Slow motion for degenerate potentials

Fabrice Bethuel

Laboratoire Jacques-Louis Lions,
Université Pierre et Marie Curie – Paris 6

Singular limits problems in nonlinear PDE's

Collège de France, décembre 2013

joint work with Didier Smets (Paris 6)

Reaction-diffusion equations of gradient type

We investigate the behavior of solutions v of reaction-diffusion equations of gradient type

$$\partial_t v_\varepsilon - \frac{\partial^2 v_\varepsilon}{\partial x^2} = -\varepsilon^{-2}\nabla V_\varepsilon(v). \qquad (RDG_\varepsilon)$$

The function v is a function of the space variable $x \in \mathbb{R}$ and the time variable $t \geq 0$ and takes values in some euclidean space $\mathbb{R}^k$, so that (RDG) is a system of k scalar partial differential equations.

Here $< 0 < \varepsilon \leq 1$ denotes a small parameter representing a typical lenght. It is kind of virtual since it can be scaled out and put equal to 1 by the change of variables

$$v(x,t) = v_\varepsilon(\varepsilon^{-1}x, \varepsilon^{-2}t)$$

so that v satisfies (RDG) with $\varepsilon = 1$.

Equation (RDG) is the L^2 gradient-flow of the energy $\mathcal{E}$ defined by

$$\mathcal{E}_\varepsilon(u) = \int_{\mathbb{R}} e_\varepsilon(u) = \int_{\mathbb{R}} \varepsilon \frac{|\dot{u}|^2}{2} + \frac{V(u)}{\varepsilon}, \text{ for } u : \mathbb{R} \mapsto \mathbb{R}^k.$$

The properties of the flow (RDG) strongly depend on the potential V .

Throughout we assume that

- V is smooth from $\mathbb{R}^k$ to $\mathbb{R}$,
- V tends to infinity at infinity, so that it is bounded below

$$V \geq 0.$$

An intuitive guess is that the flow drives to mimimizers of the potential :

- if V is strictly convex, the solution should tend to the unique minimizer of the potential V.
- Here we consider the case where there are several mimimizers for the potential $V \rightsquigarrow$ Transitions between minimizers

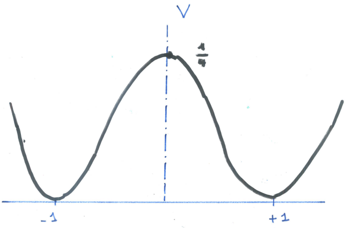
Multiple-well potentials

We assume in this talk that V is has a finite number of and at least two distinct minimizers.

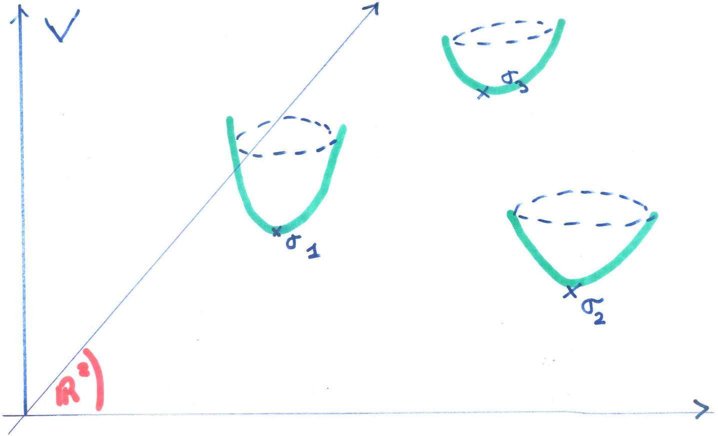
A classical example in the scalar case (Allen-Cahn) $k = 1$

$$V(u) = \frac{(1-u^2)^2}{4}, \qquad (AC)$$

whose minimizers are $+1$ and -1.



The picture for systems



Assumptions on V

(H_1) inf $V = 0$ and the set of minimizers $\Sigma \equiv \{y \in \mathbb{R}^k, V(y) = 0\}$

is a finite set, with **at least two** distinct elements, that is

$$\Sigma = \{\sigma_1, ..., \sigma_q\}, q \geq 2, \ \sigma_i \in \mathbb{R}^k, \forall i = 1, ..., q.$$

(H_∞) There exists constant $\alpha_0 > 0$ and $R_0 > 0$ such that

$$y \cdot \nabla V(y) \geq \alpha_0 |y|^2, \ \text{if } |y| > R_0.$$

Stationary solutions

Simple solutions to (RDG) are provided by stationary ones, that is solutions of the form

$$v(x,t) = u(x), \ \forall x \in \mathbb{R},$$

where the profil $u : \mathbb{R} \mapsto \mathbb{R}^k$ is a solution of the ODE

$$-u_{xx} = -\nabla V(u). \qquad (ODE)$$

For instance :

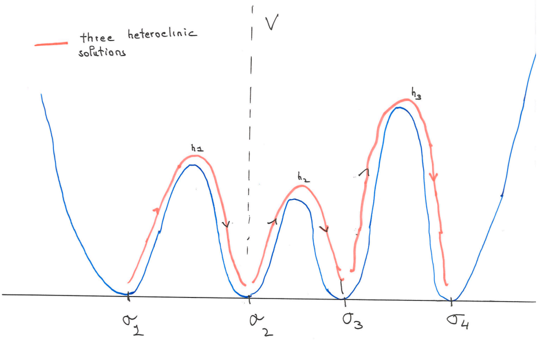
- Constant functions $v(x,t) = \sigma$, where σ is a critical point of V
- Stationary fronts. Solutions to (ODE) tending, as $x \to \pm\infty$ to critical points of the potential V.

