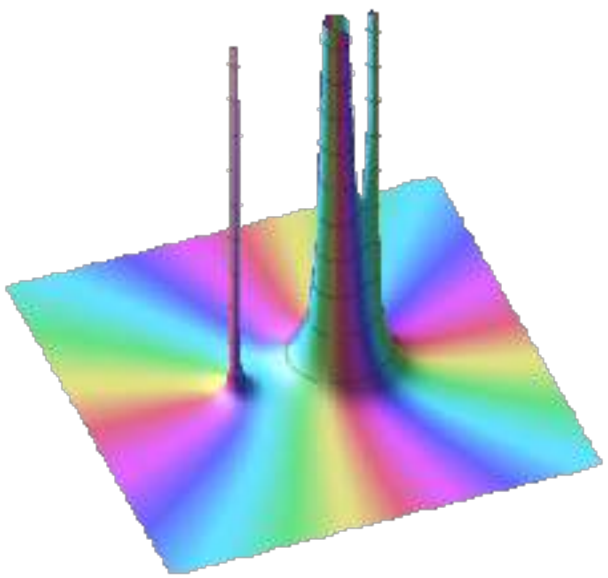
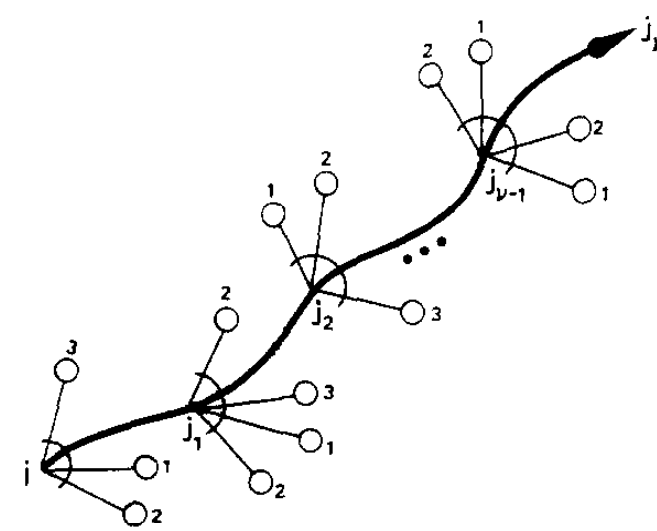
On (Effective) Analytic Combinatorics



