Boundary kernels for dissipative systems

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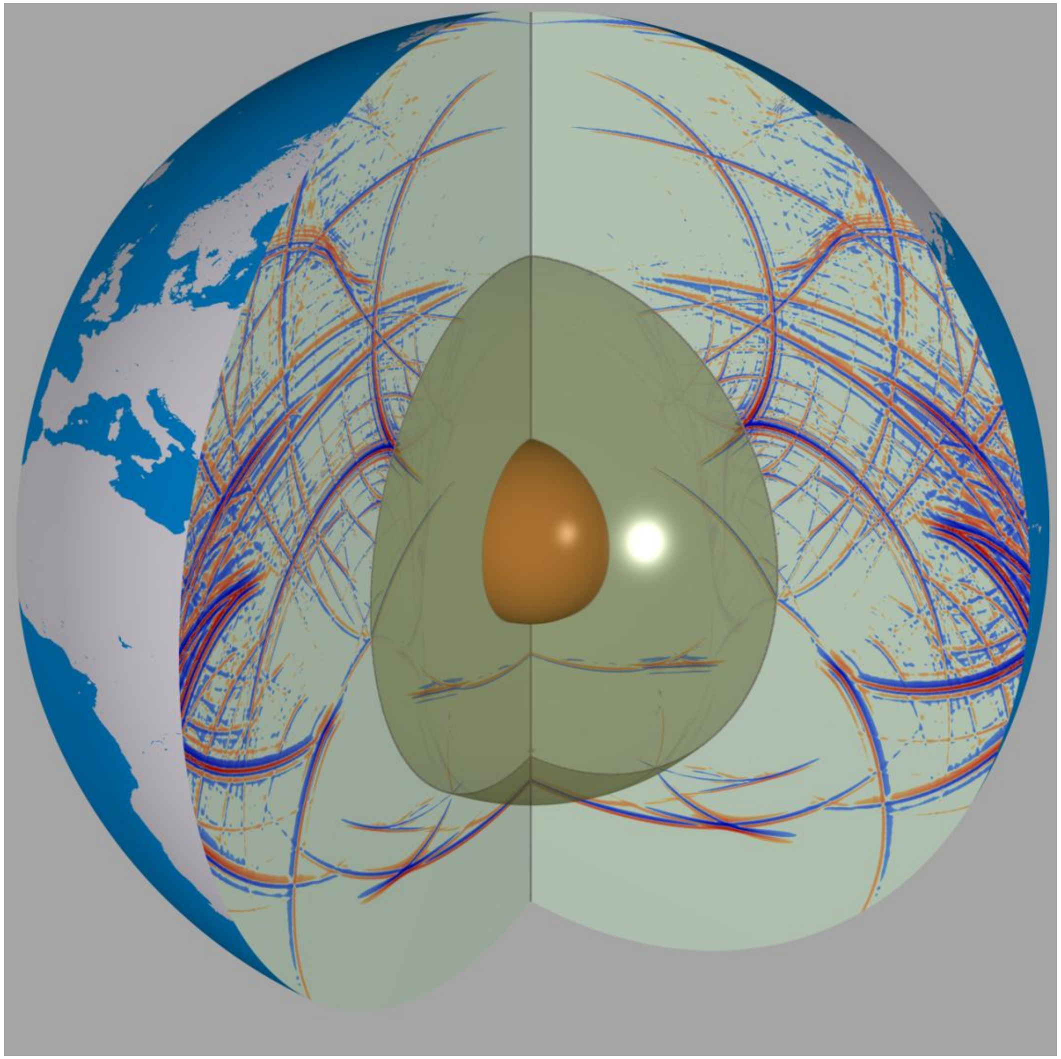
Motivation:

To develop mathematical study on wave propagation in a system with boundary.



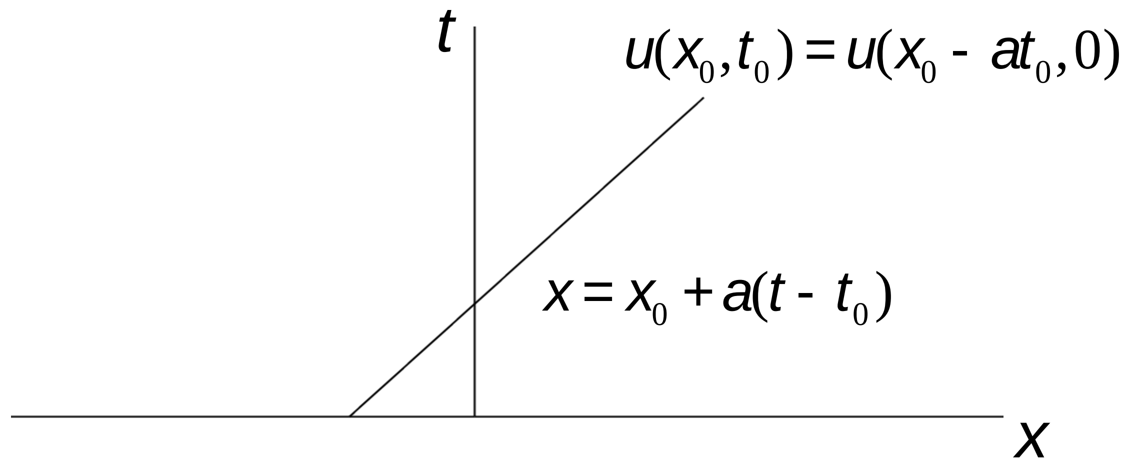
- Cauchy's problem in whole space
- Initial boundary value problem in half space domain.

http://seismology.geophysik.uni-muenchen.de/images/earth_big.jpg

Some Simple Examples.