Conservation of energy for (ODE) implies $V(u(+\infty)) = V(u(-\infty))$.

Heteroclinic solutions

We focus next the attention on heteroclinic solutions to (ODE) joining two distinct minimizers σ_i and σ_j.

This is a difficult topic in general (see e. g works by N. Alikakos and collaborators). However, it is an exercice in the scalar case $k = 1$.



Indeed, in the scalar case $k = 1$ equation (ODE) may be integrated explicitely thanks to the method of separation of variables.

Separation of variables

Conservation of energy for (ODE) yields $\frac{\dot{u}^2}{2} = V(u)$. Set

$$\gamma_i(u) = \int_{z_i}^{u} \frac{ds}{\sqrt{2V(s)}}, \quad z_i \text{ given and fixed in } (\sigma_i, \sigma_{i+1})$$

Define

$$\zeta_i^+(x) = \gamma_i^{-1}(x)$$

from $\mathbb{R}$ to (σ_i, σ_{i+1}) and $\zeta_i^-(x) = \zeta_i^{-1}(-x)$. We verify that $\zeta_i^+(\cdot)$ and $\zeta_i^-(\cdot)$ solve (ODE) and hence (RDG).

Lemma

Let u be a solution to (ODE) such that $u(x_0) \in (\sigma_i, \sigma_{i+1})$, for some x_0, and some $i \in 1, \dots q-1$. Then

$$u(x) = \zeta_i^+(x-a), \forall x \in I, \quad \text{or} \quad u(x) = \zeta_i^-(x-a), \forall x \in I,$$

for some $a \in \mathbb{R}$.

Fronts

In the context of the reaction-diffusion equation $(RDG)_\varepsilon$, heteroclinic stationary solutions or their perturbations are often termed fronts.
Notice that, if we set for $a \in \mathbb{R}$

$$\xi^{\pm}_{i,a,\varepsilon}(\cdot) = \xi^{\pm}\left(\frac{\cdot - a}{\varepsilon}\right)$$

Then $\xi^{\pm}_{i,a,\varepsilon,}$ is a stationary solution to $(RDG)_\varepsilon$. Notice that

$$\xi^{\pm}_{i,a,\varepsilon} \to H^{\pm}_{i,a,}$$

where $H^{\pm}_{i,a,}$ is a step function joining σ_i to σ_{i+1} with a transition at the front point a , for instance

$$\begin{cases} H^{+}_{i,a}(x) = \sigma_{i+1}, \text{ for } x > a \\ H^{+}_{i,a}(x) = \sigma_i \quad \text{for } x < a. \end{cases}$$

The case of degenerate potentials

Since the points σ_i are minimizers for the potential, we have

$$D^2 V(\sigma_i) \geq 0.$$

In this talk, we will focus on the case the potentials are degenerate, that is

$$D^2 V(\sigma_i) = 0.$$

More precisely, We assume that for all $i \neq j$ in $\{1, \cdots, q\}$ there exists a number $\theta_i > 1$ and numbers $\lambda_i^{\pm}$ such that near σ_i

$$(\mathrm{H}_2) \qquad \lambda_i^- |y - \sigma_i|^{2(\theta_i - 1)} \mathrm{Id} \leq D^2 V(y) \leq \lambda_i^+ |y - \sigma_i|^{2(\theta_i - 1)} \mathrm{Id}$$

We set

$$\theta \equiv \mathrm{Max}\{\theta_i, i = 1, \ldots q\}.$$

degenerate potentials in the scaler case

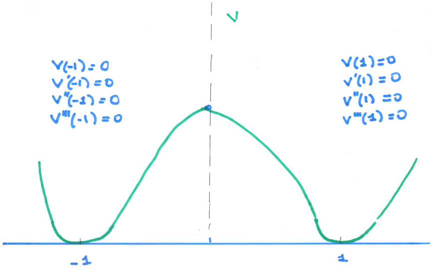
For scalar potentials, the degeneracy implies $V''(\sigma_i) = 0$, θ_i is related to the order of vanishing of the derivatives near σ_i.

$$\frac{d^j}{du^j} V(\sigma_i) = 0 \ \text{ for } j = 1, \dots, 2\theta - 1 \ \ \textit{and} \ \frac{d^{2\theta}}{du^{2\theta}} V(\sigma_i) \neq 0$$

For instance, for

$$V(u) = \frac{1}{4}(|u|^2 - 1)^4.$$

we have $\sigma_1 = -1$, $\sigma_2 = 1$ and $\theta = \theta_1 = \theta_2 = 2$.



