Quantum bounds and fluctuation-dissipation relations

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"Strange Metals, SYK Models and Beyond"

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Outline

- Quantum bounds and Planckian scales
- Fluctuation-dissipation theorem: from frequency to time
- OTOCs as two-point function: bound on chaos via FDT
- Blurring on a Planckian scale

Quantum bounds

The most famous bound: the uncertainty principle

$$\Delta x \, \Delta p \geq \frac{1}{2}\hbar$$

Most distinctive feature of how quantum mechanics differs from the classical world

More recently new bounds on physical time scales

Towards a bound on chaos

Lyapunov exponent

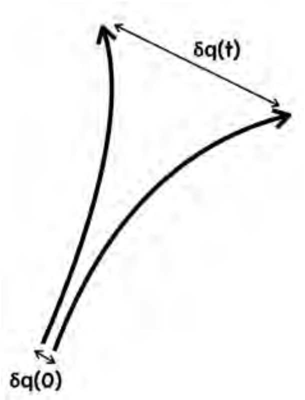
Classical

$$\left|\frac{\delta q(t)}{\delta q(0)}\right|^2 = |\{q(t), p(0)\}|^2 \simeq e^{\lambda t}$$

Quantum

$$\left\langle \left(i[q(t), p(0)] \right)^2 \right\rangle \simeq \hbar^2 e^{\lambda t}$$

Larkin and Ovchinikov (1969)



NB: Focus on operators/observables, not on wavefunction

Out-of-Time-Order Correlations

$$-\langle [A(t), B(0)]^2 \rangle = \langle A(t)B(0)B(0)A(t) \rangle + \langle B(0)A(t)A(t)B(0) \rangle$$

$$-\langle A(t)B(0)A(t)B(0) \rangle - \langle B(0)A(t)B(0)A(t) \rangle$$

4-point Out-of-Time-Order Correlation (OTOC)

$$\langle A(t)B(0)A(t)B(0) \rangle$$

$$\langle \bullet \rangle = \frac{1}{Z} \mathrm{Tr}\left[e^{-\beta H} \bullet\right]$$

Bound on chaos

Regularized OTOC

$$\frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H/4}A(t)e^{-\beta H/4}B(0)e^{-\beta H/4}A(t)e^{-\beta H/4}B(0)\right] \simeq a - \epsilon e^{\lambda t}$$

for $t_d \ll t \ll t_{Ehr}$

$$\lambda \leq \frac{2\pi T}{\hbar}$$

Maldacena, Shenker and Stanford (2016)

Sachdev-Ye-Kitaev (SYK) model

$$H_{SYK} = -\sum_{ijkl} J_{ijkl}\chi_i\chi_j\chi_k\chi_l$$

$\{\chi_i, \chi_j\} = 2\delta_{ij}$ $\qquad$ J_{ijk}Gaussian i.i.d. variables

Sachdev, Ye (1993)
Kitaev (2015)

- Asymptotically saturates the bound

Planckian time scales and transport coefficients

$$\tau \propto \tau_P = \frac{\hbar}{T}$$

- Same dependence on T for very different systems
- No details of the system $\rightarrow$ Universality
- Shortest possible time scale in many-body systems?
- Conjectured bounds on transport coefficients (e.g. viscosity)

Chowdhury, Georges, Parcollet, Sachdev (2021)
Hartnoll, Mackenzie (2021)

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Correlations and responses

$$S(t) = \frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}A(t)A(0)\right] = C(t) - i\hbar R(t)$$

Symmetrized correlation function

$$C(t) = \frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}\frac{1}{2}\{A(t), A(0)\}\right]$$

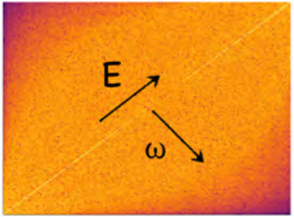
Response function (Kubo formula)

$$\chi(t) = \theta(t)2R(t) = \theta(t)\frac{i}{Z\hbar}\mathrm{Tr}\left[e^{-\beta H}[A(t), A(0)]\right]$$