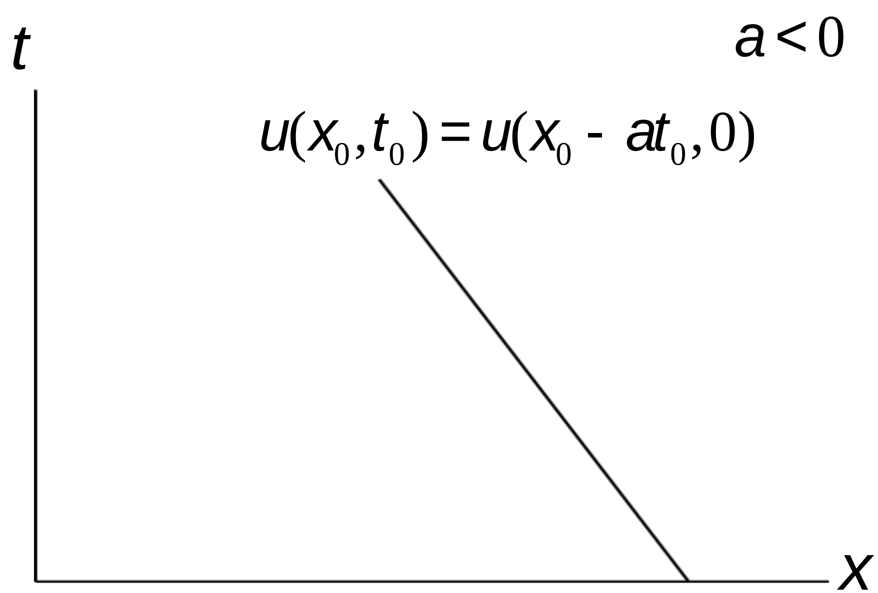
Linear Transport Equation.

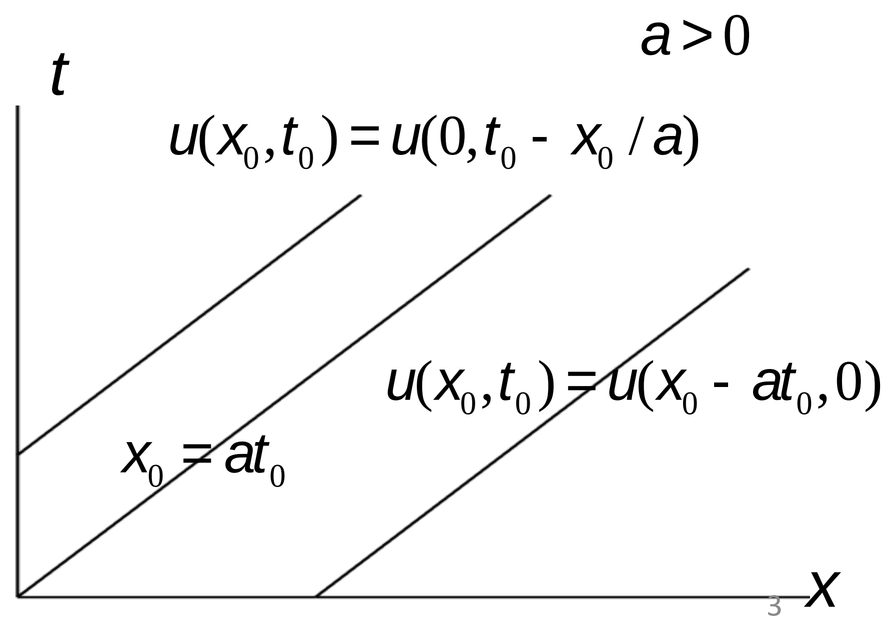
$$u_t + a u_x = 0.$$



Linear Transport Equation in half space.

$$u_t + a u_x = 0, \quad x, t > 0.$$

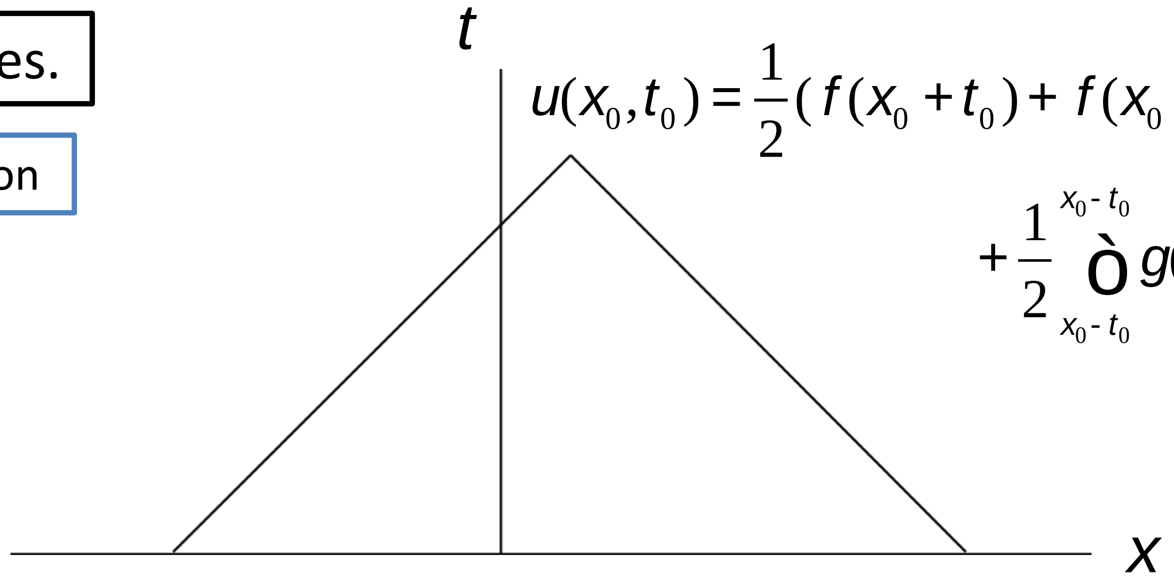




Some Simple Examples.

D'Alermbert Wave Equation

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$



$$u(x_0,t_0) = \frac{1}{2}\left(f(x_0+t_0) + f(x_0 - t_0)\right) + \frac{1}{2}\int_{x_0-t_0}^{x_0-t_0} g(y)\,dy$$

D'Alermbert Wave Equation in Half Space

$$\begin{cases} u_{tt} - u_{xx} = 0, \quad x > 0. \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \\ u(0,t) = 0. \end{cases}$$

$$\begin{cases} u_{tt} - u_{xx} = 0, \quad x > 0. \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \\ u_x(0,t) = 0. \end{cases}$$

(Odd Extension)

(Even Extension)