A fundamental remark

In the scalar case, a fundamental difference between degenerate and non-degenerate potentials is seen at the level of the heteroclinic solutions, and their expansions near infinity :

- for degenerate potentials we have the algebraic decay

$$\begin{cases} \zeta_i^+(x) = \sigma_i + B_i^- |x - A_i^-|^{-\frac{1}{\theta-1}} + \underset{x\to-\infty}{O}\left(|x - A_i|^{-\frac{\theta}{\theta-1}})\right) \\ \zeta_i^+(x) = \sigma_{i+1} - B_i^+ |x - A_i^+|^{-\frac{1}{\theta-1}} + \underset{x\to+\infty}{O}\left(|x - A_{i+1}|^{-\frac{\theta}{\theta-1}})\right) \end{cases}$$

- whereas for non-degenerate potential we have an exponential decay

$$\begin{cases} \zeta_i^+(x) = \sigma_i + B_i^- \exp(\sqrt{\lambda_i}\, x) + \underset{x\to-\infty}{O}\left(\exp(2\sqrt{\lambda_i}\, x)\right) \\ \zeta_i^+(x) = \sigma_{i+1} - B_i^+ \exp(-\sqrt{\lambda_{i+1}}\, x) + \underset{x\to+\infty}{O}\left(\exp -(2\sqrt{\lambda_{i+1}}\, x)\right) \end{cases}$$

$$\lambda_i \equiv V''(\sigma_i).$$

The general principle for the reaction-diffusion equation

After this lenghty digression, we come back to the evolution equation

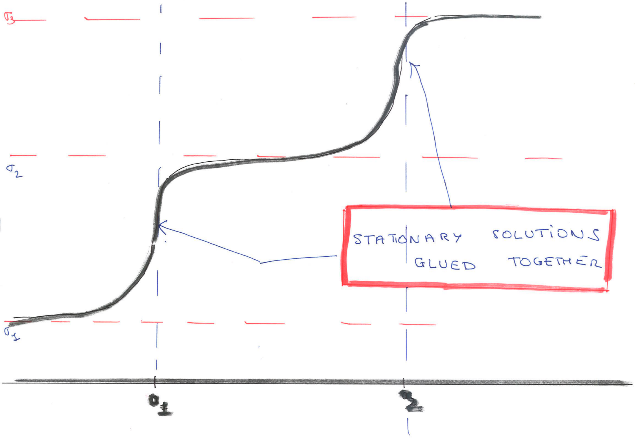
$$\partial_t v_\varepsilon - \frac{\partial^2 v_\varepsilon}{\partial x^2} = -\varepsilon^{-2} \nabla V_\varepsilon(v). \qquad (RDG)$$

The general principle we wish to establish imay be stated as follows :

The solution to (RDG) relaxes after a suitable time to a chain of stationary solutions which :

- interact algebraically weakly, hence stationary solutions or fronts are metastable
- renormalizing time, an equation may be derived for the front points in the limit $\varepsilon \to 0$, and in the scalar case.

.



STATIONARY SOLUTIONS
GLUED TOGETHER

Assumptions on the initial datum

The main assumption on the initial datum $v_0^\varepsilon(\cdot) = v(\varepsilon\cdot, 0)$ is that its energy is bounded. Given an arbitrary constant $M_0 > 0$, we assume that

$$(\mathrm{H}_0^\varepsilon) \qquad \mathcal{E}_\varepsilon(v_0^\varepsilon) \leq M_0 < +\infty.$$

In view of the classical energy identity

$$\mathcal{E}_\varepsilon(v^\varepsilon(\cdot, T_2)) + \varepsilon \int_{T_1}^{T_2} \int_{\mathbb{R}} \left| \frac{\partial v^\varepsilon}{\partial t} \right|^2 (x,t) dx\, dt = \mathcal{E}_\varepsilon(v^\varepsilon(\cdot, T_1)) \quad \forall\, 0 \leq T_1 \leq T_2\,,$$

So that, $\forall t > 0$,

$$\mathcal{E}_\varepsilon\left(v(\cdot, t)\right) \leq M_0.$$

In particular for every $t \geq 0$, we have $V(v(x,t)) \to 0$ as $|x| \to \infty$. It follows that

$$v^\varepsilon(x,t) \to \sigma_\pm \text{ as } x \to \pm\infty,$$

where $\sigma_\pm \in \Sigma$ does not depend on t.

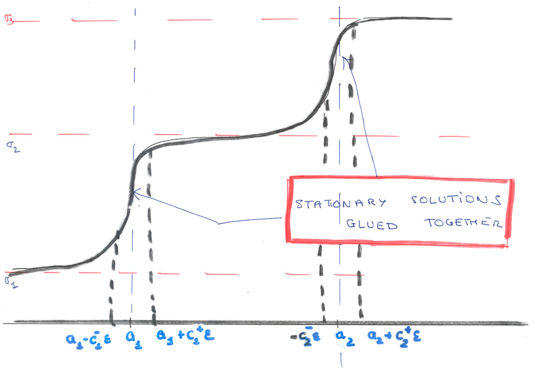
Front sets

For a given small parameter $\mu_0 > 0$ we introduce, for a map $u : \mathbb{R} \mapsto \mathbb{R}^k$ the front set $\mathcal{D}(u)$ defined by

$$\mathcal{D}(u) \equiv \{x, \text{dist}(u(x), \Sigma) > \mu_0\}.$$

We choose $\mu_0 > 0$ sufficiently so small so that, for $i = 1, \ldots, q$,

$$\Sigma \cap B(\sigma_i, \mu_0) = \{\sigma_i\}.$$



In the example on the figure the front set is of the form

$$\mathcal{D}(u) = \bigcup_{i=1}^{2} [a_i - c_i^- \varepsilon, a_i + c_i^+ \varepsilon].$$

A covering Lemma

A similar result may actually be deduced from the bound on the energy.

