A $(5/3 + \epsilon)$-Approximation for Unsplittable Flow on a Path: Placing Small Tasks into Boxes

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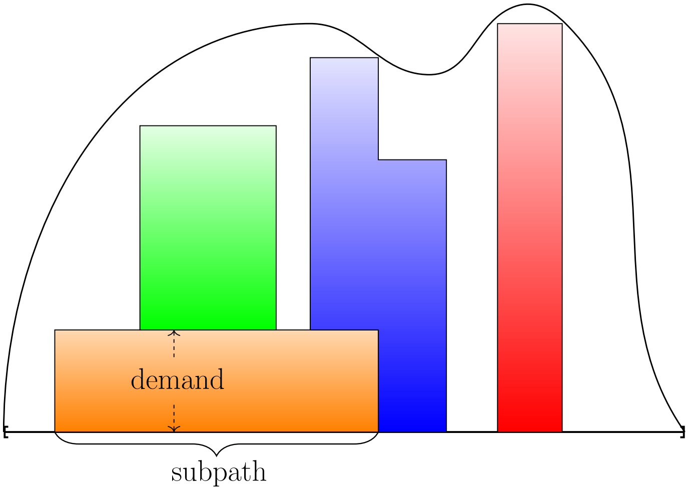
Symposium on Theory of Computing (STOC), 2018

Unsplittable Flow on a Path (UFP)

A task: subpath, demand, weight



Polynomial time:

- $O(\log n)$ [Bansal, Friggstad, Khandekar, Salavatipour, SODA 2009]
- $7 + \epsilon$ [Bonsma, Schulz, Wiese, FOCS 2011]
- $2 + \epsilon$ [Anagnostopoulos, Grandoni, Leonardi, Wiese, SODA 2014]
- $1 + \epsilon$ when weight/demand is bounded
  [Batra, Garg, Kumar, Mömke, Wiese, SODA 2015]
- $1 + \epsilon$ when all tasks share a common edge
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Quasi-polynomial time:

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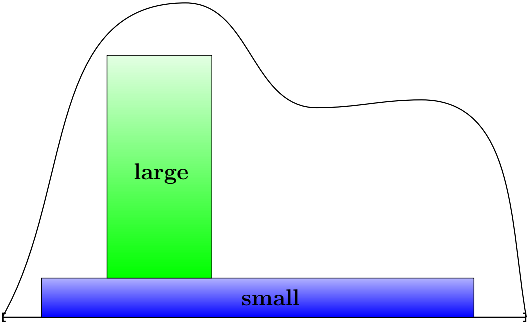
Open Question

Is there a polynomial-time approximation scheme for UFP?

Our Result

Polynomial-time $(5/3+\epsilon)$**-approximation for UFP**

Idea: combine large tasks and small tasks together



Previous techniques:

1. Dynamic programming: large tasks
2. Linear programming: small tasks

$$\boxed{1} + \boxed{2} \implies (2+\epsilon)\text{-approximation}$$

Q: How to achieve better-than-2 approximation?