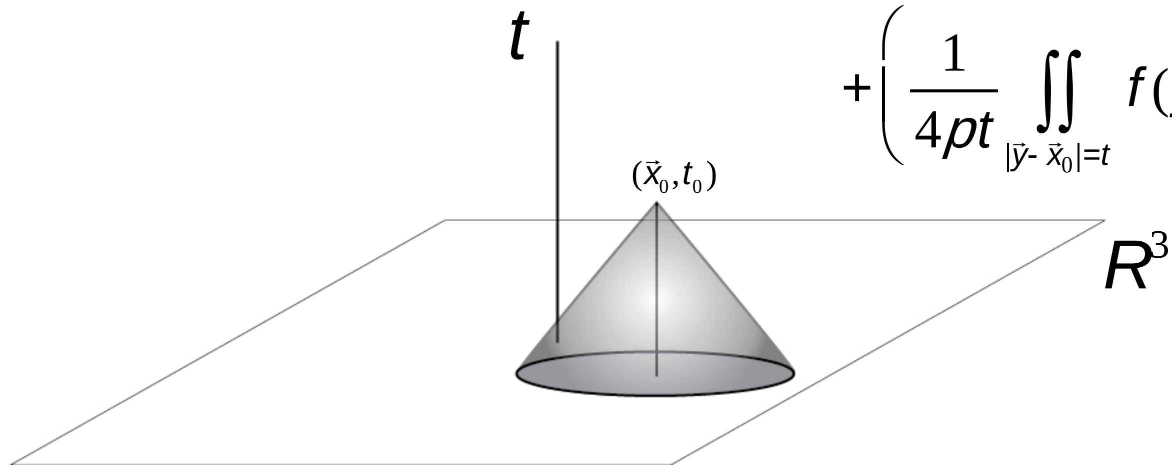
Some Simple Examples.

D'Alermbert Wave Equation in 3D

$$\begin{cases} u_{tt} - \mathrm{D}u = 0 \\ u(\vec{x},0) = f(\vec{x}) \\ u_t(\vec{x},0) = g(\vec{x}) \end{cases}$$

Kirchhoff's formula.

$$u(\vec{x}_0,t_0) = \frac{1}{4pt}\iint_{|\vec{y}-\vec{x}_0|=t} g(\vec{y})\,dS_{\vec{y}} + \left(\frac{1}{4pt}\iint_{|\vec{y}-\vec{x}_0|=t} f(\vec{y})\,dS_{\vec{y}}\right)_t$$



D'Alermbert Wave Equation in Half Space

$$\begin{cases} u_{tt} - \mathrm{D}u = 0, \qquad x^1 > 0 \\ u(\vec{x},0) = f(\vec{x}) \\ u_t(\vec{x},0) = g(\vec{x}) \\ u(0,x^2,x^3,t) = 0. \end{cases}$$

(Odd Extension)

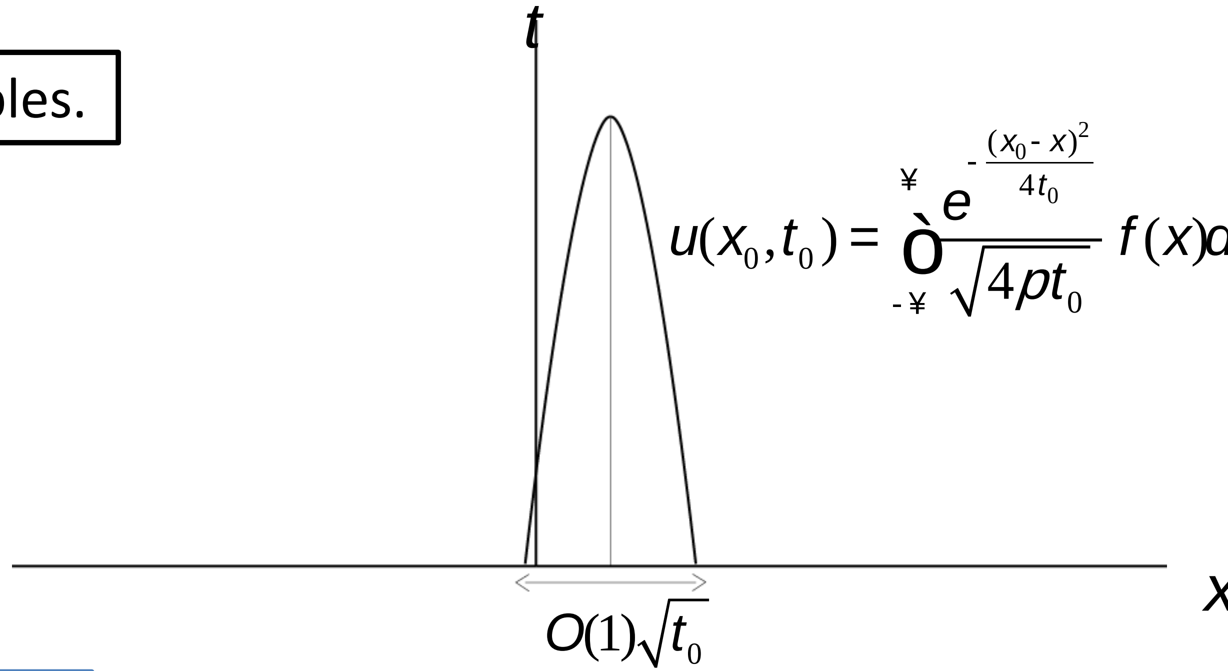
$$\begin{cases} u_{tt} - \mathrm{D}u = 0, \qquad x^1 > 0 \\ u(\vec{x},0) = f(\vec{x}) \\ u_t(\vec{x},0) = g(\vec{x}) \\ u_{x^1}(0,x^2,x^3,t) = 0. \end{cases}$$

(Even Extension)

Some Simple Examples.

Heat Equation

$$\begin{cases} u_t - u_{xx} = 0 \\ u(x,0) = f(x) \end{cases}$$



$$u(x_0,t_0) = \int_{-\infty}^{\infty} \frac{e^{-\frac{(x_0-x)^2}{4t_0}}}{\sqrt{4\pi t_0}} f(x)dx$$

Heat Equation in Half Space

$$\begin{cases} u_t - u_{xx} = 0, \qquad x > 0. \\ u(x,0) = f(x) \\ u(0,t) = 0. \end{cases}$$

(Odd Extension)

$$\begin{cases} u_t - u_{xx} = 0, \qquad x > 0. \\ u(x,0) = f(x) \\ u_x(0,t) = 0. \end{cases}$$

(Even Extension)