Proposition

Assume that the map u *satisfies* $\mathcal{E}(u) \leq M_0$*, and let* $\mu_0 > 0$ *be given. There exists* ℓ *points* $x_1, ..., x_\ell$ *such that*

$$\mathcal{D}(v) \subset \cup_{i=1}^{\ell} [x_i - \varepsilon, x_i + \varepsilon],$$

where the number ℓ *of points* x_i *is bounded by* $\ell \leq \ell_0 = \frac{3M_0}{\eta_1}$.

The measure of the front set is hence in of order ε, a small neighborhood of order ε of the points x_i.

Remarks on compactness

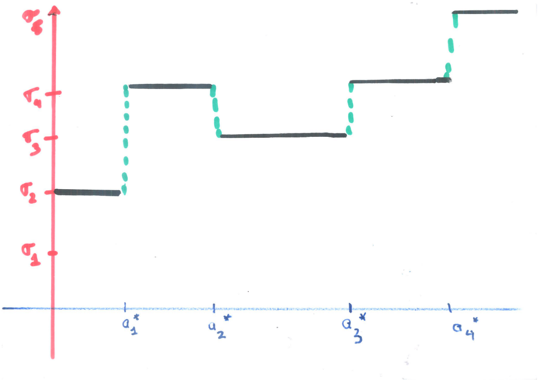
Let $(u_\varepsilon)_{\varepsilon>0}$ be a family of functions on $\mathbb{R}$ with $\mathcal{E}_\varepsilon(u_\varepsilon) \leq M_0$. It is classical tha up to a subsequence

$$u_\varepsilon \to u_\star \text{ in } L^1(\mathbb{R}),$$

where $u_\star$ takes values in Σ and is a step function : for finite number of points $-\infty \equiv a_0^* < a_1^* \ldots < a_{\ell*}^* < a_{\ell*+1}^* \equiv +\infty$ and

$$u_\star = \sigma_{i(k)}^+ \text{ on } \ (a_k^*, a_{k+1}^*) \ \text{ for } k = 0, \ldots, \ell^*, \text{ with } \sigma_{i(k)}^+ \in \Sigma.$$

Setting $\sigma_{i(k)}^- = \sigma_{i(k-1)}^+$, a transition occurs at a_k^* between $\sigma_{i(k)}^-$ and $\sigma_{i(k)}^+$. The points a_k^* are the limits as ε goes to 0 of the points x_i^ε.



On this figure

$$\sigma^-_{i(1)} = \sigma_2, \; \sigma^+_{i(1)} = \sigma^-_{i(2)} = \sigma_4, \quad \sigma^+_{i(2)} = \sigma^-_{i(3)} = \sigma_3,$$

$$\sigma^+_{i(3)} = \sigma^-_{i(4)} = \sigma_4, \; \sigma^+_{i(4)} = \sigma_5.$$

An upper bound on the motion of front sets

Theorem (B-Smets, 2011)

Assume (H_2) holds. Let $\varepsilon > 0$ be given and consider a solution v_ε to (RDG_ε). Assume that $v_\varepsilon(\cdot, 0)$ satisfies the energy bound (H_0^ε). If $r \geq \alpha_0 \varepsilon$, then

$$\mathcal{D}^\varepsilon(T + \Delta T_r) \subset \mathcal{D}^\varepsilon(T) + [-r, r]$$

provided

$$0 \leq \frac{\Delta T_r}{r^2} \leq \rho_0 \left(\frac{r}{\varepsilon}\right)^\omega,$$

where

$$\omega = \frac{\theta + 1}{\theta - 1}.$$

Comments

1. The motion of the front set is slow for large values of r : its average speed should not exceed

$$c(r) = \frac{r}{\Delta T_r} = \mathrm{c}_0 \rho_0^{-1} r^{-(\omega+1)} \varepsilon^{\omega}$$

   For large r this speed is algebraically small.

2. In contrast, the speed is exponentially small, of order $\exp(-cd/\varepsilon)$ in the non-degenerate case :
   see e.g Carr-Pego (1989)for the scalar case, B-Orlandi-Smets (2011) for the non-degenerate case.

3. May possibly be used to perform a renormalization procedure which yields a non trivial limit for ε to 0, accelerating time by a factor

$$\varepsilon^{-\omega}$$

[Renormalization impossible in the non degenerate case, exponential factors are not jointly commensurable when ε tends to 0.]

We consider the accelerated time $s = \varepsilon^{-\omega} t$ and the map

$$\mathfrak{v}_\varepsilon(x, s) = v_\varepsilon(x, s\varepsilon^{-\omega}),$$

Setting $\mathfrak{D}_\varepsilon(s) = \mathcal{D}(\mathfrak{v}_\varepsilon(\cdot, s))$ we have hence

$$\mathfrak{D}_\varepsilon(s + \Delta s) \subset \mathfrak{D}_\varepsilon(s) + [-r, r], \quad \text{provided that } \; 0 \leq \Delta s \leq \rho_0 r^{\omega+2}.$$

This last result is valid for systems : however, the rest of the talk is devited to the scalar case, where a precise motion law for the front points can be derived. Hence $k = 1$ from now on.