Bruno Salvy

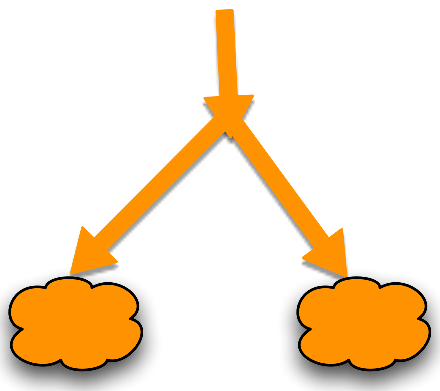
Inria & ENS de Lyon

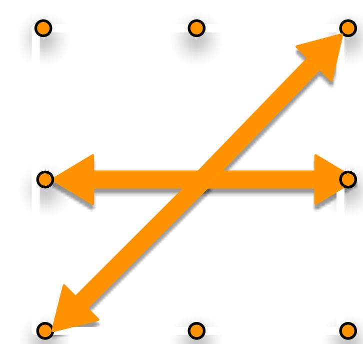


Séminaire Algorithmes

Dec. 2017

Combinatorics, Randomness and Analysis

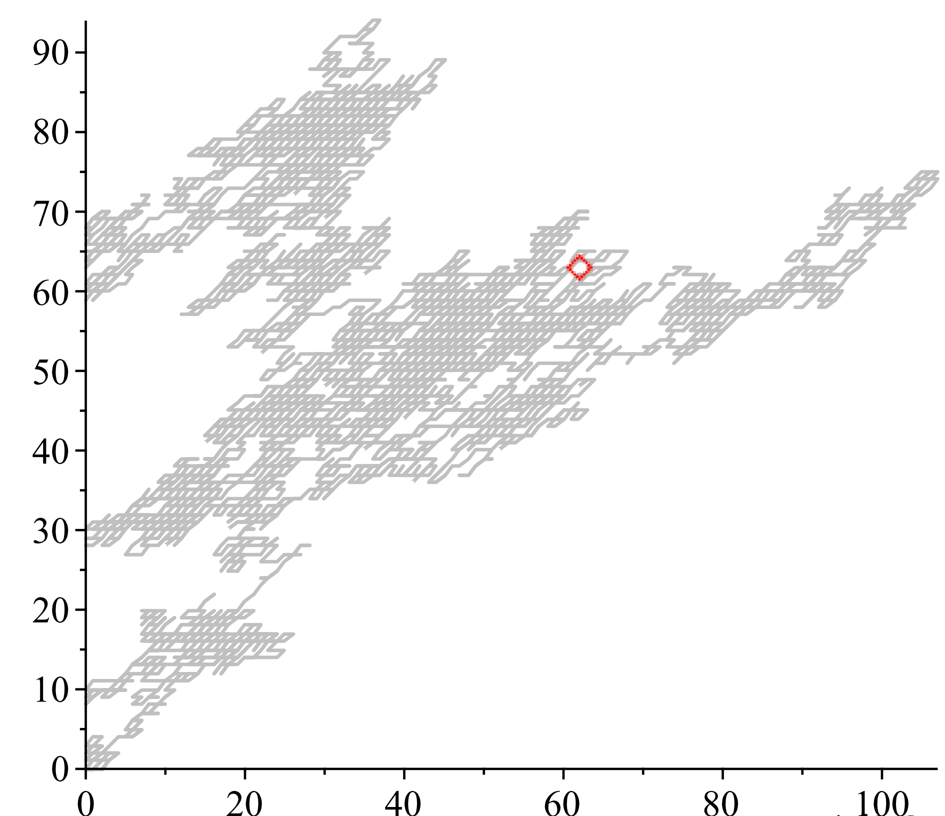




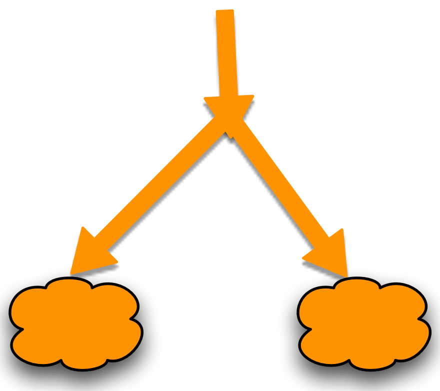
From **simple local** rules, a **global** structure arises**.**