Fundamental Solution, Green's Identity, and Green's function for a Convent Heat Equation

$$\text{Green's function for } (\partial_t + \partial_x - \partial_{xx})u(x,t) = 0, \quad x > 0, \ t > 0, \ u(0,t) = 0.$$

$$(\partial_t + \partial_x - \partial_{xx})u(x,t) = 0, \quad x > 0, \ t > 0,$$

$$u(x,0) = \delta(x - y),$$

$$u(0,t) = 0.$$

$$\textit{Fundamental Solution}: \quad g(x,t) = e^{-\frac{(x-t)^2}{4t}} / \sqrt{4\pi t}$$

$$\textit{Green's idendity for } v(x,t) \equiv u(x,t) - g(x-y,t):$$

$$v(x,t) = \int_0^t g(x,t-\tau)(v(0,\tau) - v_x(0,\tau))d\tau - \int_0^t g_x(x,t-\tau)v(0,\tau)d\tau$$

where $v(0,\tau) = -g(-y,\tau).$

Dirichlet-Neumann Relationship

$$(\partial_t + \partial_x - \partial_{xx})v(x,t) = 0, \quad x > 0, \quad t > 0,$$

$$v(x,0) = 0,$$

$$v(0,t) = -g(-y,t).$$

Laplace-Laplace transformation:

$$x > 0, \; s>0,$$

$$V(x,s) \equiv \int_0^\infty e^{-st} v(x,t)dt, \quad W(\xi,s) \equiv \int_0^\infty e^{-\xi x} v(x,s)dx.$$

$$W(\xi,s) = \frac{(1-\xi)V(0,s) - V_x(0,s)}{s+\xi-\xi^2}; \quad \xi_\pm = \frac{1 \pm \sqrt{1+4s}}{2}$$

$$V(x,s) = e^{\xi_+ x}\frac{(1-\xi_+)V(0,s) - V_x(0,s)}{-(\xi_+ - \xi_-)} + e^{\xi_- x}\frac{(1-\xi_-)V(0,s) - V_x(0,s)}{-(\xi_- - \xi_+)}$$

$$(1-\xi_+)V(0,s) - V_x(0,s) = 0.$$

$$v_x(0,t) = -\frac{1}{2}\left(\frac{e^{-\frac{t}{16}}}{\sqrt{\pi t}} - Erfc\left(\frac{\sqrt{t}}{4}\right)\right) * v_t(0,t)$$

Dirichlet-Neumann Relationships is valid for boundary value problems.

$$(\partial_t + \partial_x - \partial_{xx}) v(x,t) = 0, \quad x > 0, \quad t > 0,$$

$$v(x,0) = 0$$

$$V(x,s) \equiv \int_0^\infty e^{-st} v(x,t)\,dt, \quad \Omega(x,s) \equiv \int_0^\infty e^{-\xi x} v(x,s)\,dx$$

$$(1 - \xi_+) V(0,s) - V_x(0,s) = 0.$$

Example: given Robin Boundary Condition $\alpha v(0,t) + v_x(0,t) = f(t).$

$$V(0,s) = L(f)/(1 + \alpha - \xi_+), \quad L(f) = \int_0^\infty e^{-st} f(t)\,dt.$$

$$v(0,t) = -e^{\frac{\alpha^2+\alpha}{4}t} * \left(\frac{3+2\alpha}{8}\delta(t) + \frac{1}{2}\left(\frac{1}{\sqrt{\pi t}} - \frac{1}{4} Erfc\left(\frac{\sqrt{t}}{4}\right)\right) * \partial_t \right) * f(t)$$

Example: Linear Compressible Navier-Stokes Equation in Half-Space

$$\begin{cases} \partial_t r + \partial_x u = 0, \\ \partial_t u + \partial_x r = \partial_{xx} u, \end{cases} \quad \begin{cases} u(x,0) = r(x,0) = 0, & x > 0, \quad t > 0. \\ u(0,t) = u_b(t), \end{cases} ,$$

$$\textit{Fundamental Solution}[\textit{Liu - Zeng}, 1994]: \quad \mathrm{G}(x,t) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

$$|g_{11} - e^{-t} d(x)|, \ |g_{jk}| \le O(1) \left\{ \left(e^{-\frac{(x-t)^2}{Ct}} + e^{-\frac{(x+t)^2}{Ct}} \right) / \sqrt{t} + e^{-\frac{|x|+t}{C}} \right\}, \quad (j,k) \neq (1,1).$$

Laplace-Laplace transformation:

$$\begin{cases} R(x,s) = \int_0^\infty e^{-st} r(x,t)\,dt, \\ T(x,s) = \int_0^\infty e^{-xx} R(x,s)\,dx \end{cases} \quad \begin{cases} U(x,s) = \int_0^\infty e^{-st} u(x,t)\,dt, \\ W(z,s) = \int_0^\infty e^{-xx} U(x,s)\,dx, \end{cases}$$

(Conti.)

Laplace-Laplace transformation:

$$\begin{cases} R(x,s) = \int_0^\infty e^{-st} r(x,t)\,dt, \\ T(x,s) = \int_0^\infty e^{-xx} R(x,s)\,dx \end{cases} \qquad \begin{cases} U(x,s) = \int_0^\infty e^{-st} u(x,t)\,dt, \\ W(z,s) = \int_0^\infty e^{-xx} U(x,s)\,dx, \end{cases}$$

$$\begin{pmatrix} s & x \\ x & s - x^2 \end{pmatrix} \begin{pmatrix} T \\ W \end{pmatrix} = \begin{pmatrix} U(0,s) \\ R(0,s) - xU(0,s) - U_x(0,s) \end{pmatrix}, \quad sR(0,s) + U_x(0,s) = 0.$$

$$W(x,s) = \frac{U_x(0,s) + xU(0,s)}{x^2 - \dfrac{s^2}{1+s}}, \qquad U(x,s) = e^{x_+ x} \frac{U_x(0,s) + x_+ U(0,s)}{x_+ - x_-} + e^{x_- x} \frac{U_x(0,s) + x_- U(0,s)}{x_- - x_+}$$

$$x_\pm = \pm s / \sqrt{1+s}.$$

Dirichlet-Neumann Relationship

$$U_x(0,s) + x_+ U(0,s) = 0.$$

$$u_x(0,t) = -\frac{e^{-t}}{\sqrt{pt}} * u_t(0,t).$$

The method for solving initial-boundary value problems:

$$u_t = P(\partial_x)u, \quad u(x,t) \in R^n, \; x > 0.$$

- Subtract (fundamental solution) *(initial data)
- Take Laplace-Laplace transformation.
- Compute the roots of the characteristic polynomial

$$\det(s - P(x)) = 0 \; \textit{with} \; s>0, \; \mathrm{Re}(x) > 0.$$

- Express the Dirichlet-Neumann relation to obtain the full boundary data
- Use the fundamental solution, Green's Identity, and full boundary data to reconstruct the solution in the interior of the space domain.