An intermediate regulated function

$$F(t) = \frac{1}{Z}\mathrm{Tr}\left[e^{-\frac{1}{2}\beta H}A(t)e^{-\frac{1}{2}\beta H}A(0)\right]$$

Meaning in ETH



$$A_{\alpha\beta} = \mathscr{A}(E)\delta_{\alpha\beta} + e^{-S(E)/2} f(E,\omega) R_{\alpha\beta}$$

$$E = (\epsilon_\alpha + \epsilon_\beta)/2 \qquad \omega = \epsilon_\alpha - \epsilon_\beta$$

M. Srednicki (1999)

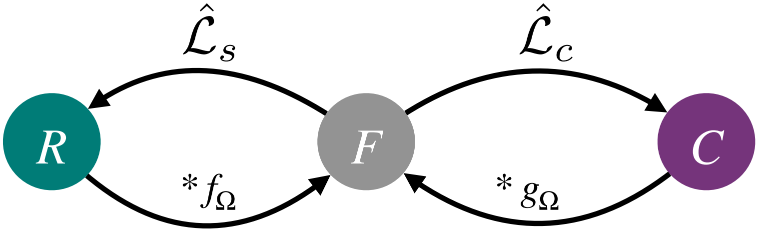
FDT in the frequency domain

$$\hbar \mathrm{Im} R(\omega) = \tanh\left(\frac{\hbar\beta\omega}{2}\right) C(\omega)$$

$$C(\omega) = \cosh\left(\frac{\hbar\beta\omega}{2}\right) F(\omega)$$

$$\hbar \mathrm{Im} R(\omega) = \sinh\left(\frac{\hbar\beta\omega}{2}\right) F(\omega)$$

FDT in the time domain domain



Going from one function to the other via convolutions or differential operators

FDT in the time domain domain (1)

$$C(t) = \cos\left(\frac{\hbar\beta}{2}\frac{d}{dt}\right)F(t)$$

$$R(t) = -\frac{1}{\hbar}\sin\left(\frac{\hbar\beta}{2}\frac{d}{dt}\right)F(t)$$

Classically $\hbar \to 0$

$$C(t) = F(t) \qquad \chi(t) = \theta(t)2R(t) = -\theta(t)\beta\partial_t C(t) \tag{1}$$

FDT in the time domain domain (2)

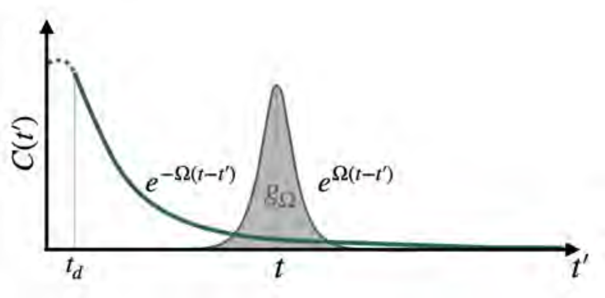
$$F(t) = \int \mathrm{d}t' g_\Omega(t - t') C(t')$$

Blurring function

$$g_\Omega(t) = FT\left[\frac{1}{\cosh \hbar\beta\omega/2}\right] = \frac{\Omega}{\pi}\frac{1}{\cosh \Omega t}$$

Planckian scale $\qquad \Omega = \dfrac{\pi T}{\hbar} = \dfrac{1}{\tau_\Omega}$

Blurring functions



- Smooths out the details on a scale $\simeq \hbar/T$
- Suppression of high frequency components $F(\omega) = \frac{C(\omega)}{\cosh \beta\omega/2}$
- $g_\Omega(t) \to \delta(t)$ for $\hbar \to 0$

t-FDT with (integrated) response

$$F(t) = \int_{-\infty}^{\infty} \mathrm{d}t' f(t-t') R(t')$$

$$= \frac{\psi(\infty)}{\beta} + \int_0^{\infty} \left[\tilde{f}_{2\Omega}(t-t') + \tilde{f}_{2\Omega}(t+t') \right] \psi(t')$$

$$\psi(t) = \int_0^t \chi(t') = 2 \int_0^t R(t')$$

$$f(t) = FT \left[\frac{1}{\sinh \hbar\beta\omega/2} \right] = \frac{1}{\beta} \tanh \Omega t$$