Compactness assumptions on the initial data

We assume that there exists a family points $a_1^0, \dots a_\ell^0$, such that $a_1^0 \dots, a_\ell^0$ and

$$\begin{cases} \mathfrak{D}_\varepsilon(0) \to \{a_k^0\}_{k\in J},\ J = 1 \dots, \ell \text{ in the sense of the Hausdorff distance} \\ v_\varepsilon(0) \to v_0^* \ \text{ in } \ L^1_{\mathrm{loc}}(\mathbb{R}) \end{cases}$$

Where v_0^* is of the form

$$v_0^* = \sigma^+_{i(k)} \text{ on } \ (a_k^0, a_{k+1}^0) \ \text{ for } k = 0, \dots, \ell, \text{ with } \sigma^+_{i(k)} \in \Sigma.$$

we impose the additional condition

$(\mathrm{H}_{\min})$ $$|\sigma^+_{i(k)} - \sigma^-_{i(k)}| = 1.$$

The limiting equation for the front points

Consider the system of ordinary differential equation for $k = 1, \dots, \ell$

$$\frac{d}{ds} a_k(s) = -\Gamma_{i(k-1)^+}[(a_k(s) - a_{k-1}(s)]^{-\omega} + \Gamma_{i(k)^+}[(a_k(s) - a_{k+1}(s)]^{-\omega} \quad \text{for } k \in J,$$

where, for constants $\mathcal{A}_\theta$ and $\mathcal{B}_\theta$ depending only on θ

$$\begin{cases} \Gamma_{i(k)^+} = 2^\omega \left(\lambda^+_{i(k)}\right)^{-\frac{1}{\theta-1}} \mathcal{A}_\theta \ \text{ if } \dagger_k = -\dagger_{k+1} \\ \Gamma_{i(k)^+} = -2^\omega \left(\lambda^+_{i(k)}\right)^{-\frac{1}{\theta-1}} \mathcal{B}_\theta \ \text{ if } \dagger_k = \dagger_{k+1}, \textit{with} \end{cases}$$

$$\begin{cases} \dagger_k = + \ \text{ if } \sigma_{i(k)^+} = \sigma_{i(k-1)^+} + 1 \\ \dagger_k = - \ \text{ if } \sigma_{i(k)^+} = \sigma_{i(k-1)^+} - 1. \end{cases}$$

The equation is supplemented with the initial time condition

$$a_k(0) = a_k^0.$$

Let $0 < S^* \leq +\infty$ be the maximal time of existence for this equation.

Motion law in the scalar case $k = 1$

Theorem (B-Smets, 2012)

Assume that the inital data $(v_\varepsilon(0))_{0<\varepsilon<1}$ satisfy the previous conditions. Then, given any $0 < s < S^*$, we have

$$\bigcup_{0\leq s\leq S} \mathfrak{D}_\varepsilon(s) \to \bigcup_{0\leq s\leq S} \{a_k(s)\}_{k\in J}$$

where $a_k(\cdot)_{k\in J}$ is the solution to the system of ordinary differential equations. Moreover, we have

$$\mathfrak{v}_\varepsilon(s) \to \sigma^+_{i(k)}$$

uniformly on every compact subset of $\bigcup_{0\leq s\leq S}(a_k(s), a_{k+1}(s))_{k\in J}$.

Comments on the limiting equation

Each point $a_k(t)$ is only moved by its interaction with its nearest neighbors, $a_{k-1}(t)$ and $a_{k+1}(t)$, and that this interaction, which is a decreasing function of the mutual distance between the points.

If

$$\sigma_{j^-(k-1)} < \sigma_{j^+(k-1)} = \sigma_{j^-(k)} < \sigma_{j^+(k)}$$

then, the interaction is repulsive.

If

$$\sigma_{j^-(k-1)} < \sigma_{j^+(k-1)} = \sigma_{j^-(k)} > \sigma_{j^+(k)} = \sigma_{j^-(k-1)}$$

then, the interaction is attractive, leading to collisions, which remove fronts from the collection.

We give provide a few elements in the proof for the motion law.

Chains of almost stationary solutions

In the scalar case, the solution v_ε to $(\text{PGL})_\varepsilon$ becomes close to a of chain of stationary solutions, denoted here below $\zeta_i^{\pm}$, translated at points $a_k(t)$, which are :

- well-separated
- suitably glued together

The accuracy of approximation is described thanks to a parameter $\delta > \alpha_* \varepsilon$ homogeneous to a lenght.

More precisely, we say that the well-preparedness condition $WP(\delta, t)$ holds iff

- (WP1)For each $k \in J(t_0)$ there exists a symbol $\dagger_k \in \{+, -\}$ such that

$$\left\| v_\varepsilon(\cdot, t_0) - \zeta_{i(k)}^{\dagger_k}\left(\frac{\cdot - a_k(t)}{\varepsilon}\right)\right\|_{C^1_\varepsilon(I_k)} \leq \exp\left(-\rho_1 \frac{\delta}{\varepsilon}\right),$$