The method for an initial-boundary value problem:

$$u_t = P(\partial_{x^1}, \partial_{x^2}, .., \partial_{x^m})u, \quad u(x^1, .., x^m, t) \in R^n, \; x^1 > 0.$$

- Subtract (fundamental solution)* (initial data)
- Take Laplace-Laplace transformation.
- Compute the roots of the characteristic polynomial

$$\det(s - P(x, ih^2, ..., ih^m)) = 0 \; with \; s>0, \; \mathrm{Re}(x) > 0, h^j \in R.$$

- Express the Dirichlet-Neumann relation to obtain the full boundary data.
- Use the Fundamental solution, Green’s identity, and full boundary datum to reconstruct the solution in the interior of space domain.

Example: A 2x2 System in half 2-D Space domain

$$\left((\partial_t + \Lambda \partial_x) + \begin{pmatrix} \partial_x & \partial_y \\ \partial_y & -\partial_x \end{pmatrix} - \Delta \right) \begin{pmatrix} u(x,y,t) \\ v(x,y,t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{matrix} t > 0 \\ x > 0 \\ y \in R \end{matrix} \quad \Lambda \in (0,1).$$

$$u(0,y,t) = u_b(y,t); \quad v(0,y,t) = v_b(y,t); \quad u(x,y,0) = v(x,y,0) = 0.$$

$$U(x,\eta,s) = \int_0^\infty e^{-st} \int_{-\infty}^\infty e^{-iy\eta} u(x,y,t)\,dy\,dt; \quad \widehat{U}(\xi,\eta,s) = \int_0^\infty e^{-\xi x} U(x,\eta,s)\,dx.$$

$$V(x,\eta,s) = \int_0^\infty e^{-st} \int_{-\infty}^\infty e^{-iy\eta} v(x,y,t)\,dy\,dt; \quad \widehat{V}(\xi,\eta,s) = \int_0^\infty e^{-\xi x} V(x,\eta,s)\,dx.$$

$$U_b(\eta,s) = \int_0^\infty e^{-st} \int_{-\infty}^\infty e^{-iy\eta} u_b(y,t)\,dy\,dt.$$

$$V_b(\eta,s) = \int_0^\infty e^{-st} \int_{-\infty}^\infty e^{-iy\eta} v_b(y,t)\,dy\,dt.$$

$$\begin{pmatrix} s + (\Lambda + 1)\xi - \xi^2 + \eta^2 & i\eta \\ i\eta & s + (\Lambda - 1)\xi - \xi^2 + \eta^2 \end{pmatrix} \begin{pmatrix} \widehat{U} \\ \widehat{V} \end{pmatrix} = \left. \begin{pmatrix} (-\xi + \Lambda + 1)U_b - U_x \\ (-\xi + \Lambda - 1)V_b - V_x \end{pmatrix} \right|_{x=0},$$

(Conti.)

$$\begin{pmatrix} \widehat{U} \\ \widehat{V} \end{pmatrix} = \frac{1}{p(x,ih,s)} \begin{pmatrix} s+(\Lambda-1)x-x^2+h^2 & -ih \\ -ih & s+(\Lambda+1)x-x^2+h^2 \end{pmatrix} \begin{pmatrix} (-x+\Lambda+1)U_b-U_x \\ (-x+\Lambda-1)V_b-V_x \end{pmatrix}\Bigg|_{x=0},$$

Characteristic Polynomial: $p(x,ih,s)=(s+\Lambda x-x^2+h^2)^2-x^2+h^2$

$$x_1=\sqrt{\left((s+\Lambda x_1+\frac{1}{4})^{1/2}+\frac{1}{2}\right)^2+h^2},\ x_2=\sqrt{\left((s+\Lambda x_2+\frac{1}{4})^{1/2}-\frac{1}{2}\right)^2+h^2},\ x_3=-\sqrt{\left((s+\Lambda x_3+\frac{1}{4})^{1/2}-\frac{1}{2}\right)^2+h^2},$$
$$x_4=-\sqrt{\left((s+\Lambda x_4+\frac{1}{4})^{1/2}+\frac{1}{2}\right)^2+h^2}.$$

For $s>0,\ h\in R$, the roots are $x=x_j(h,s)$, j=1, 2, 3, 4

$\mathrm{Re}(x_1),\ \mathrm{Re}(x_2)>0>\mathrm{Re}(x_3),\ \mathrm{Re}(x_4)$.

Dirichlet-Neumann Relationship

$$\begin{cases} (s+(\Lambda-1)x_1-(x_1)^2)((-x_1+\Lambda+1)U_b-U_x)-ih((-x_1+\Lambda-1)V_b-V_x)\big|_{x=0}=0 \\ (s+(\Lambda-1)x_2-(x_2)^2)((-x_2+\Lambda+1)U_b-U_x)-ih((-x_2+\Lambda-1)V_b-V_x)\big|_{x=0}=0 \end{cases}$$

(Conti.)

The Boundary Kernel in the Laplace domain

$$\left.\begin{pmatrix} U_x \\ V_x \end{pmatrix}\right|_{x=0} = \begin{pmatrix} K_{11}(h,s;x_1,x_2) & K_{12}(h,s;x_1,x_2) \\ K_{21}(h,s;x_1,x_2) & K_{22}(h,s;x_1,x_2) \end{pmatrix} \begin{pmatrix} U_b \\ V_b \end{pmatrix},$$

$$K_{11}(h,s;x_1,x_2) = \frac{-1-(x_1)^2 - 2x_2 - (x_2)^2 - x_1(2+x_2) + s + h^2 + L^2}{1 + x_1 + x_2 - L}$$

$$K_{12}(h,s;x_1,x_2) = -\frac{ih}{1 + x_1 + x_2 - L}$$

$$K_{21}(h,s;x_1,x_2) = -i\,\frac{(-(x_1)^2 + s + h^2 + x_1(-1+L))(-(x_2)^2 + s + h^2 + x_2(-1+L))}{1 + x_1 + x_2 - L}$$

$$K_{22}(h,s;x_1,x_2) = \frac{-1-(x_1)^2 - 2x_2 - (x_2)^2 - x_1(2+x_2) + s + h^2 + L^2}{1 + x_1 + x_2 - L}$$

Inversion to Space-Time variables

1. Analytic extend variable s in $x_i(h,s)$ from $s > 0$ to sÎ SÌ C via $p(x_i, h, s) = 0$.
2. Classify the roots of $p(x,h,s) = 0$ into two types:

   Non-Characteristic root: If $x_i(h,s)$ is analytic in s for $|s|, |h| \ll 1$.

