# Propagation d'interfaces avec termes non locaux

P. Cardaliaguet

Univ. Brest

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Joint works with G. Barles (Tours), O. Alvarez (Rouen), O. Ley (Tours), R. Monneau (CERMICS), A. Monteillet (Brest).

# Outline


1. Introduction
2. Inclusion preserving flows
   - A typical example : a gradient flow for Bernoulli problem
   - Existence and uniqueness of solutions
   - Link with the energy
3. Fronts without inclusion principle
   - Weak solutions
   - A uniqueness result for Fitzhugh-Nagumo type system
4. Other results and open questions

# Geometric equations

We are interested in geometric equations governing the movement of a family $K = \{K(t)\}_{t\in[0,T]}$ of compact subsets of $\mathbb{R}^N$ :

$$V_{x,t} = f(x, t, \nu_{x,t}, A_{x,t}, K).$$

- $V_{x,t}$ is the normal velocity of a point $x$ of $\partial K(t)$ at time $t$.
- $\nu_{x,t}$ is the unit exterior normal to $K(t)$ at $x \in \partial K(t)$.
- $A_{x,t} = [-\frac{\partial \nu_i}{\partial x_j}(x, t)]$ is the curvature matrix of $K(t)$ at $x \in \partial K(t)$.
- $K \mapsto f(x, t, \nu_{x,t}, A_{x,t}, K)$ is a **non-local dependence** in the whole front $K$ (up to time $t$).

# Motivations

Such evolution equations appear in several areas :

- Cristal growth,
- Elasticity,
- Biology,
- Finance,
- Shape optimization design,
- Image processing, ....

For these problems,

- Existence and uniqueness of classical solutions can be obtained by methods of differential geometry.
  (cf. Huisken, Escher-Simonnet, ...)
- However the front develops (often) singularities in finite time.
- Aim :

  - define the front after the onset of singularities,

  - study its properties

# Geometric flows with local velocity law

For local evolutions of the form

$$(*) \qquad V^{\Omega}_{t,x} = F(t, x, \nu_x, H_x) \qquad \forall x \in \partial\Omega(t),\ \forall t \geq 0,$$

Evans & Spruck (1991), Chen, Giga & Goto (1991) have defined a notion of generalized solution by using the level-set approach and techniques of viscosity solutions.

Similar but more geometric approaches have been developed by Soner (1993), Barles-Soner-Souganidis (1993), Belletini-Novaga (1995), Barles-Souganidis (1998)...

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# Case of velocities monotone in $K$

## Assumption

The function $f$ is monotone with respect to $K$ :

if $K \subset K'$ and $x \in \partial K \cap \partial K'$, then

$$f(x, t, \nu, A, K) \leq f(x, t, \nu, A, K'),$$

In this case, existence and uniqueness of solution can be obtained by comparison arguments and viscosity solutions techniques.

**Contributions :** Andrews, Barles, Caffarelli, C., Da Lio, Feldman, Kim, Ishii, Lederman, Ley, Mikami, Monneau, Rouy, Slepcev, Vazquez, Wolanski...

## Main difficulty

The velocity $f(x, t, \nu, A, K)$ is in general only defined for sets $K$ with smooth boundaries and for $x \in \partial K$.

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# Definition of the flow

Throughout this section we study the

## Geometric flow

$$V_{t,x}^{K} = -1 + \lambda |\nabla u_S^{K(t)}(x)|^2 \qquad (GF_\lambda)$$

where
$u_S^K$ is the solution to

$$\begin{cases} -\Delta u = 0 & \text{in } K(t)\backslash S \\ u = 1 & \text{on } \partial S \\ u = 0 & \text{on } \partial K \end{cases}$$

# Motivation

This equation appears in the numerical analysis of Bernoulli free boundary problems.

Bernoulli free boundary problem amounts to find a domain $K \in \mathbb{R}^N$ containing a given compact set $S \subset \mathbb{R}^N$ and minimizing the energy

$$\mathcal{E}_\lambda(K) = Vol(K) + \lambda \mathrm{cap}_S(K)$$

where

$$\mathrm{cap}_S(K) = \inf\left\{ \int_{K \setminus S} |\nabla u|^2 \; ; \; u \in H_0^1(K), \; u = 1 \ \text{ on } S \right\}$$

and $\lambda > 0$ is a fixed given parameter.