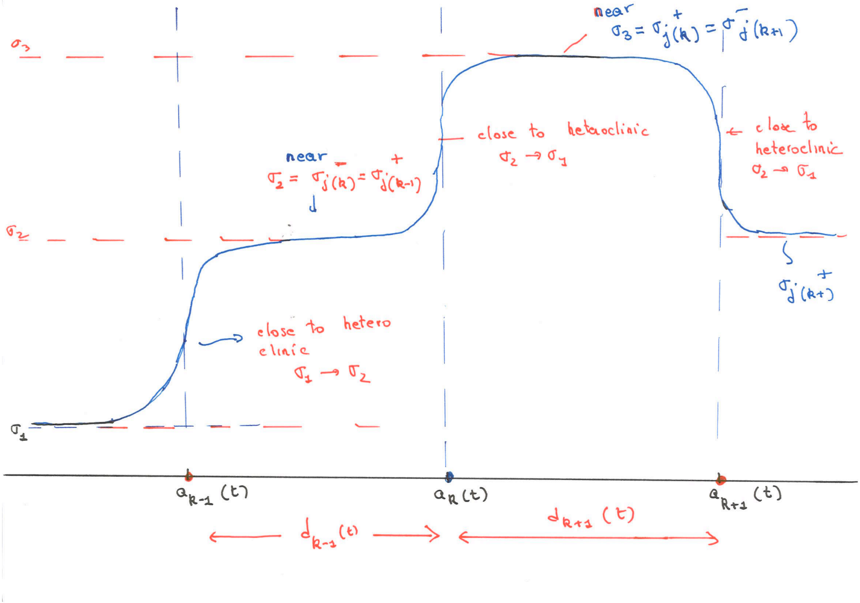
where $I_k = ([a_k(t_0) - \delta, a_k(t_0) + \delta]$, for each $k \in J(t_0)$.

- (WP2) Set $\Omega(t_0) = \mathbb{R} \setminus \bigcup_{k=1}^{\ell(t_0)} I_k$. We have the energy estimate

$$\int_{\Omega(t_0)} e_\varepsilon\left(v_\varepsilon(\cdot, t_0)\right) dx \leq CM_0 \exp\left(-\rho_1 \frac{\delta}{\varepsilon}\right).$$

Two orders of magnitude for δ will be considered, namely

$$\begin{cases} \delta^{\varepsilon}_{\log} = \dfrac{\omega}{2\rho_1}\varepsilon \log \dfrac{1}{\varepsilon} \\ \delta^{\varepsilon}_{\log\log} = \dfrac{\omega}{2\rho_1}\varepsilon \log\left(\log \dfrac{\omega}{2\rho_1 \varepsilon}\right). \end{cases}$$



Notice that

$$|j^-(k) - j^+(k)| = 1.$$

Evolution towards chains of fronts

It follows from the parabolic nature of the equation that the dynamics drives to prepared chains of fronts.

Proposition

Given any time $s \geq 0$ there existes some time $s_\varepsilon > 0$ such that

$$|s - s_\varepsilon| \leq \mathrm{c_0}^2 \varepsilon^2 M_0$$

and such that $\mathcal{WP}_\varepsilon(\delta^{\varepsilon}_{\log}, s_\varepsilon)$ holds. Moreover $\mathcal{WP}_\varepsilon(\delta^{\varepsilon}_{\mathrm{loglog}}, s')$ holds for any $s_\varepsilon + \varepsilon^2 \leq s' \leq \Gamma_0^\varepsilon(s)$, where

$$\Gamma_0^\varepsilon(s) = \inf\{s' \geq s, \ \ d^\varepsilon_{\min}(s) \leq \frac{1}{2}\mathrm{c}_2 \varepsilon^{\frac{2}{\omega+2}}\}.$$

The localized energy

Let χ be a smooth function with compact support on $\mathbb{R}$. Set, for $s \geq 0$ for $s \geq 0$

$$\mathcal{I}_\varepsilon(s, \chi) = \int_{\mathbb{R}} e_\varepsilon \left(\mathfrak{v}_\varepsilon(x, s)\right) \chi(x) dx.$$

If $\mathcal{WP}_\varepsilon(\delta^\varepsilon_{\rm loglog}, s)$ holds then

$$\left| \mathcal{I}_\varepsilon(s, \chi) - \sum_{k \in J} \chi(a_k^\varepsilon(s)) \mathfrak{S}_{i(k)} \right| \leq CM_0 \left(\frac{\varepsilon}{\delta^\varepsilon_{\rm loglog}} \right)^\omega \left[\|\chi\|_\infty + \delta^\varepsilon_{\rm loglog} \|\chi'\|_\infty \right],$$

where $\mathfrak{S}_{i(k)}$ stands for the energy of the corresponding stationary fronts. Hence the evolution of $\mathcal{I}_\varepsilon(s, \chi)$ yields the motion law for the points $a_k^\varepsilon(s)$.

Evolution for localized energies

$$\frac{d}{dt}\int_{\mathbb{R}} \chi(x)\, e_\varepsilon(v_\varepsilon)dx = -\int_{\mathbb{R}\times\{t\}} \varepsilon\chi(x)|\partial_t v_\varepsilon|^2 dx + \mathcal{F}_S(t,\chi,v_\varepsilon), \quad (LEI)$$

where, the term $\mathcal{F}_S$, is given by

$$\mathcal{F}_S(t,\chi,v_\varepsilon) = \int_{\mathbb{R}\times\{t\}} \left(\left[\varepsilon\frac{\dot{v}^2}{2} - \frac{V(v)}{\varepsilon}\right]\ddot{\chi}\right) dx.$$

The first term is local dissipation, the second is a flux. The quantity

$$\xi(x) \equiv [\varepsilon\frac{\dot{v}^2}{2} - \frac{V(v)}{\varepsilon}], \qquad |\xi| \leq e_\varepsilon(v_\varepsilon)$$

is referred to as **the discrepancy**. For solutions of the equation (ODE)

$$-u^\varepsilon_{xx} + \frac{1}{\varepsilon^2}\nabla V(u^\varepsilon) = 0$$

ξ is constant, and vanishes if the interval is $\mathbb{R}$.