   Characteristic root: otherwise.

$x_1(h,s)$ and $x_4(h,s)$: Non-Characteristic.

$x_2(h,s)$ and $x_3(h,s)$: Characteristic.

$\xi_i(\eta, s) \sim O(1)\sqrt{s}$ as $s \to \infty$.

(Conti.)

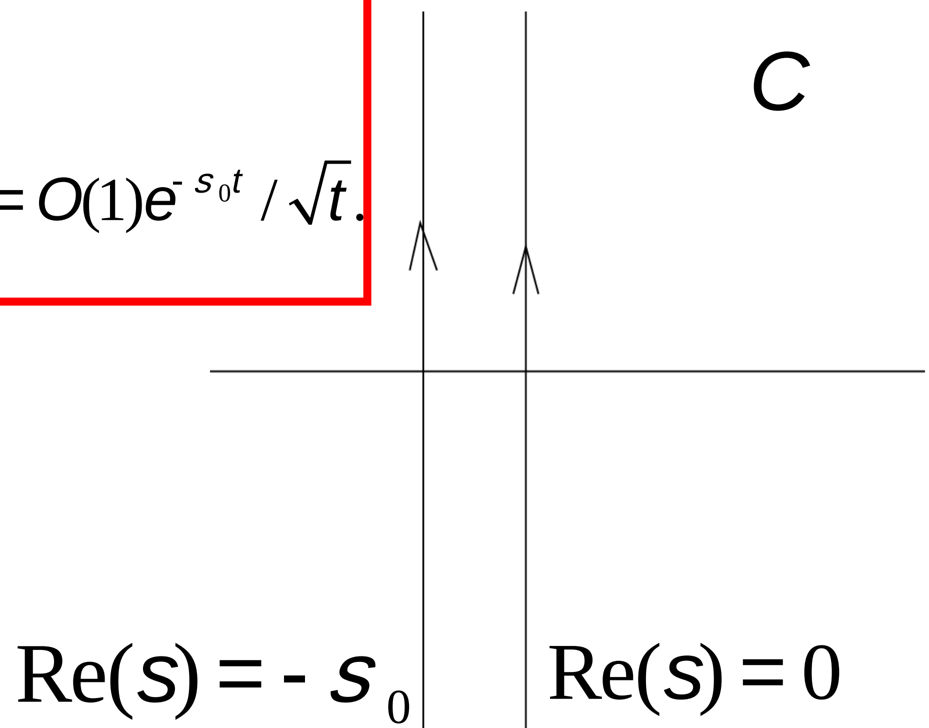
Non-Characteristic root.

$$x_1(h,s) = \frac{x_1(h,s) - x_1(h,0)}{s} s + x_1(h,0)$$

$$\leftrightarrow L^{-1}\left[\frac{x_1(h,s) - x_1(h,0)}{s}\right] * \partial_t + x_1(h,0)\delta(t).$$

$$L^{-1}\left[\frac{x_1(h,s) - x_1(h,0)}{s}\right](t)$$

$$= \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=0} e^{st}\left[\frac{x_1(h,s) - x_1(h,0)}{s}\right] ds = O(1)e^{-s_0 t} / \sqrt{t}.$$



(Conti.)

Characteristic root.

Comparison on the Laplace variable and the Fourier variable in the normal direction.

The Laplace variable in the normal direction.

$$p(x,h,s) = \det\begin{pmatrix} s+(L+1)x - x^2 + h^2 & ih \\ ih & s+(L-1)x - x^2 + h^2 \end{pmatrix} = 0.$$

$$x = x_2(h,s) \leftrightarrow e^{x_2(h,s)x}$$

The wave train is written in x and s variables.

The Fourier variable in the normal direction.

$$0 = \det\begin{pmatrix} s + (L+1)iV + V^2 + h^2 & ih \\ ih & s + (L-1)iV + V^2 + h^2 \end{pmatrix}.$$

$$x_2(h,s(V)) = iV \leftrightarrow e^{iVx}$$

The wave train is written in x and V variables.

s is the spectrum w.r.t. (V,h).

(Conti.)

The connection between the Laplace-Fourier variable.

Laplace-Fourier Path

Definition: $\{s(V)\in C \mid p(iV,h,s(V))=0,\quad V\in R\}$

Cauchy's integral formula.

The key transform

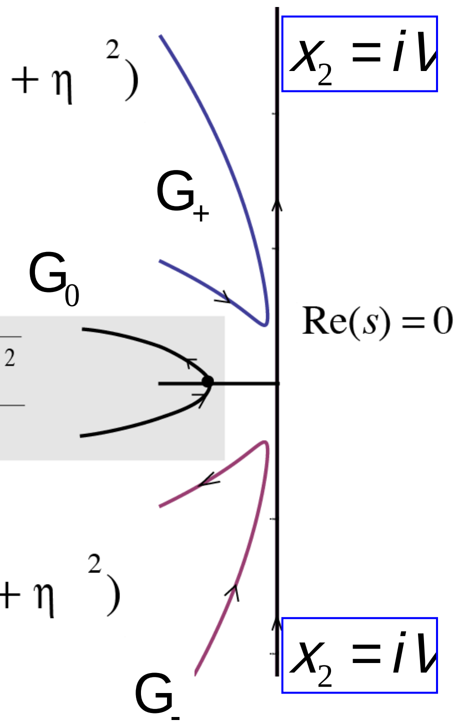
$$\int_{-\infty}^{\infty} e^{iyh}\left(\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=0} e^{st}\frac{\partial x_2(h,s)}{\partial s}ds\right)dh=$$

$$\int_{-\infty}^{\infty} e^{iyh}\left(\frac{1}{2\pi i}\int_{G_+ + G_0 + G_-} e^{st}\frac{\partial x_2(h,s)}{\partial s}ds\right)dh$$

$$s=i(-\Lambda\zeta+\sqrt{\zeta^2+\eta^2})-(\zeta^2+\eta^2)$$

$$s=-\frac{1+4\zeta^2+2\Lambda\sqrt{(1+i2\zeta)^2+4\eta^2}}{4}$$

$$s=i(-\Lambda\zeta-\sqrt{\zeta^2+\eta^2})-(\zeta^2+\eta^2)$$



$$\int_{-\infty}^{\infty} e^{iyh}\left(\frac{1}{2\pi i}\int_{G_+ + G_-} e^{st}\frac{\partial x_2(h,s)}{\partial s}ds\right)dh$$

$$=\int_{-\infty}^{\infty} e^{iyh}\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\cos\left(\left(V^2+h^2\right)^{1/2}t\right)e^{(-i\Lambda V-(V^2+h^2))t}\,dV\right)dh=\frac{1}{\pi}\iint_{(V,h)\in R^2} e^{iyh-i\Lambda Vt-(V^2+h^2)t}\cos\left(\left(V^2+h^2\right)^{1/2}t\right)dV\,dh$$

