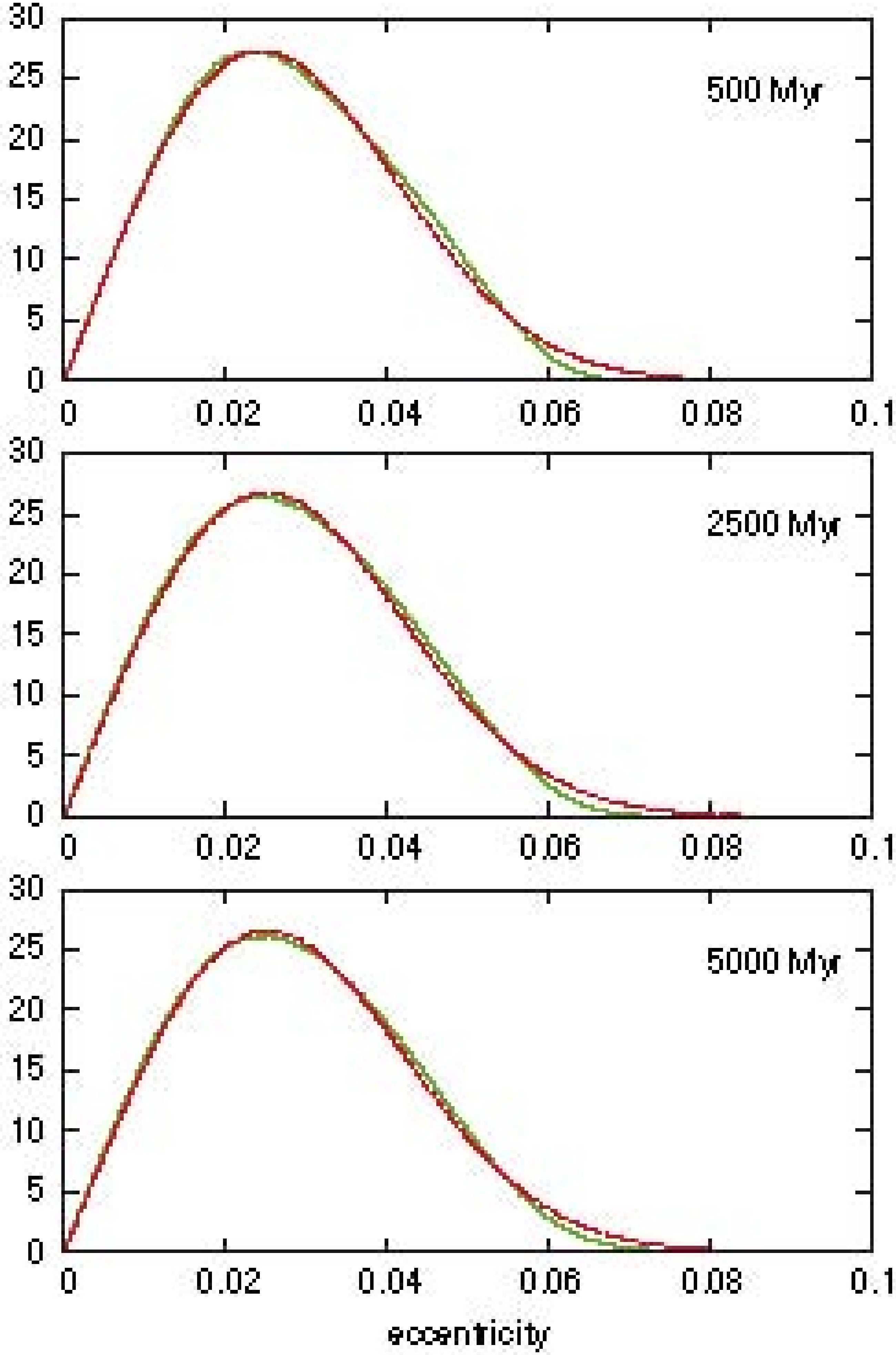


Rice density

$$f_{\sigma,m}(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2+m^2}{2\sigma^2}\right) I_0\left(\frac{xm}{\sigma^2}\right)$$