A: **Dynamic programming with boxes:** large tasks + 1/3 of small tasks

Combined with $\boxed{2} \implies (5/3+\epsilon)$-approximation

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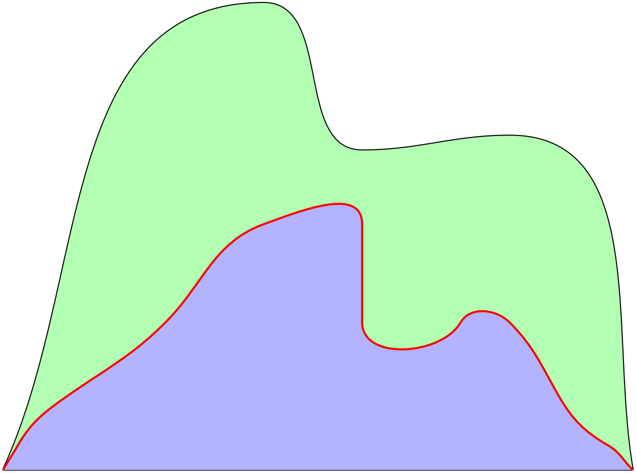
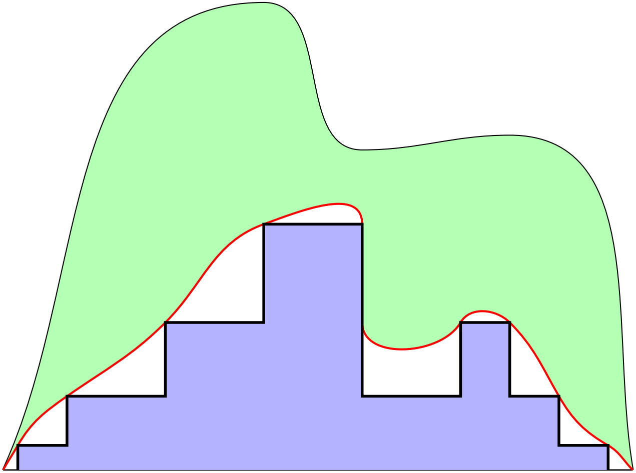
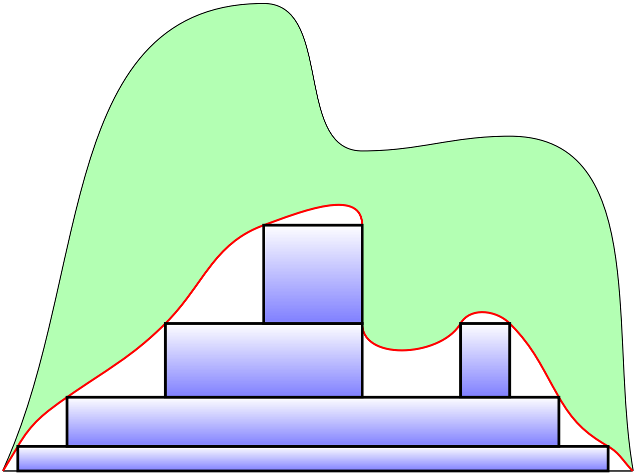
Difficulty: Unknown separation between **space for large tasks** and **space for small tasks** in the optimal solution



Preprocessing: Round down the separation profile to powers of $1+\epsilon$

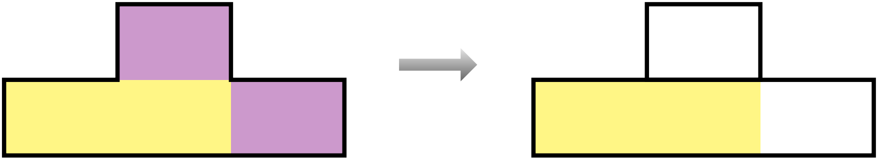


Main idea: Decompose the space for small tasks into **boxes**



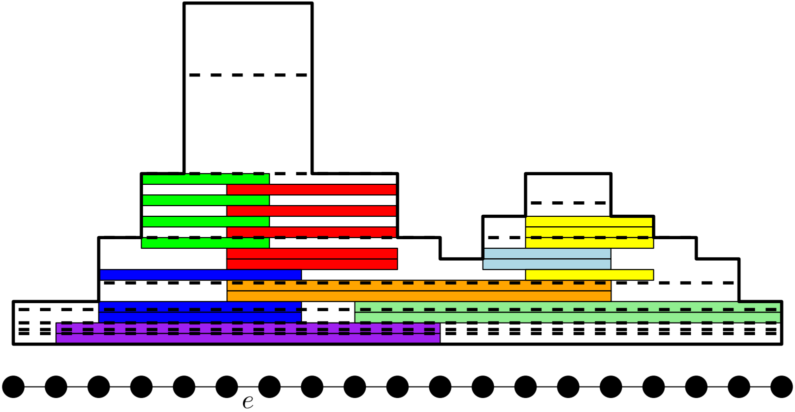
Q: What factor of small tasks do we lose by introducing boxes?

A: At most 2.



Horizontal slicing lemma

We are given a set of tasks such that, on each edge e, the total demand of the tasks using e is at most half of the capacity on the edge e. We are allowed to slice each task horizontally into pieces of unit height. Then we can pack all the resulting slices geometrically within the capacity profile.