Blurring function $\tilde{f}_{2\Omega}(t) = \dfrac{\Omega}{(\cosh \hbar\Omega t)^2}$

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OTOCs as two point functions (1)

$$\frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H/2}A(t)A(0)e^{-\beta H/2}A(t)A(0)\right]$$

$$=\frac{1}{Z}\sum_{ij}e^{-\frac{\beta}{2}(E_i+E_j)}\langle ij|A(t)\otimes A(t)A(0)\otimes A(0)|ji\rangle$$

$$=\frac{1}{Z}\mathrm{Tr}\left[e^{-\beta_2\mathbb{H}}\mathbb{A}(t)\mathbb{A}(0)\mathbb{P}\right]\propto S_{\mathbb{A},\mathbb{AP}}(t)$$

$$\mathbb{H}=H\otimes I+I\otimes H \qquad \mathbb{A}=A\otimes A \qquad \mathbb{P}|ij\rangle=|ji\rangle \qquad \beta_2=\beta/2$$

OTOCs as two point functions (2)

$$F_{\mathbb{A},\mathbb{A}\mathbb{P}}(t) \propto \text{Tr}\left[e^{-\frac{\beta_2}{2}\mathbb{H}}\mathbb{A}(t)e^{-\frac{\beta_2}{2}\mathbb{H}}\mathbb{A}(0)\mathbb{P}\right] \to MSS$$

$$C_{\mathbb{A},\mathbb{A}\mathbb{P}}(t) \propto \text{Tr}\left[e^{-\beta_2\mathbb{H}}\left\{\mathbb{A}(t),\mathbb{A}(0)\mathbb{P}\right\}\right]$$

$$R_{\mathbb{A},\mathbb{A}\mathbb{P}}(t) \propto \text{Tr}\left[e^{-\beta_2\mathbb{H}}\left[\mathbb{A}(t),\mathbb{A}(0)\mathbb{P}\right]\right]$$

Behaviour of C and F

$$F_{\mathbb{A},\mathbb{AP}}(t) \simeq a - \epsilon e^{\lambda t} \qquad \text{for } t_d \ll t \ll t_{Ehr}$$

$$C_{\mathbb{A},\mathbb{AP}}(t) \simeq b - \delta e^{\lambda t} \qquad \text{for } t_d \ll t \ll t_{Ehr}$$

$$0 < \delta \ll 1 \qquad 0 < \epsilon \ll 1$$

Exponential behavior from positive defined squared commutator

Bound on chaos from FDT

t-FDT $$C(t) = \cos\left(\frac{\hbar\beta}{2}\frac{d}{dt}\right) F(t)$$

If $F(t) = \epsilon\, e^{\lambda t}$ then $C(t) = \cos\left(\frac{\hbar\beta\lambda}{2}\right) F(t)$

Sign argument

$$\cos\left(\frac{\hbar\beta_2\lambda}{2}\right) \geq 0$$ for $\lambda \leq \frac{2\pi T}{\hbar}$

Tsuji, Shitara, Ueda (2018)

Pappalardi, Foini, Kurchan (2021)

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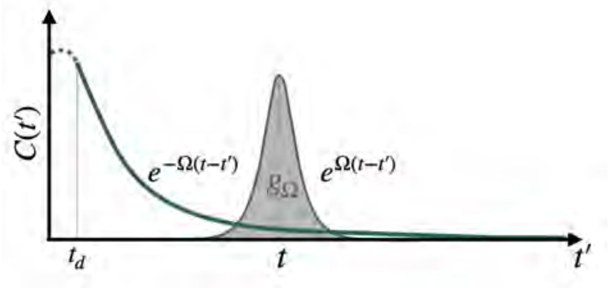
FDT as a blurring

$$F(t) = \int \mathrm{d}t' g_\Omega(t-t') C(t') \qquad g_\Omega(t) = \frac{\Omega}{\pi}\frac{1}{\cosh \Omega t}$$



Suppose $C(t) \propto e^{-at} \quad t \gg t_d$

- If $a < \Omega$ integrand peaked at $t' \simeq t$
- If $a > \Omega$ dominated by short times