$$m = 0.1875$$
$$\sigma^2 = 0.00207 + 0.00104\,T$$



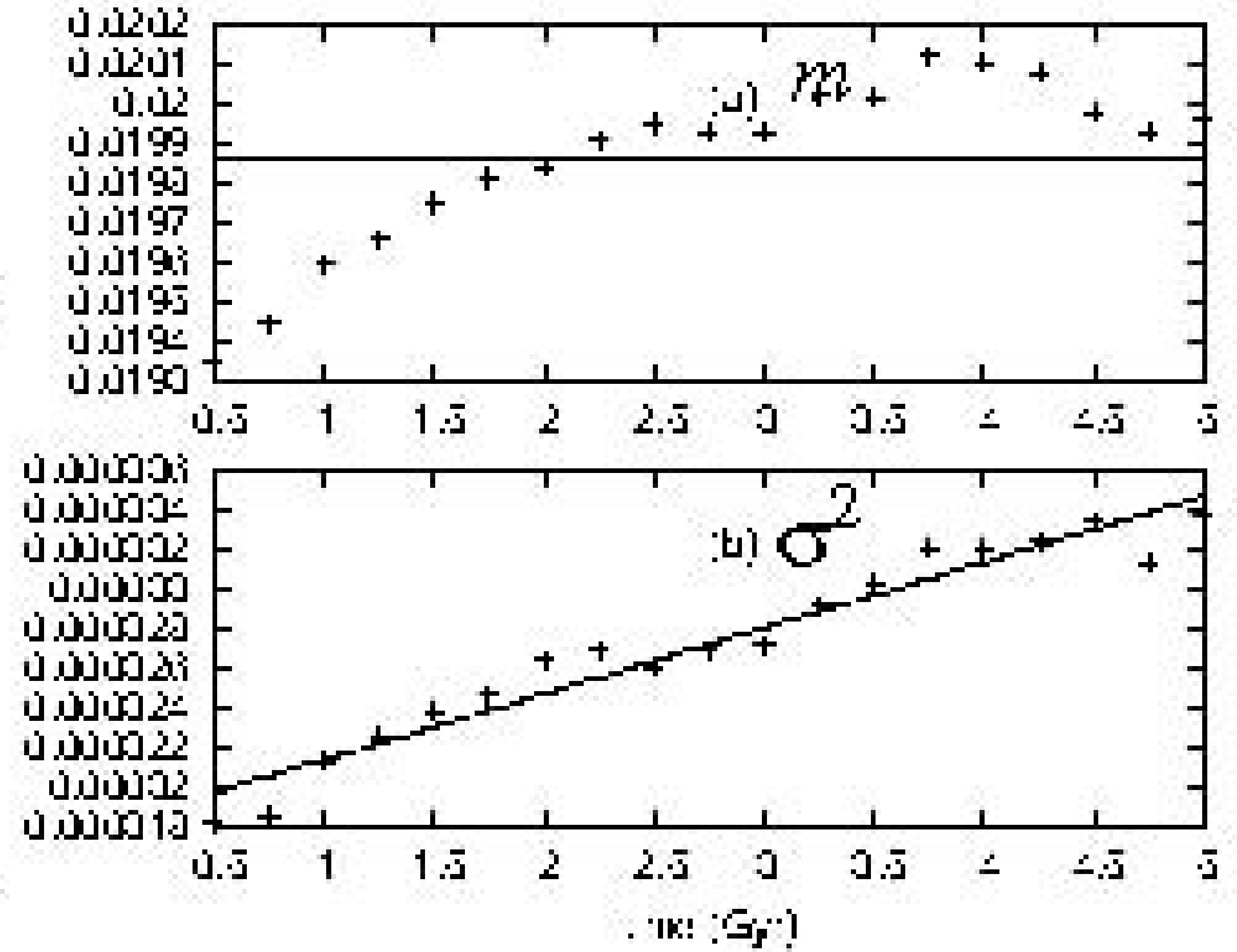


The Earth

Rice density

$$f_{\sigma,m}(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2+m^2}{2\sigma^2}\right) I_0\left(\frac{xm}{\sigma^2}\right)$$