Algorithm to compute small tasks within boxes

- Guess boxes **bottom-up** using *dynamic programming*
- Fill each box with small tasks using *linear programming*

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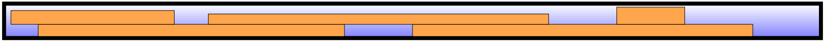

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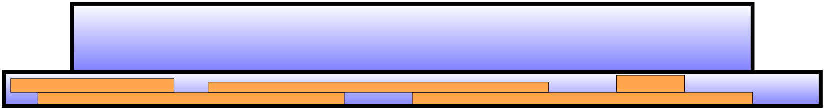

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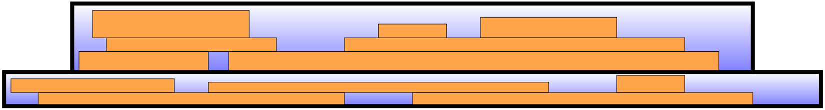

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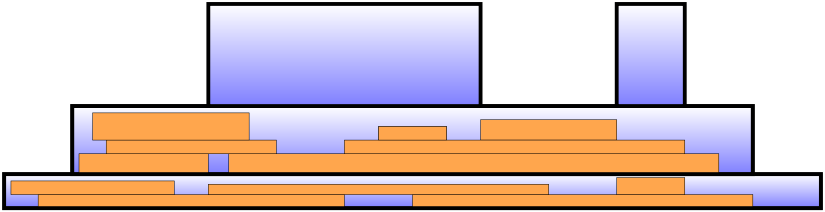

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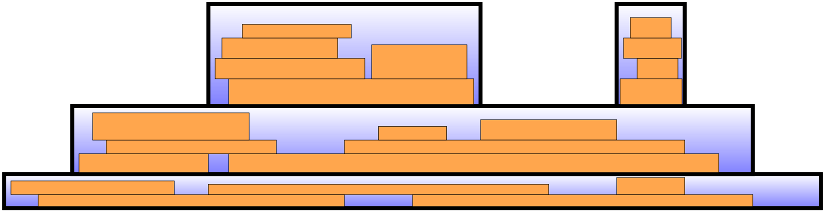

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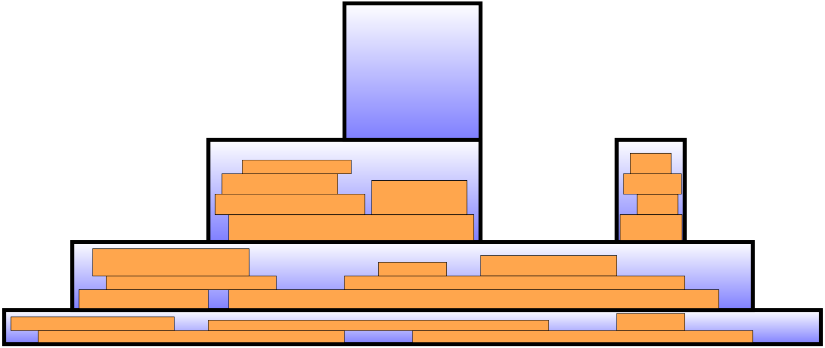

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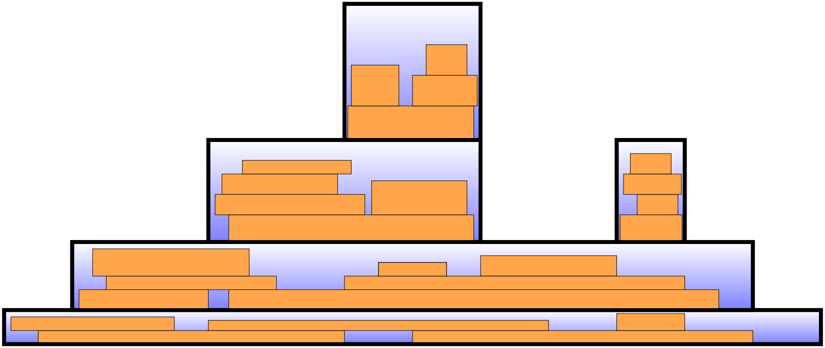
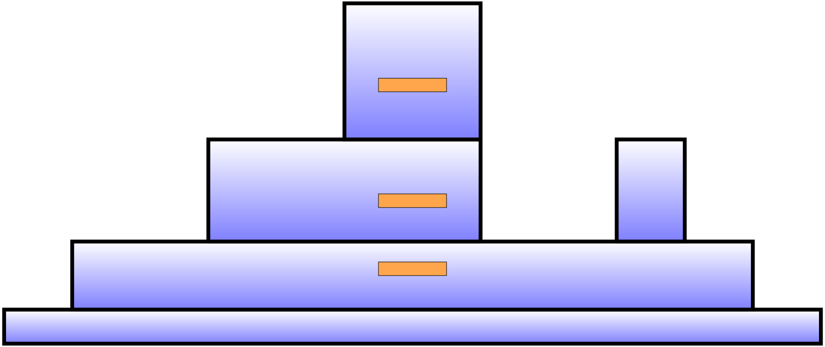
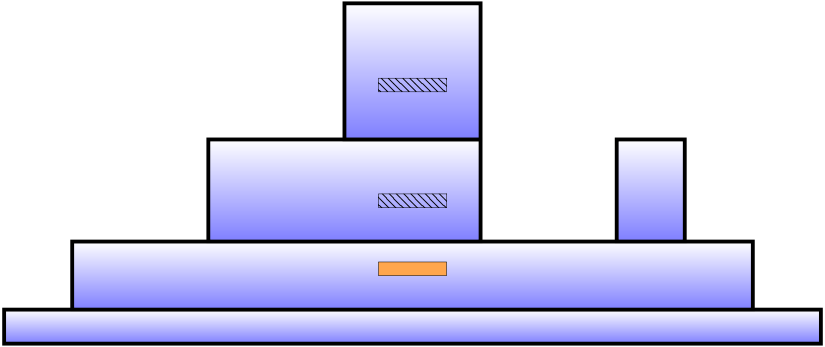