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# Motivation (continued)

Our geometric flow

$$V_{t,x}^{K} = -1 + \lambda |\nabla u_S^{K(t)}(x)|^2 \qquad (GF_\lambda)$$

can be interpreted as a gradient flow for the Bernoulli problem.

# The gradient flow

For this, let us recall Hadamard derivation formula :

$$D\mathcal{E}_\lambda(\phi) = \int_{\partial K} \langle \phi(x), \nu_x \rangle - \lambda \int_{\partial K} |\nabla u_S^K(x)|^2 \langle \phi(x), \nu_x \rangle$$

So, if $K(t)$ is a solution of our geometric flow, we get

$$\frac{d}{dt}\mathcal{E}_\lambda(K(t)) = \int_{\partial K(t)} V_{t,x}^K (1 - \lambda |\nabla u|^2) dx$$

$$= \int_{\partial K(t)} -(1 - \lambda |\nabla u|^2)^2 dx \;\leq\; 0$$

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## Key fact

The velocity is increasing : if $K_1 \subset K_2$ and $x \in \partial K_1 \cap \partial K_2$, then

$$-1 + \lambda |\nabla u_S^{K_1}(x)|^2 \leq -1 + \lambda |\nabla u_S^{K_2}(x)|^2 \; .$$

This leads to a notion of sub- and supersolutions in the viscosity sense.

# Inclusion principle

## Theorem (C., Ley 2007)

*Let* $0 \leq \lambda_1 < \lambda_2$ *be fixed.*
*If* $\mathcal{K}_1$ *is a subsolution of* $(GF_{\lambda_1})$ *and* $\mathcal{K}_2$ *is a supersolution of* $(GF_{\lambda_2})$ *with*

$$\mathcal{K}_1(0) \subset\subset \mathcal{K}_2(0) ,$$

*then*

$$\forall t \in [0, T), \qquad \mathcal{K}_1(t) \subset\subset \mathcal{K}_2(t) .$$

# Consequences of the inclusion principle

## Proposition (Existence)

*For any initial position $K_0$ with $S \subset Int(K_0)$ and $K_0$ bounded, there is at least one solution to $(GF_\lambda)$ : there is one largest solution and one smallest one.*

No uniqueness in general. However :

## Proposition (generic uniqueness)

*Let $(K_\lambda^0)_{\lambda>0}$ be a strictly increasing family of initial positions. Then the solution of $(GF_\lambda)$ with initial position $K_\lambda^0$ has a unique solution but for an enumerable number of $\lambda$'s.*

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# Proof of the inclusion principle

Let $\mathcal{K}_1$ be a subsolution of $(GF_{\lambda_1})$ and $\mathcal{K}_2$ be a supersolution of $(GF_{\lambda_2})$ with

$$\mathcal{K}_1(0) \subset\subset \mathcal{K}_2(0) \ ,$$

Let $\delta > 0$ small and

$$t_\delta = \inf\{t \geq 0 \mid \mathrm{dist}(K_1(t), \mathbb{R}^N \backslash K_2(t)) \leq \delta\}$$

and $x_\delta \in \partial K_1(t_\delta)$, $y_\delta \in K_2(t_\delta)$ s.t.

$$|x_\delta - y_\delta| = \mathrm{dist}(K_1(t), \mathbb{R}^N \backslash K_2(t))$$

Key point : Compare

$$\text{“}f(x_\delta, K_1(t))\text{" and “}f(y_\delta, K_2(t))\text{"}$$

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# Proof of the inclusion principle (2)

This comparison relies on Ilmanen interposition Lemma :

Let $K_1, K_2 \subset \mathbb{R}^N$ with $K_1$ compact, $K_2$ closed and $K_1 \cap \overline{\mathbb{R}^N \backslash K_2} = \emptyset$. Let $y_1 \in K_1$ and $y_2 \in \partial K_2$ s.t.

$$|y_1 - y_2| = \mathrm{dist}(K_1, \partial K_2) \ .$$

## Theorem (Ilmanen, 1993)

*There is an open set $\Sigma_1$ with a $\mathcal{C}^{1,1}$ boundary, s.t.*

$$K_1 \subset \Sigma \qquad \text{with} \qquad y_1 \in \partial\Sigma \cap \partial K_1$$