$$m = 0.1875$$
$$\sigma^2 = 0.00032 + 0.000006\,T$$



Eccentricity of Mercury (1001 solutions)

secular equations

	500	1000	2000	3000	4000	5000
0.35	14	44	127	228	328	426
0.4	2	8	37	81	153	219
0.5	-	-	1	8	28	48
0.6	-	-	-	2	10	21
0.7	-	-	-	1	8	14
0.8	-	-	-	1	8	12
0.9	-	-	-	-	6	9

Excentricity of Mercury (1001 solutions)

Secular Equations- No relativity

	500	1000	2000	3000	4000	5000
0.35	130	341	558	692	763	812
0.4	75	249	449	589	684	747
0.5	24	118	306	442	552	640
0.6	16	76	238	364	476	564
0.7	14	67	218	343	454	541
0.8	12	63	209	331	442	531
0.9	12	61	202	325	441	530

Full System

8 Planets + Pluto

General relativity

Averaged Lunar contribution (Boué & Laskar, 2006)

Tidal dissipation

Fitted to INPOP06 (Fienga et al, 2008)

(Laskar & Gastineau, Nature, 2009)

Search for a small probability ~1%

2500 solutions over 5 Ga

CI : differences of 0.38 mm in Mercury's semi-major axis

(real uncertainty ~ a few meters)

6-7 million CPU hours

JADE

1536 Intel E5472 nodes : 12288 cores

147 Tflop/s

14th of TOP500 (nov. 2008)

Test period : August-December 2008

Eccentricity of Mercury (No relativity)

Secular system (1000 sol)

direct equations (200 sol)