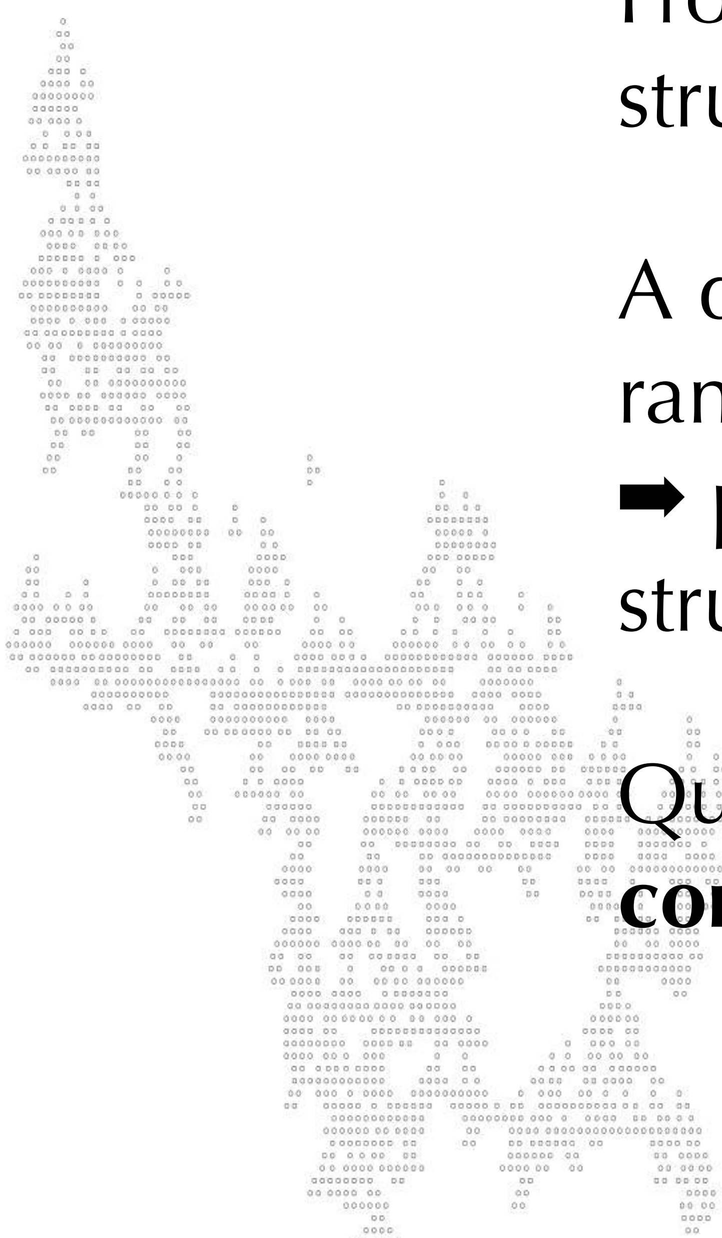
A quest for **universality** in random discrete structures**:**
➡ **probabilistic complexity** of structures and algorithms.

Quantitative results using **complex analysis.**



Combinatorics, Randomness and Analysis





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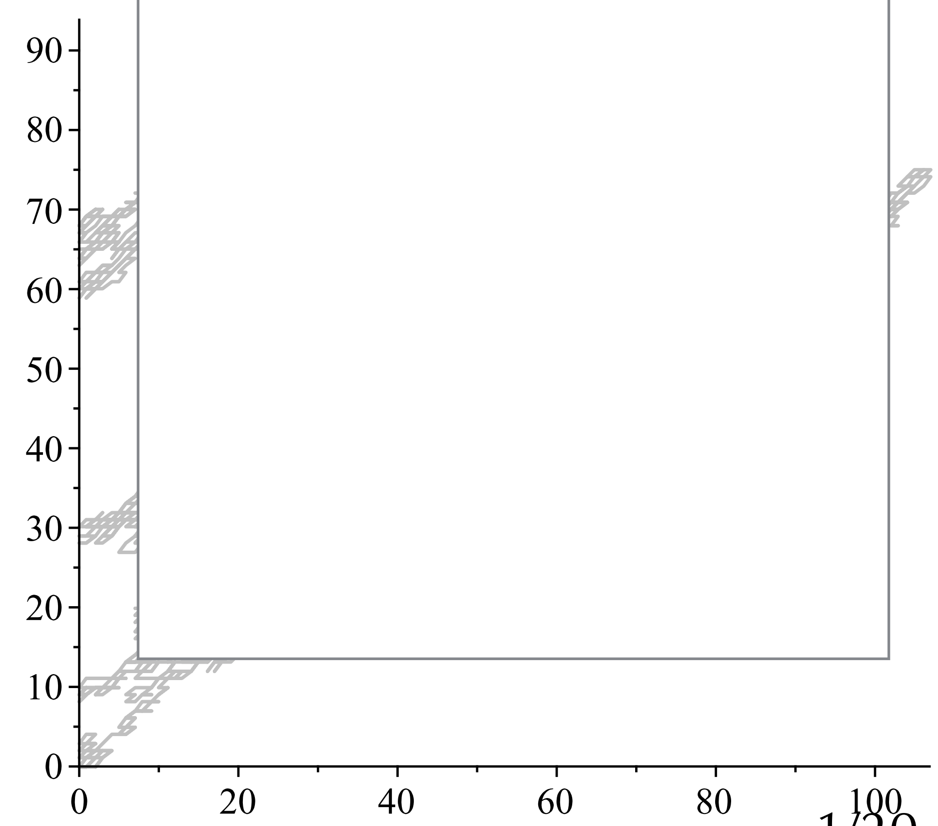
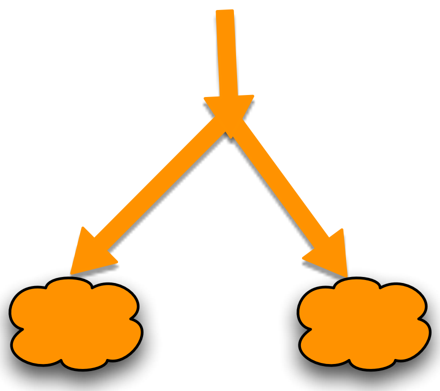