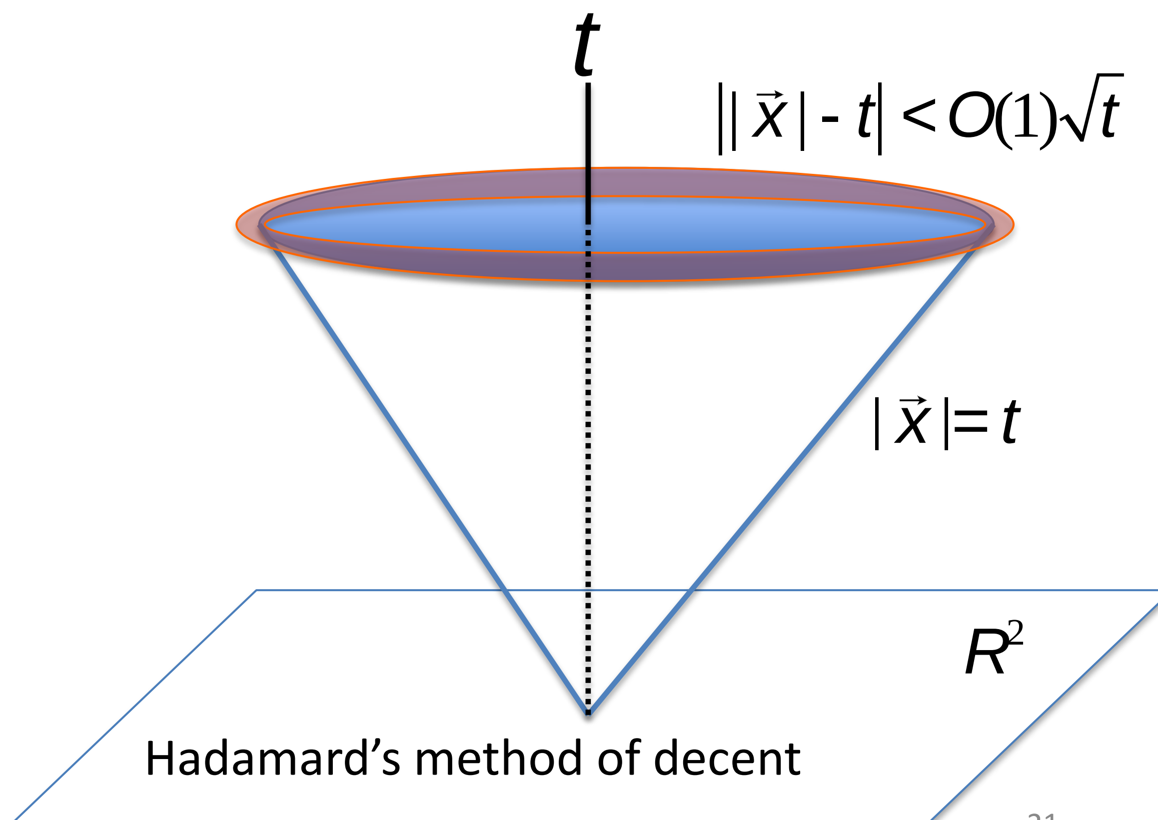
(Conti.)

More precisely,

$$\frac{1}{p}\iint_{(V,h)\in R^2} e^{ixV+iyh-i\mathrm{L}Vt-\left(V^2+h^2\right)t}\cos\left(\left(V^2+h^2\right)^{1/2}t\right)dVdh = W(x-\mathrm{L}t,y,t),$$

where g satisfies

$$\begin{cases}(\partial_t^2-\partial_x^2-\partial_y^2)W=0\\ W(x,y,0)=\dfrac{e^{-\frac{x^2+y^2}{4t}}}{4pt}\\ \partial_t W(x,y,0)=0.\end{cases}$$



The divisor in the Boundary Kernel in the Laplace domain:

$$\frac{1}{1+x_1+x_2-L}$$

$x_1(h,s)$ and $x_4(h,s)$: Non-Characteristic.
$x_2(h,s)$ and $x_3(h,s)$: Characteristic.

$$q(x,h,s)=\frac{p(x,h,s)}{(x-x_1)(x-x_4)}=x^2+q_1(h,s)x+q_0(h,s)=(x-x_2)(x-x_3).$$

$$\frac{1}{1+x_1+x_2-L}=\frac{1+x_1+x_3-L}{\left(1+x_1+x_2-L\right)\left(1+x_1+x_3-L\right)}$$

$$=\frac{1+x_1-q_1(h,s)-x_2-L}{\left(1+x_1-L\right)^2-\left(1+x_1-L\right)q_1(h,s)+q_0(h,s)}$$

$$=\frac{1+x_1-q_1(h,s)-L}{\left(1+x_1-L\right)^2-\left(1+x_1-L\right)q_1(h,s)+q_0(h,s)}$$

$$-\frac{-x_2}{\left(1+x_1-L\right)^2-\left(1+x_1-L\right)q_1(h,s)+q_0(h,s)}$$

Recombination of the boundary kernel

$$K_{11}(h,s;x_1,x_2)=\frac{-1-(x_1)^2-2x_2-(x_2)^2-x_1(2+x_2)+s+h^2+L^2}{1+x_1+x_2-L}$$

$$K_{12}(h,s;x_1,x_2)=-\frac{ih}{1+x_1+x_2-L}$$

$$K_{21}(h,s;x_1,x_2)=-i\frac{(-(x_1)^2+s+h^2+x_1(-1+L))(-(x_2)^2+s+h^2+x_2(-1+L))}{1+x_1+x_2-L}$$

$$K_{22}(h,s;x_1,x_2)=-\frac{-1+s+h^2+x_2(-1+L)+2L-L^2+x_1(-1+x_2+L)}{1+x_1+x_2-L}$$