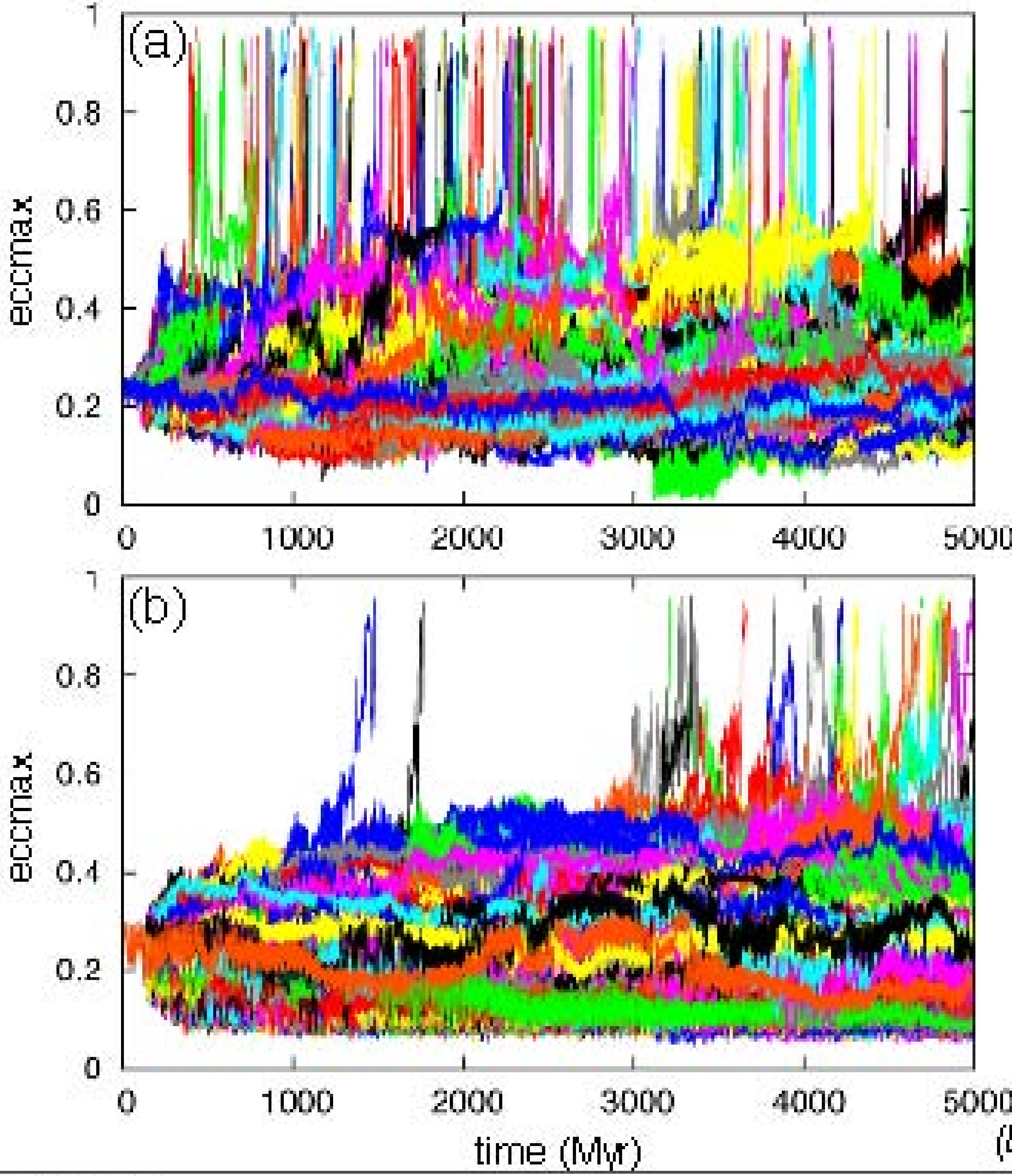
	500		1000		2000		3000		4000		5000	
0.35	164	130	423	341	627	558	766	692	826	763	886	812
0.4	80	75	313	249	527	449	667	589	736	684	831	747
0.5	25	24	124	118	333	306	517	442	612	552	687	640
0.6	15	16	95	76	274	238	433	364	547	476	612	564
0.7	15	14	90	67	264	218	423	343	527	454	602	541
0.8	15	12	90	63	259	209	423	331	527	442	602	531
0.9	10	12	85	61	259	202	423	325	527	441	602	530

Eccentricity of Mercury with relativity

Secular system (1000 sol)

direct equations (2501 sol)

	500		1000		2000		3000		4000		5000	
0.35	30	14	91	44	202	127	318	228	418	328	492	426
0.4	3	2	20	8	67	37	126	81	189	153	255	219
0.5	-	-	-	-	3	1	10	8	20	28	40	48
0.6	-	-	-	-	1	-	2	2	5	10	10	21
0.7	-	-	-	-	1	-	1	1	4	8	9	14
0.8	-	-	-	-	1	-	1	1	4	8	8	12
0.9	-	-	-	-	1	-	1	-	3	6	8	9