*and*

$$\Sigma_2 := \Sigma_1 + y_2 - y_1 \subset K_2 \qquad \text{with} \qquad y_2 \in \partial K_2 \cap \partial \Sigma_2 \ .$$

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$$\mathcal{E}_\lambda(K) = \mathrm{Vol}(K) + \lambda \mathrm{cap}_S(K)$$

The gradient flow of $\mathcal{E}_\lambda$ is formaly

$$V_{t,x} = -1 + \lambda |\nabla u_S^{K(t)}(x)|^2 \qquad (GF)$$

Given a viscosity solution $\mathcal{K}$ of $(GF_\lambda)$ we want to show that

$$t \to \mathcal{E}_\lambda(\mathcal{K}(t)) \quad \text{is nonincreasing.}$$

**Main difficulty :**

The notion of viscosity solutions has very little to do with energy estimates.

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**Technique :**

Approximation of the solution $\mathcal{K}$ by minimizing movements.

Minimizing movements = limits of discrete schemes of Euler type

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# Discrete minimizing movements

Let $h > 0$ be the time-step in the discretization.
Let $K_0$ be an initial condition with $S \subset\subset K_0$. We define the sequence $(K_n^h)_{n\in\mathbb{N}}$ by induction :

- $K_0^h := K_0$
- If $K_n^h$ is defined, then $K_{n+1}^h$ is a minimum of the penelized energy

$$J_h(K_n^h, K) = \mathcal{E}_\lambda(K) + \frac{1}{h}\int_{K_n^h \Delta K} d_{\partial K_n^h}$$

The sequence $(K_n^h)_{n\in\mathbb{N}}$ is called a discrete minimizing movement.

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# Heuristic explanation

If $K_{n+1}^h$ is a minimum of the penalized energy

$$J_h(K_n^h, K) = \mathcal{E}_\lambda(K) + \frac{1}{h}\int_{K_n^h \Delta K} d_{\partial K_n^h}$$

then $K_{n+1}^h$ (formally) satisfies the Euler equation

$$-1 + \lambda |\nabla u_S^{K_{n+1}^h(t)}(x)|^2 + \frac{1}{h} d^s_{\partial K_n^h}(x) = 0 \qquad \forall x \in \partial K_{n+1}^h$$

where the term $\frac{1}{h} d^s_{\partial K_n^h}(x)$ appears as a discrete normal velocity.

This is an implicit Euler scheme.

# Link with the viscosity solution

Let $(K^h_n)_n$ be a discrete minimizing movement. Let $\mathcal{K}^*$ and $\mathcal{K}_*$ be the upper and lower half-relaxed limits of $(K^h_n)_n$ :

$$\mathcal{K}^*(t) := \left\{ x \in \mathbb{R}^N \mid \begin{array}{l} \exists h_k \to 0^+,\ n_k \to +\infty,\ x_k \in K^{h_k}_{n_k}, \\ \text{with } x_k \to x \text{ and } h_k n_k \to t \end{array} \right\},$$

and

$$\mathbb{R}^N \backslash \mathcal{K}_*(t) = \left\{ x \in \mathbb{R}^N \mid \begin{array}{l} \exists h_k \to 0^+,\ n_k \to +\infty,\ x_k \notin K^{h_k}_{n_k}, \\ \text{with } x_k \to x \text{ and } h_k n_k \to t \end{array} \right\}.$$

**Lemma**

*$t \to \mathcal{K}^*(t)$ and $t \to \widehat{\mathcal{K}_*}(t)$ are respectively sub- and super solutions of $(GF_\lambda)$.*

The proof heavily relies of the fine properties of minimizers of $J^h$ (Alt, Caffarelli, 1980).