$x_1(h,s)$ and $x_4(h,s)$: Non-Characteristic

$x_2(h,s)$ and $x_3(h,s)$: Characteristic

$$(x-x_2)(x-x_3)=x^2+q_1(h,s)x+q_0(h,s)$$

$$K_{11}(h,s;x_1,x_2)=A_{11}(h,s)+B_{11}(h,s)x_2$$

$$K_{12}(h,s;x_1,x_2)=A_{12}(h,s)+B_{12}(h,s)x_2$$

$$K_{21}(h,s;x_1,x_2)=A_{21}(h,s)+B_{21}(h,s)x_2$$

$$K_{22}(h,s;x_1,x_2)=A_{22}(h,s)+B_{22}(h,s)x_2$$

where $A_{ij}(h,s),\ B_{ij}(h,s)$ satisfy

$\$h_0,\ s_0>0\ \ '\ \ "\ |h|<h_0$

$A_{ij}(h,s)$ and $B_{ij}(h,s)$ are analytic for

$s\in\{z\,|\,\mathrm{Re}(s)>-s_0\}$.

Long wave component of the boundary kernel on the boundary

Let $V_b(h,s)$ *and* $\hat{V}_b(h,t)$ *be the Fourier-Laplace transformation and Fourier transformation of an input function* $v_b(y,t)$.

For $(i,j) \neq (2,1)$,

$$L^{-1}\left[\int_{-h_0}^{h_0} e^{iyh} K_{ij}(h,s;x_1,x_2)V_b(h,s)\,dh\right](t)$$

$$\leq O(1)\int_0^{\max(\frac{t}{2},t-1)} \left(\frac{|W(x,z,t-\tau)|}{t-\tau} \underset{(x,z)}{*} \frac{e^{-\frac{x^2+z^2}{4(t-\tau)}}}{t-\tau} \right)\Bigg|_{(x,z)=(-\Lambda(t-\tau),y)} \underset{y}{*} |v_b(y,\tau)|\,d\tau$$

$$+O(1)\max_{\substack{\tau\in(\max(t-1,\frac{t}{2}),t)\\ |h|<h_0}} |\hat{V}_b(h,\tau)| + O(1)\int_0^t e^{-\frac{t-\tau}{C_0}}\left(\left\|\partial_t v_b(\cdot,\tau)\right\|_{L^1_y} + \left\|v_b(\cdot,\tau)\right\|_{L^1_y}\right)d\tau.$$

Energy estimates for short wave components and the global wave structure.

(Short wave component.)

$$\left((\partial_t + \Lambda \partial_x) + \begin{pmatrix} \partial_x & i\eta \\ i\eta & -\partial_x \end{pmatrix} - (\partial_x^2 - \eta^2) \right) \begin{pmatrix} \hat{u}(x,\eta,t) \\ \hat{v}(x,\eta,t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{matrix} t > 0 \\ x > 0 \\ |\eta| \ge \eta_0 \end{matrix}$$

$$\hat{u}(0,\eta,t) = \hat{u}_b(\eta,t); \quad \hat{v}(0,\eta,t) = \hat{v}_b(\eta,t); \quad \hat{u}(x,\eta,0) = \hat{v}(x,\eta,0) = 0.$$

$$\int_0^t \left(\left| \partial_t^j \partial_x \hat{u}(0,\eta,\tau) \right|^2 + \left| \partial_t^j \partial_x \hat{v}(0,\eta,\tau) \right|^2 \right) e^{-\eta^2(t-\tau)} d\tau$$

$$\le O(1) \sum_{k=0}^{1} \int_0^t \left(\left| \partial_t^{k+j} \hat{u}_b(\eta,\tau) \right|^2 + \left| \partial_t^{k+j} \hat{v}_b(\eta,\tau) \right|^2 \right) e^{-\eta^2(t-\tau)} d\tau, \quad j = 0,1.$$

(conti.)

(Space-time exponential decay.)

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} e^{-a(y-bt)}\begin{pmatrix} u \\ v \end{pmatrix}\cdot\left((\partial_t + L\,\partial_x) + \begin{pmatrix} \partial_x & \partial_y \\ \partial_y & -\partial_x \end{pmatrix} - D\right)\begin{pmatrix} u \\ v \end{pmatrix} dxdy = 0, \quad a, b > 0.$$

$$\int_0^t\int_{-\infty}^{\infty}\left(\left|\bar{\partial}^g \partial_x u(0,y,t)\right|^2 + \left|\bar{\partial}^g \partial_x v(0,y,t)\right|^2\right) e^{a(y-bt)-\frac{a(b-a-1)}{4}(t-t)}\,dydt$$

$$\le O(1)\sum_{k=0}^{1}\int_0^t\int_{-\infty}^{\infty}\left(\left|\bar{\partial}^g \partial_t^k u_b(y,t)\right|^2 + \left|\bar{\partial}^g \partial_t^k v_b(y,t)\right|^2\right) e^{a(y-bt)-\frac{a(b-a-1)}{4}(t-t)}\,dydt,$$

$$\text{where } g = (g^1, g^2),\ \bar{\partial}^g = \partial_t^{g^1}\partial_y^{g^2},\quad g^1 + g^2 \ge 2.$$

$$\left|\partial_x u(0,y,t)\right| + \left|\partial_x v(0,y,t)\right| \le e^{-\frac{a|y|}{4}-\frac{a(b-a-1)t}{8}}$$

$$\cdot O(1)\left(\sum_{k=0}^{1}\int_0^t\int_{-\infty}^{\infty}\left(\left|\bar{\partial}^g \partial_t^k u_b(y,t)\right|^2 + \left|\bar{\partial}^g \partial_t^k v_b(y,t)\right|^2\right) e^{a(y-bt)-\frac{a(b-a-1)}{4}(t-t)}\,dydt\right)^{1/2}$$

$$\text{for } y > 2bt.$$

The conclusion from the example.

$$\left((\partial_t + L\,\partial_x) + \begin{pmatrix} \partial_x & \partial_y \\ \partial_y & -\partial_x \end{pmatrix} - D \right) \begin{pmatrix} u(x,y,t) \\ v(x,y,t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$