

TWO TOPICS IN NONCOMMUTATIVE GEOMETRY

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My lectures were devoted to two topics in Noncommutative Geometry on which I have been working in the last several years. The common thread between them resides in a specific feature of noncommutative symmetry, which underlies both themes, albeit in a different manner.

1. HOPF ALGEBRA COHOMOLOGY AND DIFF-EQUIVARIANT CHARACTERISTIC CLASSES

The geometric significance of Hopf algebras as well as the Hopf algebra version of cyclic cohomology emerged from the work of A. Connes and myself on the local index formula for transversely elliptic operators on foliations [2, 3]. While the transverse characteristic classes of foliations are well described in terms of Gelfand-Fuks Lie algebra cohomology, in the K -homological context the appropriate tool turned out to be the Hopf cyclic cohomology. Moreover, the Hopf-algebraic approach widens the scope of reachable applications, as illustrated by our work on modular Hecke algebras [4], which revealed hidden geometric structures underlying classical constructs in the theory of modular forms.

After a brief review of the above mentioned origins and motivation for the subject, I outlined the subsequent progress made in my joint work with B. Rangipour. In [9] we have extended the construction of the Hopf algebra $\mathcal{H}_n$ associated to the general pseudogroup of local diffeomorphisms of $\mathbb{R}^n$ to all infinite primitive Lie-Cartan pseudogroups of local diffeomorphisms. The Hopf algebra $\mathcal{H}_\Pi$ associated to such a pseudogroup Π is essentially a “repackaging” of the infinite-dimensional Lie algebra of Π . Like $\mathcal{H}_n$ it arises naturally through its tautological action on the étale groupoid associated to Π . On the other hand, $\mathcal{H}_\Pi$ can be reconstructed by a “bending mechanism” out of the Lie algebra of Π , in the form of a bicrossed product of two Hopf algebras of classical type – the universal enveloping algebra of a finite-dimensional Lie sub algebra and the Hopf algebra of regular functions on a formal pronilpotent group.

The bicrossed product realization played a critical role in the explicit determination of the Hopf cyclic cohomology of these Hopf algebras. In our paper with A. Connes [3], the cyclic cohomology of the Hopf algebra $\mathcal{H}_n$ was shown to be isomorphic to the Gelfand-Fuks cohomology, bypassing any direct computation (except for $n = 1$). To gain more insight into the computational aspects, in the work with B. Rangipour [8, 9, 10] we relied on the bicrossed product construction in order to disassemble the original cyclic bicomplex defining the Hopf cyclic cohomology of the Hopf algebras $\mathcal{H}_\Pi$ and then reassemble it into a series of progressively more manageable quasi-isomorphic cohomological models. These are bicocyclic bicomplexes which mix Lie algebra cohomology with coefficients and coalgebra cohomology with coefficients, with the essential distinction (by comparison with Lie algebra cohomology) that the coefficients always “act back”. The refinement of the Hopf cyclic

cohomological apparatus allowed us to explicitly identify, as a Hopf cyclic complex, the image of the canonical homomorphism from the Gelfand-Fuks complex to the Bott complex of Diff-equivariant cohomology.

Relying on the latter implementation of the Hopf cyclic complex for the Hopf algebra $\mathcal{H}_n$, I have recently succeeded to construct explicit representative cocycles for all the classes in the periodic Hopf cyclic cohomology of $\mathcal{H}_n$ as well as in its cohomology relative to the Lie subalgebras $\mathfrak{o}_n$ and $\mathfrak{gl}_n$. I concluded my exposition by showing how this concrete realization of the universal Hopf cyclic characteristic classes of foliations can be essentially “imported” from the Bott-Chern-Weil construction of characteristic classes of foliations in terms of Diff-equivariant cohomology.

2. SPECTRAL FUNCTIONALS AND THE GEOMETRY OF NONCOMMUTATIVE TORI

In Noncommutative Geometry the paradigm of space as a manifold formed of points labeled by numerical coordinates is replaced by one of a much more general nature, in which the coordinates are operator-valued and not required to commute, as in quantum physics. The general topological content of such a space resides in its representation as a C^* -algebra, while the extra refinement of a topo-geometric structure is captured, in Connes’ *spectral triple* template, by adding to the datum of the Hilbert space, in which the algebra of coordinates $\mathcal{A}$ is represented by bounded operators, that of an unbounded self-adjoint operator D playing the role of the inverse line element, i.e. of the Dirac operator, and having bounded commutators with the operator-valued coordinates. For the global treatment of these spaces there are well-developed algebraic and analytic tools available, having been successfully adapted and upgraded from their classical topological context. On the other hand the local geometric concepts are much less transparent, inasmuch as they can only be accessed via spectral functionals related to the high frequency behavior of the spectrum of D coupled with the action of the algebra of coordinates (as illustrated for example by the local index formula [2]). In particular the notion of *intrinsic curvature*, which lies at the very core of Geometry, remains quite difficult to grasp. As the most primary form of classical curvature arises for Riemann surfaces, it was natural to look first at noncommutative 2-tori, the simplest and best understood examples of noncommutative manifolds.

The setup for the computation of the scalar curvature of the noncommutative 2-torus $\mathbb{T}_\theta^2$, $\theta \notin \mathbb{Q}$, equipped with a translation-invariant conformal structure was developed in the work of Connes-Tretkoff [6], initiated in late 1980’s. They proved the analogue of the Gauss-Bonnet formula for the conformal metric, which required only the total integral of the curvature. The full calculation of the curvature was completed in [5] with the assistance of Mathematica software, and was independently checked in [7] using a different software. The curvature formulae involve second order (outer) derivatives of the Weyl factor, and as a new and crucial ingredient they also involve the modular operator of the non-tracial weight associated to the Weyl factor.

In my talks I briefly reviewed Connes’ pseudodifferential calculus for C^* -dynamical systems [1], which is the main analytical tool behind the curvature computation. I then proceeded to present the main new results obtained in our joint paper [5]. First of all, we succeeded to express by a closed formula the Ray-Singer log-determinant of D^2 , issue which was left open in [6]. The gradient of the log-determinant functional was then shown to coincide with the local curvature, which arises as a sum of

two terms, each involving two functions in the modular operator, of one and respectively two variables. Computing the gradient by two different methods led to the proof of a deep internal consistency relation between these two distinct constituents, and at the same time elucidated the meaning of the intricate two operator-variable function. As a third fundamental result, we established the analogue of the classical result which asserts that in every conformal class the maximum value of the determinant of the Laplacian for metrics of a fixed area is uniquely attained at the constant curvature metric. The key ingredient for the proof is the positivity of a certain operator-valued function. Restricted to real scalars this is a generating function for Bernoulli numbers, known to play a prominent role in the theory of characteristic classes of deformations, where it is merely used however as a formal power series.

In the last part of my exposition I outlined an extension of Connes' pseudodifferential calculus, obtained in joint work with M. Lesch, adapted to incorporate the Morita equivalence, a purely noncommutative feature. This should allow to extend the Ray-Singer log-determinant functional to Heisenberg bimodules over pairs of Morita equivalent noncommutative tori, and relate it to the Yang-Mills functional of Connes-Rieffel in order to show that its extreme values occur only at the Hermitian metrics of constant curvature.

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