# Link with the viscosity solution

Let $(K_n^h)_n$ be a discrete minimizing movement. Let $\mathcal{K}^*$ and $\mathcal{K}_*$ be the upper and lower half-relaxed limits of $(K_n^h)_n$ :

$$\mathcal{K}^*(t) := \left\{ x \in \mathbb{R}^N \mid \begin{array}{l} \exists h_k \to 0^+, \; n_k \to +\infty, \; x_k \in K_{n_k}^{h_k}, \\ \text{with } x_k \to x \text{ and } h_k n_k \to t \end{array} \right\} ,$$

and

$$\mathbb{R}^N \backslash \mathcal{K}_*(t) = \left\{ x \in \mathbb{R}^N \mid \begin{array}{l} \exists h_k \to 0^+, \; n_k \to +\infty, \; x_k \notin K_{n_k}^{h_k}, \\ \text{with } x_k \to x \text{ and } h_k n_k \to t \end{array} \right\} .$$

**Lemma**

*$t \to \mathcal{K}^*(t)$ and $t \to \widehat{\mathcal{K}_*}(t)$ are respectively sub- and super solutions of $(GF_\lambda)$.*

The proof heavily relies of the fine properties of minimizers of $J^h$ (Alt, Caffarelli, 1980).

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# Link with the viscosity solution

**Corollary**

*If $(GF)$ has a unique solution $\mathcal{K}$ starting from $K_0$, then*

$$\overline{\mathcal{K}_*} = \mathcal{K}^* = \mathcal{K} \ .$$

Since $\mathcal{E}(K_n^h) \leq \mathcal{E}(K_0)$, one gets from the lower semi-continuity of $\mathcal{E}$ :

$$\mathcal{E}_\lambda(\mathcal{K}(t)) \leq \mathcal{E}_\lambda(K_0) \qquad \text{a.e.}$$

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# The behavior of the energy

The semi-group property of the flow then implies that the energy is non decreasing along the flow :

**Theorem (C., Ley, to appear)**

*If $(GF_\lambda)$ has a unique solution $\mathcal{K}$ starting from $K_0$, then there is $\mathcal{T} \subset [0,+\infty)$ of full measure s.t.*

$$\mathcal{E}_\lambda(\mathcal{K}(t)) \leq \mathcal{E}_\lambda(\mathcal{K}(s)) \qquad \forall s,t \in \mathcal{T},\ s < t\ .$$

# Generalization and open problems

- Inclusion principle, existence, generic uniqueness can be extended to more general velocities law with with curvature dependence (C., Ley, 2007).
- Recent works focus on homogenization (Kim, 2007, Kim and Mellet, 2007).

# Outline


1. Introduction
2. Inclusion preserving flows
   - A typical example : a gradient flow for Bernoulli problem
   - Existence and uniqueness of solutions
   - Link with the energy
3. Fronts without inclusion principle
   - Weak solutions
   - A uniqueness result for Fitzhugh-Nagumo type system
4. Other results and open questions

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## Aim

From now on we geometric flows

$$(f.p.p.) \qquad V_{x,t} = f(x, t, \nu_{x,t}, A_{x,t}, K).$$

without inclusion principle :

$$K \subset K' \text{ does not imply}$$

$$f(x, t, \nu, A, K) \leq f(x, t, \nu, A, K').$$

$\Rightarrow$ The previous techniques for building a viscosity solution fail.

## Main assumption

$f = f(x, t, \nu, A, K)$ is defined for any bounded $K$.

# Level-set equation

If we represent $K$ as the 0 super level-set of some regular function $u$ :

$$K(t) = \{x \in \mathbb{R}^N;\ u(x,t) \geq 0\},$$

the level-set equation corresponding to (f.p.p.) is

$$u_t(x,t) = f\left(x, t, -\frac{Du}{|Du|}, \frac{1}{|Du|}\left(I - \frac{Du\,Du^T}{|Du|^2}\right) D^2u, \{u \geq 0\}\right) |Du(x,t)|$$

$$= F(x, t, Du, D^2u, \mathbf{1}_{\{u \geq 0\}}). \qquad (1)$$

with initial condition $u_0 \in UC(\mathbb{R}^N)$ such that

$$K_0 = \{u_0 \geq 0\}, \quad \overset{\circ}{K_0} = \{u_0 > 0\}.$$

# Initial condition

Let $K_0$ be the initial (compact) set, and let $u_0$ be a UC function such that

$$K_0 = \{u_0 \geq 0\}, \quad \overset{\circ}{K_0} = \{u_0 > 0\}.$$

# Motivation of the definition of weak solutions

Let us investigate the stability of solutions : Suppose that $u_\varepsilon$ are solutions to

$$\begin{cases} u_t(x,t) = F(x,t,Du,D^2u,\mathbf{1}_{\{u\geq 0\}}) \\ u_\varepsilon(x,0) = u_0^\varepsilon(x), \end{cases}$$

with $u_0^\varepsilon \to u_0$ uniformly.