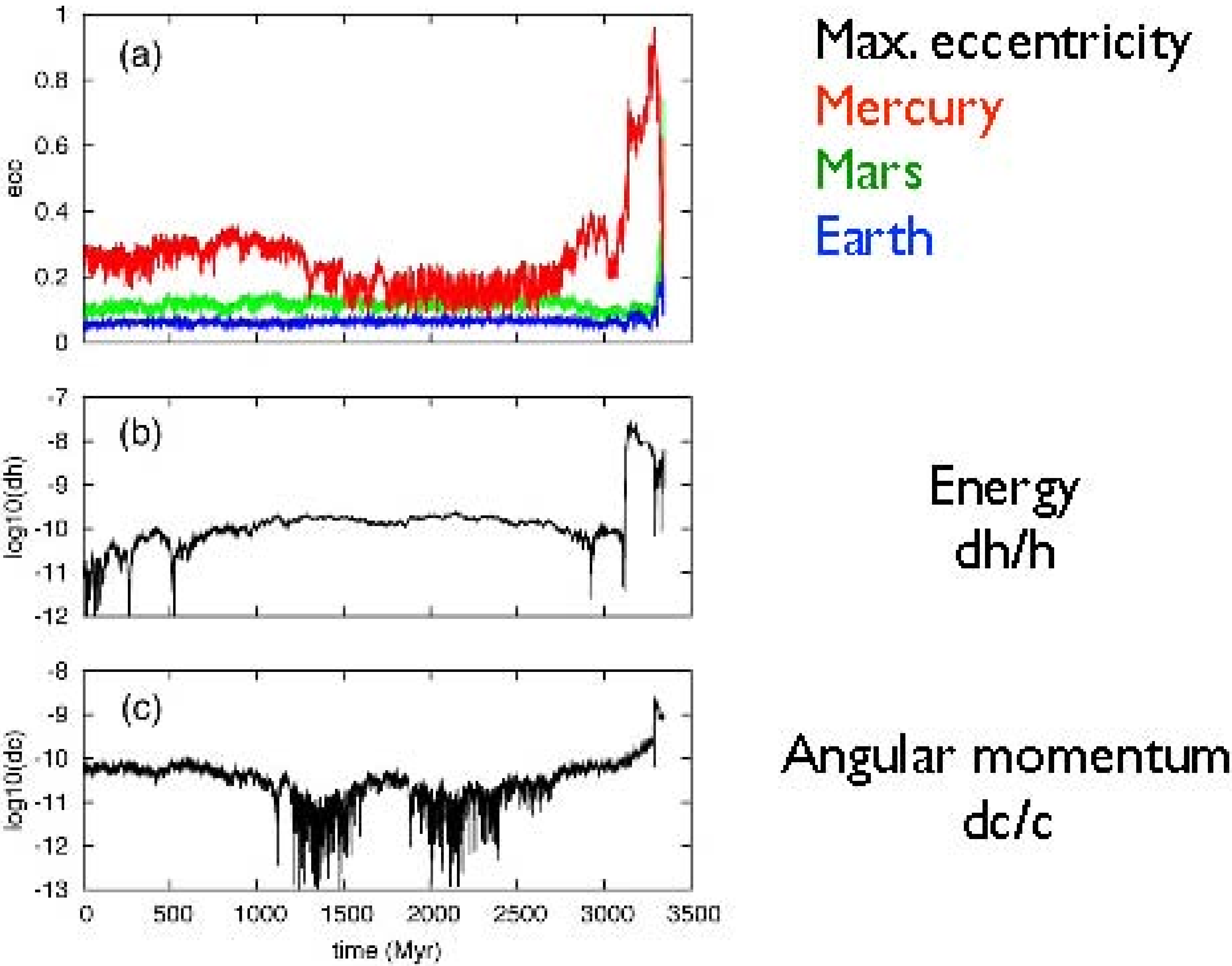


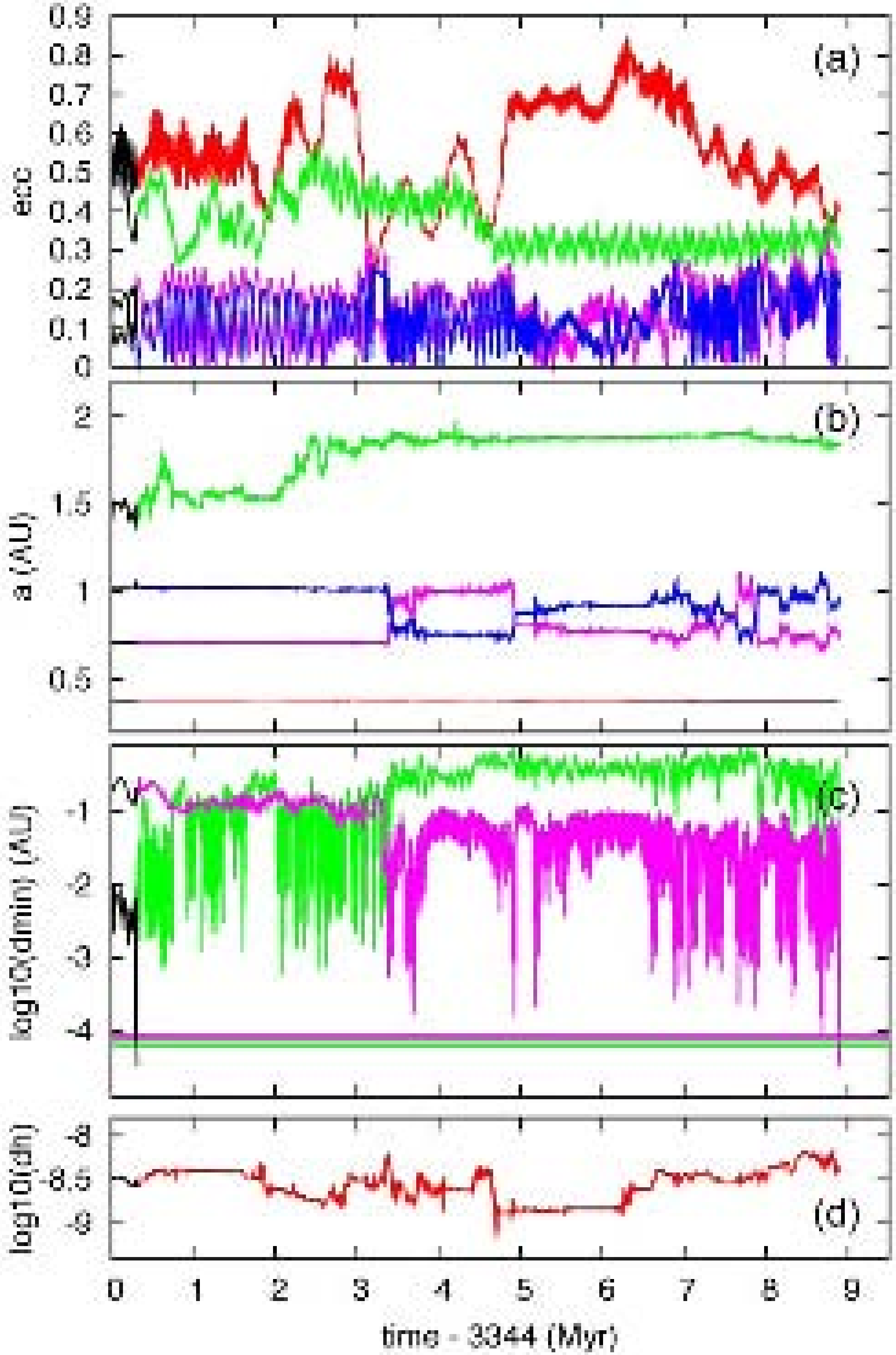
Mercury's eccentricity

201 sol.
No Relativity
(3.8 cm)

2501 sol.
With Relativity
(0.38 mm)

(Laskar & Gastineau, Nature, 2009)


(a)
(b)
(c)
ecc
log10(dh)
log10(dc)
time (Myr)
0
500
1000
1500
2000
2500
3000
3500
Max. eccentricity
Mercury
Mars
Earth
Energy
dh/h
Angular momentum
dc/c



Eccentricity.
Mercury, Venus, Earth, Mars.

Semi-major axis.
Mercury, Venus, Earth, Mars.

distances.
Venus-Earth, Mars-Earth.

Energy conservation