Relating motion law and discrepancy

Set $\mathfrak{F}_\varepsilon(s_1, s_2, \chi) \equiv \varepsilon^{-\omega} \int_{s_1}^{s_2} \mathcal{F}_S(s, \chi, v_\varepsilon) ds.$

Lemma

Assume that condition $\mathcal{WP}_\varepsilon(\delta^{\varepsilon}_{\mathrm{loglog}}, s)$ holds for any $s \in (s_1, s_2)$. Then we have the estimate

$$\left| \sum_{k \in J} \left[\chi(a_k^\varepsilon(s_2)) - \chi(a_k^\varepsilon(s_1)) \right] \mathfrak{S}_{i(k)} - \mathfrak{F}_\varepsilon(s_1, s_2, \chi) \right| \leq$$

$$CM_0 \left(\frac{\omega}{2\rho_1} \log \left(\log \frac{\omega}{2\rho_1 \varepsilon} \right) \right)^{-\omega} \left[\|\chi\|_{L^\infty(\mathbb{R})} + \varepsilon \|\chi'\|_{L^\infty(\mathbb{R})} \right]. \tag{6}$$

If the test function χ is choosen to be affine near a given front point a_{k_0} and zero near the other fronts, then the first term on the l.h.s yields a measure of the motion of a_{k_0} between times s_1 and s_2 whereas the second, $\mathfrak{F}_\varepsilon(s_1, s_2, \chi)$ is a good approximation of the measure of this motion.

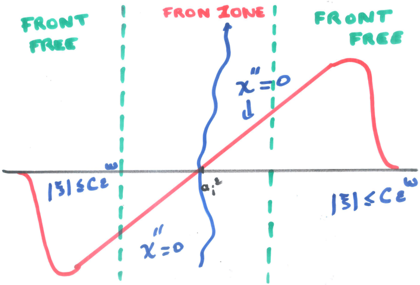
This suggest that

$$(a^{\varepsilon}_{k_0}(s_2)) - (a^{\varepsilon}_{k_0}(s_1)) \simeq \frac{1}{\chi'(a^{\varepsilon}_{k_0})\mathfrak{S}_{i(k_0)}} \mathfrak{F}_{\varepsilon}(s_1, s_2, \chi).$$

The computation of

$$\mathfrak{F}_{\varepsilon}(s_1, s_2, \chi) = \int_{\mathbb{R}\times[s_1,s_2]} \ddot{\chi}\xi dxds$$

is performed with test functions χ having vanishing second derivatives far from the front set.



The choice of test functions

Estimates off the front set

This is a major ingredient in our proofs :

Proposition

Let v_ε be a solution to $(PGL)_\varepsilon$ satisfying assumption (H_0), let $x_0 \in \mathbb{R}$, $r > 0$ and $S_0 > s_0 \geq 0$ be such that

$$\mathfrak{v}_\varepsilon(y,s) \in B(\sigma_i, \mu_0) \quad \text{for all} \quad (y,s) \in [x_0 - 3r/4, x_0 + 3r/4] \times [s_0, S_0]$$

then, for $s_0 < s \leq S_0$ and $x \in [x_0 - r/2, x_0 + r/2]$

$$\begin{cases} \varepsilon^{-\omega} \displaystyle\int_{x_0 - r/2}^{x_0 + r/2} e_\varepsilon(\mathfrak{v}_\varepsilon(x,s))\, dx \leq C \left(1 + \varepsilon^{\frac{\omega\,\theta}{\theta-1}} \left(\frac{r^2}{s - s_0}\right)^{\frac{\theta}{\theta-1}}\right) \left(\frac{1}{r}\right)^{\omega} \\ |\mathfrak{v}_\varepsilon(x,s) - \sigma_i| \leq C \varepsilon^{\frac{1}{\theta-1}} \left(\left(\frac{1}{r}\right)^{\frac{1}{\theta-1}} + \left(\frac{\varepsilon^\omega r^2}{(s - s_0)}\right)^{\frac{1}{\theta-1}}\right), \end{cases}$$

where $C > 0$ is some constant depending only on V.

The main argument of the proof is the construction of a suitable upper solution.

Refined estimates off the front set and the motion law

We need to provide a limit of $\varepsilon^{-\omega}\xi$ near $a^{\varepsilon}_{k+\frac{1}{2}}(s) \equiv \dfrac{a^{\varepsilon}_{k}(s) + a^{\varepsilon}_{k+1}(s)}{2}$.