Then standard estimates imply that $u_\varepsilon \to u$ in $C^0(\mathbb{R}^N \times [0,T])$, and that $\mathbf{1}_{\{u^\varepsilon \geq 0\}} \rightharpoonup \chi$ weakly-$\star$ in $L^\infty_{loc}(\mathbb{R}^N \times [0,T],[0,1])$.

Then formally we get :

$$\begin{cases} u_t(x,t) = F(x,t,Du,D^2u,\chi) \\ u(x,0) = u_0(x), \end{cases}$$

with

$$\mathbf{1}_{\{u>0\}} \leq \liminf \psi_\varepsilon(u_\varepsilon) \leq \chi \leq \limsup \psi_\varepsilon(u_\varepsilon) \leq \mathbf{1}_{\{u\geq 0\}}$$

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$$\mathbf{1}_{\{u>0\}} \leq \liminf \psi_\varepsilon(u_\varepsilon) \leq \chi \leq \limsup \psi_\varepsilon(u_\varepsilon) \leq \mathbf{1}_{\{u\geq 0\}}$$

## Definition of weak solutions

Let $u : \mathbb{R}^N \times [0, T]$ be a continuous function. We say that $u$ is a weak solution of the f.p.p. if there exists $\chi \in L^\infty(\mathbb{R}^N \times [0, T], [0, 1])$ such that :

1. $u$ is the $L^1$ viscosity solution of

$$\begin{cases} u_t(x,t) = F(x,t,Du,D^2u,\chi) & \text{in } \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases}$$

2. For all $t \in [0, T]$,

$$\mathbf{1}_{\{u(\cdot,t)>0\}} \le \chi(\cdot,t) \le \mathbf{1}_{\{u(\cdot,t)\ge 0\}}.$$

Moreover, we say that $u$ is a classical viscosity solution of the f.p.p. if in addition, for almost all $t \in [0, T]$ and almost everywhere in $\mathbb{R}^N$,

$$\{u(\cdot,t) > 0\} = \{u(\cdot,t) \ge 0\}.$$

# Reference

Similar definition (with existence results) can be found in :

- Giga, Goto, Ishii (SIAM J. Math. Anal., 1992)
- Soravia, Souganidis (SIAM J. Math. Anal., 1996)
- Hilhorst, Logak, Schätzle (Interfaces Free Bound., 2000) : phase-field approach for the evolution law : $V_{x,t} = \kappa_{x,t} - Vol(K)$
- Barles, C., Ley, Monneau (preprint, 2007) : existence of weak solutions for dislocation dynamics.
- Forcadel, Monteillet (preprint, 2007) : existence of solution for dislocation dynamics with a curvature term by minimizing movements.

# The existence result

Let us assume that

- $F = F(\cdot,\cdot,\cdot,\cdot,\chi)$ is geometric and satisfies the "usual conditions",
- $F = F(t, x, \nu, A, \cdot)$ is defined for $\chi : \mathbb{R}^N \to [0, 1])$ measurable with compact support,
- $F(\chi_n) \to F(\chi)$ locally uniformly if $\mathrm{spt}(\chi_n)$ uniformly bounded and $\chi_n \to \chi$ in $L^\infty(\mathbb{R}^N)-$weak*,
- $F = F(t, x, \nu, A, \cdot)$ satisfies suitable growth condition with respect to the support of $\chi$.

**Theorem [Barles, C., Ley, Monteillet]**

Under the above conditions, for any continuous initial condition $u_0$ with $\{u_0 \geq 0\}$ bounded, there is at least one weak solution of (f.p.p.).