Bound on the decay of R

$$C(t) \simeq A e^{-at} \qquad t \gg t_d$$

$$F(t) = \frac{A}{\cos\beta a/2} e^{-at} + c_\Omega e^{-\Omega t} + \cdots \propto e^{-t/\tau}$$

$$c_\Omega = \frac{2\Omega}{\pi} \int_{-\infty}^{\infty} \mathrm{d}t\, e^{-\Omega t} C(t)$$

$$R(t) = A \tan\beta a/2\, e^{-at} + c_\Omega e^{-\Omega t} + \cdots \propto e^{-t/\tau}$$

Therefore, if $c_\Omega \neq 0$ the rate of F and R is bounded:

$$\frac{1}{\tau} \leq \Omega = \frac{\pi T}{\hbar}$$

Bound on the decay of C

If $c_\Omega = 0$ one can repeat the same starting from $R(t)$ which decays exponentially

$$F(t) \simeq C(t) \simeq e^{-t/\tau}$$

$$r_{2\Omega} = \frac{4\Omega}{\pi} \int_{-\infty}^{\infty} \mathrm{d}t \, e^{-2\Omega t} R(t)$$

If $r_{2\Omega} \neq 0$ the rate of F and C is bounded:

$$\frac{1}{\tau} \leq 2\Omega = \frac{2\pi T}{\hbar}$$

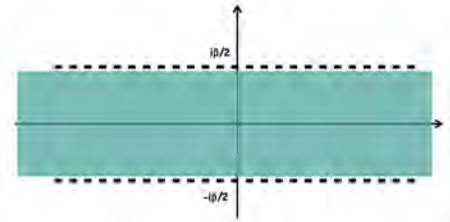
No proof of bound

$$c_\Omega = \frac{2\Omega}{\pi}\int_{-\infty}^{\infty} \mathrm{d}t\, e^{-\Omega t} C(t) = 0$$

$$r_{2\Omega} = \frac{4\Omega}{\pi}\int_{-\infty}^{\infty} \mathrm{d}t\, e^{-2\Omega t} R(t) = 0$$

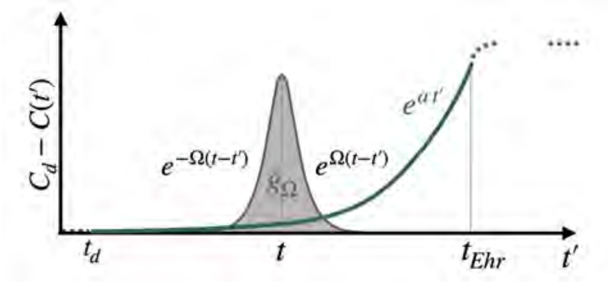
Sum rules → Relation between short and long times

Analyticity properties of $F(t)$ in the strip



Blurring for exponentially increasing functions

$$C(t) = C_d - Ae^{at} \qquad \text{for } t_d \ll t \ll t_{Ehr}$$



- If $a < \Omega$ integrand peaked at $t' \simeq t$
- If $a > \Omega$ dominated by large times

Ω with $\beta_2 \rightarrow$ bound on chaos

Conclusions

- Bound on chaos as a consequence of FDT
- Quantum FDT induces blurring on a Planckian scale

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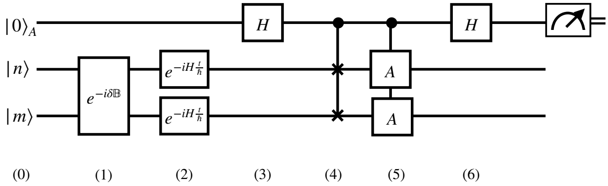
- Response function in double space as a possible probe of OTOC?



Conclusions

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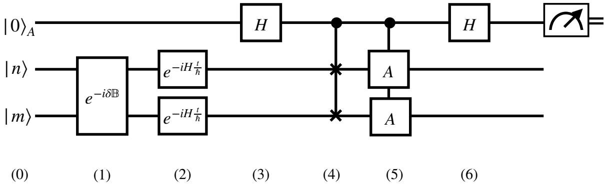

Thank you!