Consider the function $\mathfrak{W}^k_{\varepsilon} = \varepsilon^{-\frac{1}{\theta-1}}\left(\mathfrak{v}_{\varepsilon} - \sigma^{+}_{i(k)}\right)$ and expand $(PGL)_{\varepsilon}$ as

$$\varepsilon^{\omega}\frac{\partial \mathfrak{W}_{\varepsilon}}{\partial s} - \frac{\partial^2 \mathfrak{W}_{\varepsilon}}{\partial x^2} + 2\theta\lambda_{i(k)^{+}}{\mathfrak{W}_{\varepsilon}}^{2\theta-1} = O(\varepsilon^{\frac{1}{\theta-1}}).$$

Passing to the limit $\varepsilon \to 0$, we might expect that the limit $\mathfrak{W}_{*}$ solves the ordinary differential equation

$$\begin{cases} -\dfrac{\partial^2 \mathfrak{W}_{*}}{\partial x^2} + 2\theta\lambda_{i(k)^{+}}{\mathfrak{W}_{*}}^{2\theta-1} = 0 \text{ on } (a_k(s), a_{k+1}(s)), \\ \mathfrak{W}_{*}(a_k(s)) = -\mathrm{sign}(\dagger_k)\infty \text{ and } \quad \mathfrak{W}_{*}(a_{k+1}(s)) = \mathrm{sign}(\dagger_k)\infty. \end{cases}$$

The boundary conditions being a consequence of the behavior near the front points.

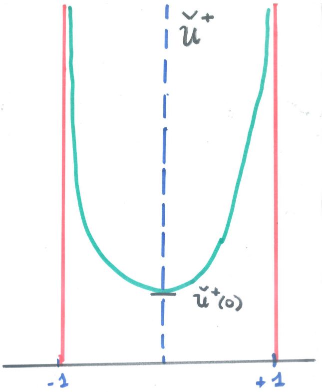
Setting $r_k(s) = \frac{1}{2}(a_{k+1}(s) - a_k(s))$ and $d_k(s) = 2r_k(s)$ we deduce

$$\begin{cases} \mathfrak{W}_*(x,s) = \pm\left(\frac{1}{r_k}\right)^{\frac{1}{\theta-1}} \left(\lambda_{i(k)^+}\right)^{-\frac{1}{2(\theta-1)}} \overset{\vee}{\mathcal{U}}^+\left(\frac{x - a_{k+\frac{1}{2}}}{r_k(s)}\right), \text{ if } \dagger_k = -\dagger_{k+1}, \\ \mathfrak{W}_*(x,s) = \pm\left(\frac{1}{r_k}\right)^{\frac{1}{\theta-1}} \left(\lambda_{i(k)^+}\right)^{-\frac{1}{2(\theta-1)}} \mathcal{U}^-\left(\frac{x - a_{k+\frac{1}{2}}}{r_k(s)}\right), \text{ if } \dagger_k = \dagger_{k+1}, \end{cases}$$

where $\overset{\vee}{\mathcal{U}}^+$ (resp $\overset{\triangleright}{\mathcal{U}}$) are the unique solutions to the problems

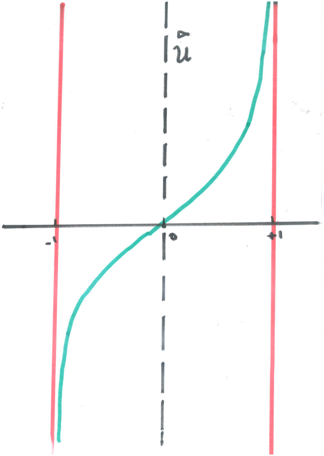
$$\begin{cases} -\mathcal{U}_{xx} + 2\theta\,\mathcal{U}^{2\theta-1} = 0 \text{ on } (-1,+1), \\ \mathcal{U}(-1) = +\infty \text{ (resp } \mathcal{U}(-1) = -\infty) \text{ and } \mathcal{U}(+1) = +\infty. \end{cases} \tag{7}$$

The attractive case $\check{\mathcal{U}}^+$



$$\xi(\check{\mathcal{U}}^+) = +\check{\mathcal{U}}^+(0)^{2\theta}$$

The repulsive case $\overset{\triangleright}{\mathcal{U}}$



$$\xi(\overset{\vee}{\mathcal{U}}^{+}) = -\frac{(\overset{\triangleright}{\mathcal{U}}_x(0))^2}{2}$$

We obtain the corresponding values of the disprecancy

$$\begin{cases} \varepsilon^{-\omega}\xi_\varepsilon(\mathfrak{v}_\varepsilon) \simeq \xi(\mathfrak{W}_*) = -\left(\lambda^+_{i(k)}\right)^{-\frac{1}{\theta-1}} (r_k(s))^{-(\omega+1)} \mathcal{A}_\theta \text{ if } \dagger_k = -\dagger_{k+1}, \\ \varepsilon^{-\omega}\xi_\varepsilon(\mathfrak{v}_\varepsilon) \simeq \xi(\mathfrak{W}_*) = \left(\lambda_{i(k)^+}\right)^{-\frac{1}{\theta-1}} (r_k(s))^{-(\omega+1)} \mathcal{B}_\theta \text{ if } \dagger_k = \dagger_{k+1}. \end{cases} \tag{8}$$

where the numbers $\mathcal{A}_\theta$ and $\mathcal{B}_\theta$ are positive and depend only on θ and are provided by the absolute value of the discrepancy of $\overset{\vee}{\mathcal{U}}^{+}$ and $\overset{\triangleright}{\mathcal{U}}$ respectively.

Thank you for your attention !

(Slowpoke Rodriguez)