# Sketch of the proof

Let us consider the set-valued mapping

$$\xi : \chi \in L^\infty(\mathbb{R}^N \times [0,T],[0,1])$$
$$\mapsto u \text{ viscosity solution of } \begin{cases} u_t(x,t) = F(x,t,\nu_{x,t},A_{x,t},\chi) \\ u(x,0) = u_0(x). \end{cases}$$
$$\mapsto \{\chi';\ \mathbf{1}_{\{u(\cdot,t)>0\}} \leq \chi'(\cdot,t) \leq \mathbf{1}_{\{u(\cdot,t)\geq 0\}} \text{ for almost all } t \in [0,T]\}.$$

Then there exists a weak solution to $(f.p.p.)$ if and only if there exists a fixed point of $\chi$ of $\xi$ in the sense that $\chi \in \xi(\chi)$.

To show the existence of a fixed point, we use

- Kakutani's fixed point theorem,
- Barles stability Lemma.

# Discussion of the definition of weak solutions

Main advantages of the definition :

- Existence result under mild conditions on the dynamics.
- Stability results.

Main drawback :

- Very strange behavior of the solution in case of fattening.

# Existence of classical viscosity solutions

Following Ley (2001), if

$$(*) \qquad f = f(x, t, \nu_{x,t}, K) \ \geq \mathbf{0}$$

then a solution $u$ of

$$\begin{cases} u_t(x,t) = F(x,t,\nu_{x,t},\chi) & \text{in } \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases}$$

satisfies for almost all $(x,t) \in \mathbb{R}^N \times [0,T]$,

$$\mathbf{1}_{\{u(\cdot,t)>0\}}(x) = \chi(x,t) = \mathbf{1}_{\{u(\cdot,t)\geq 0\}}(x).$$

## Proposition

If $(*)$ holds, any weak solution of the f.p.p. is a classical viscosity one.

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# The Fitzhugh-Nagumo type system

We now consider the following system,

$$\begin{cases} u_t(x,t) = c(v(x,t))|Du(x,t)|, \\ v_t(x,t) - \Delta v(x,t) = \mathbf{1}_{\{u(\cdot,t)\geq 0\}}(x), \\ v(\cdot,0) = 0,\ u(\cdot,0) = u_0 \end{cases} \tag{2}$$

for $(x,t) \in \mathbb{R}^N \times (0,T)$,

**Proposition [Giga, Goto, Ishii / Soravia, Souganidis]**

Assume that the function $c$ is non negative and Lipschitz continuous on $\mathbb{R}$.
Then there is at least one weak viscosity solution to the f.p.p. and any weak solution is a classical viscosity one.

# The uniqueness result

Let us assume moreover that :

- The initial set $K_0 \subset B(0, R)$ is the closure of a bounded open subset of $\mathbb{R}^N$ with $C^2$ boundary.
- There exist $\delta > 0$ and $L > 0$ such that $\delta \leq c(x) \leq L$ in $\mathbb{R}$.

**Theorem [Barles, C., Ley, Monteillet]**

Under the above assumptions, the f.p.p. has a unique solution.

# Sketch of the proof

Let $u_1$ and $u_2$ be two classical solutions of

$$\begin{cases} u_t(x,t) = c(v(x,t))|Du(x,t)|, \\ v_t(x,t) - \Delta v(x,t) = \mathbf{1}_{\{u(\cdot,t)\geq 0\}}(x). \end{cases}$$

with initial conditions $v(\cdot,0) = 0$ and $u(\cdot,0) = u_0$.

Let us set, for $i = 1, 2$ and $t \in [0, T]$,

$$K_1(t) = \{u_1(\cdot, t) \geq 0\},\ K_2(t) = \{u_2(\cdot, t) \geq 0\},$$

and

$$v_i(x, t) = \int_0^t \int_{\mathbb{R}^N} G(x - y, t - s)\mathbf{1}_{K_i(s)}(y)\, dyds$$

the solution of

$$\begin{cases} (v_i)_t - \Delta v_i = \mathbf{1}_{K_i} & \text{in } \mathbb{R}^N \times (0, T), \\ v_i(\cdot, 0) = 0 & \text{in } \mathbb{R}^N. \end{cases}$$

It suffices to prove that $K_1 = K_2$, since this implies that $v_1 = v_2$, and finally $u_1 = u_2$.