Q: How to avoid a small task being selected several times?



Q: How to avoid a small task being selected several times?

A: As soon as a small task is selected in some box, it is no longer allowed in upper boxes.



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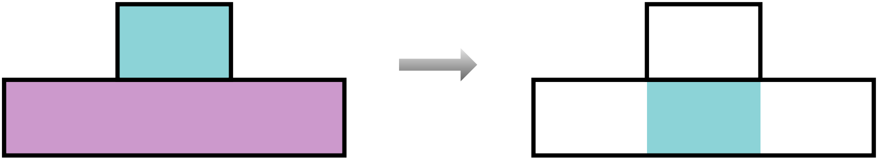

Loss of small tasks:

- factor of 2 by introducing boxes
- factor of 2 by filling boxes bottom up

Q: Do we have to lose altogether a factor of 4 of small tasks?

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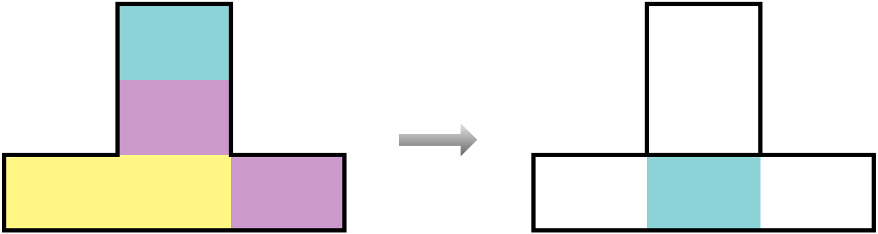
A: No. Both factors of 2 cannot happen simultaneously.

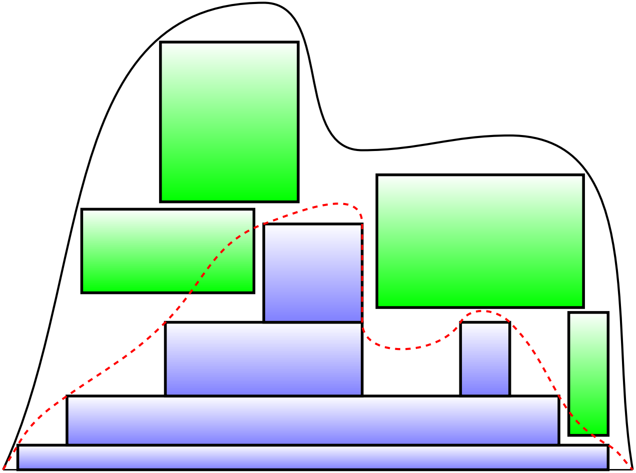
Main technical contribution

Our algorithm loses at most a factor of 3 of small tasks.



Selecting large tasks: we guess large tasks during the *dynamic program*.

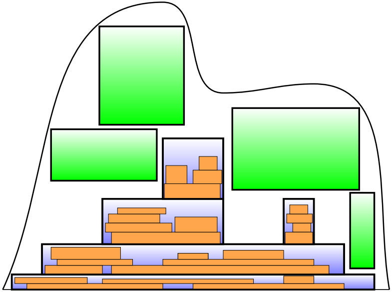
Observation: All profit from large tasks achieved when guessed correctly.



Summary

- *Dynamic programming* to guess large tasks and **boxes**
- *Linear programming* to select small tasks inside each box

Total profit: large tasks + 1/3 of small tasks



Thank you!

Our Result

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