We estimate for any $t \in [0, T]$,

$$d_{\mathcal{H}}(K_1(t), K_2(t))$$

$$\leq T\, k(N,T)\, \|c(v_1) - c(v_2)\|_{L^\infty(\mathbb{R}^N \times [0,T])}$$

$$\leq T\, k(N,T)\, \|c'\|_\infty \|v_1 - v_2\|_{L^\infty(\mathbb{R}^N \times [0,T])}.$$

where

$$|v_1(x,t) - v_2(x,t)|$$

$$\leq \left| \int_0^t \int_{\mathbb{R}^N} G(x-y, t-s)(\mathbf{1}_{K_1(s)}(y) - \mathbf{1}_{K_2(s)}(y))\, dyds \right|$$

$$\leq \int_0^t \int_{\mathbb{R}^N} G(x-y, t-s)(\mathbf{1}_{K_1(s)\setminus K_2(s)}(y) + \mathbf{1}_{K_2(s)\setminus K_1(s)}(y))\, dyds.$$

Set $\quad r = \sup_{t\in[0,T]} d_{\mathcal{H}}(K_1(t), K_2(t)) \leq LT.$

Then if $B = \bar{B}(0,1)$, we have

$$|v_1(x,t) - v_2(x,t)|$$

$$\leq \int_0^t \int_{\mathbb{R}^N} G(x-y, t-s)(\mathbf{1}_{(K_1(s) \quad )\backslash K_2(s)}(y) + \mathbf{1}_{(K_2(s) \quad )\backslash K_1(s)}(y))\, dyds.$$

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$$|v_1(x,t) - v_2(x,t)|$$

$$\leq \int_0^t \int_{\mathbb{R}^N} G(x-y, t-s)(\mathbf{1}_{(K_2(s)+rB)\setminus K_2(s)}(y) + \mathbf{1}_{(K_1(s)+rB)\setminus K_1(s)}(y))\, dyds.$$

The key is to provide the estimate

$$\int_0^t \int_{\mathbb{R}^N} G(x-y,t-s)\mathbf{1}_{(K_i(s)+rB)\setminus K_i(s)}(y)\, dyds \leq C\, r,$$

so that for any $t \in [0,T]$,

$$r = \sup_{t\in[0,T]} d_{\mathcal{H}}(K_1(t),K_2(t)) \leq T\, k(N,T)\, \|c'\|_\infty\, 2C\, r,$$

and we would obtain that $K_1 = K_2$ on $[0,T]$ for $T$ small enough.

# Interior cone property

However, the estimation

$$\int_0^t \int_{\mathbb{R}^N} G(x-y, t-s) \mathbf{1}_{(K(s)+rB)\setminus K(s)}(y)\, dyds \leq C\, r,$$

does not hold for any $K$ :

1. it requires at least that $(K(s)+rB)\setminus K(s)$ be small in $L^1(\mathbb{R}^N)$...
2. ... which is not automatic since
$$Vol((K(s)+rB)\setminus K(s)) \approx Per(K(s))\, r \ldots$$
3. ... and would not be enough since $\chi \mapsto v$ solution of $v_t - \Delta v = \chi$ is not continuous from $L^1$ to $L^\infty$.

$\rightarrow$ We need certain regularity for the sets $K_i = \{u_i \geq 0\}$.

This regularity is the **interior cone property** :

## Definition

Let $K$ be a compact subset of $\mathbb{R}^N$. We say that $K$ has the interior cone property of parameters $\rho$ and $\theta$ if $0 < \rho < \theta$ and

$$\forall\, x \in \partial K,\ \exists\, \nu \in \mathbb{S}^{N-1} \text{ such that } C_{\nu,x}^{\rho,\theta} := x + [0,\theta]\bar{B}_N(\nu, \rho/\theta) \subset K,$$

where $\bar{B}_j(x,r)$ is the closed ball of $\mathbb{R}^j$ of radius $r$ centered at $x$.

To prove our uniqueness result, we therefore need three ingredients :

1. The propagation of the interior cone property for solutions of the eikonal equation :

   $K_1(t) = \{u_1(\cdot, t) \geq 0\}$ and $K_2(t) = \{u_2(\cdot, t) \geq 0\}$ have the interior cone property for all $t \in [0, T]$, for some parameters $\rho$ and $\theta$ independent of $t$.

2. A perimeter estimate for sets having the interior cone property.

3. An estimate on the $L^\infty$ norm of the solutions of the $r$-perturbed equation

$$\begin{cases} v_t(x,t) - \Delta v(x,t) = \mathbf{1}_{(K(t)+rB)\setminus K(t)}(x) \\ v(\cdot, 0) = 0. \end{cases}$$

   in function of $r$ for such a $K$.

# Propagation of the interior cone property

## Theorem

*Let $K_0$ be the closure of a bounded open subset of $\mathbb{R}^N$ with $C^2$ boundary, and let $c : \mathbb{R}^N \times [0,T] \to \mathbb{R}^N$ satisfy the following assumptions : there exist $\delta, L, M > 0$ such that :*

$$\begin{cases} \delta \leq c \leq L, \\ c \text{ is continuous on } \mathbb{R}^N \times [0,T], \\ \forall\, t \in [0,T],\ c(\cdot,t) \text{ is differentiable in } \mathbb{R}^N \text{ with } \|Dc\|_\infty \leq M. \end{cases}$$

*Let $u$ be the unique uniformly continuous viscosity solution of*

$$\begin{cases} u_t(x,t) = c(x,t)|Du(x,t)| & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^N, \end{cases}$$

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Then there exist $\rho > 0$ and $\theta > 0$ depending only on $c$ and $K_0$ such that

$$K(t) = \{x \in \mathbb{R}^N;\ u(x,t) \geq 0\}$$

has the interior cone property of parameters $\rho$ and $\theta$ for all $t \in [0, T]$.

# Sets with the interior cone property

## Theorem

*Let $K$ be a compact subset of $\mathbb{R}^N$ having the cone property of parameters $\rho$ and $\theta$.*

*Then there exists a positive constant $C_0 = C_0(N, \rho, \theta/\rho)$ such that for all $R > 0$,*

$$\mathcal{H}^{N-1}(\partial K \cap \bar{B}(0,R)) \le C_0\, \mathcal{L}^N(K \cap \bar{B}(0, R+\rho/4)).$$

# The $r$-perturbed equation

## Theorem

*Let* $\{K(t)\}_{t\in[0,T]} \subset \bar{B}_N(0,D) \times [0,T]$ *be a bounded family of compact subsets of* $\mathbb{R}^N$ *having the interior cone property of parameters* $\rho$ *and* $\theta$ *with* $0<\rho<\theta<1$*, and let us set, for any* $x \in \mathbb{R}^N$, $t\in[0,T]$ *and* $r \geq 0$,

$$\phi(x,t,r) = \int_0^t \int_{\mathbb{R}^N} G(x-y,t-s)\, \mathbf{1}_{(K(s)+rB)\setminus K(s)}(y)\, dyds.$$

*Then for any* $r_0>0$*, there exists a constant* $C_1 = C_1(T,N,D,r_0,\rho,\theta/\rho)$ *such that for any* $x\in\mathbb{R}^N$, $t\in[0,T]$ *and* $r\in[0,r_0]$,

$$|\phi(x,t,r)| \leq C_1\, r.$$

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# Dislocation dynamics

Similar results have been obtained for a model of dislocation dynamics Alvarez, Hoch, Le Bouar, Monneau (ARMA., 2006)

$$V_{x,t} = c_0(\cdot, t) \star \mathbf{1}_{K(t)}(x) + c_1(x,t)$$

with associated level-set equation

$$u_t(x,t) = \left[c_0(\cdot, t) \star \mathbf{1}_{\{u(\cdot,t)\geq 0\}}(x) + c_1(x,t)\right] |Du(x,t)|.$$

under the assumption that

$$\|c_0(\cdot, t)\|_1 \leq c_1(x,t) \qquad \forall (x,t)\ .$$

(Alvarez, C. Monneau (2007)), Barles, Ley (2007).

# Conclusion

- For flows which preserve the inclusion, there is a natural generalization of the notion of viscosity solution.
- In the general case, existence/stability of weak solutions can be obtained under mild conditions.

  $\rightarrow$ main difficulty : the notion has very few meaning if there is fattening...

- Uniqueness holds for some first order models with positive velocity.
- Other cases are completely open.