Combinatorics, Randomness and Analysis



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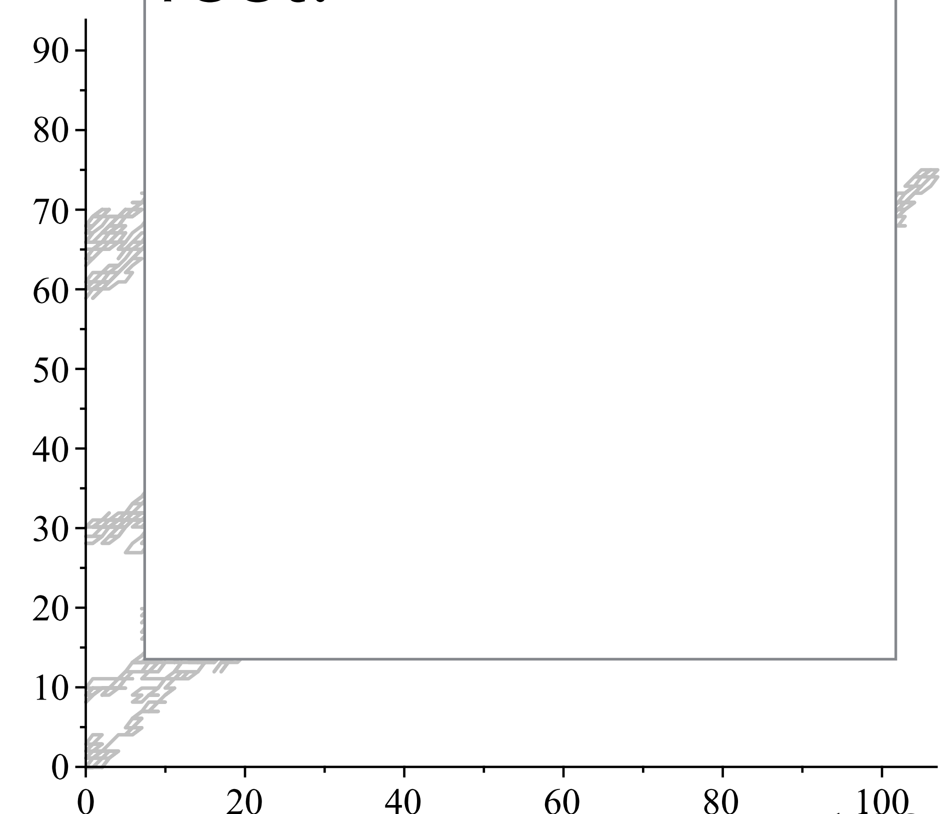

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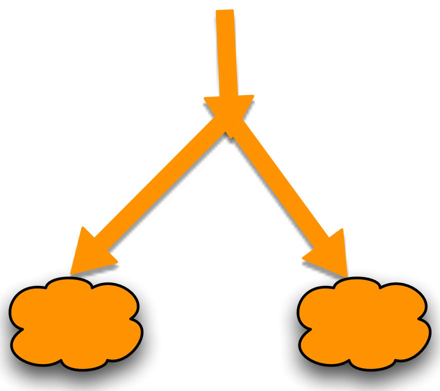

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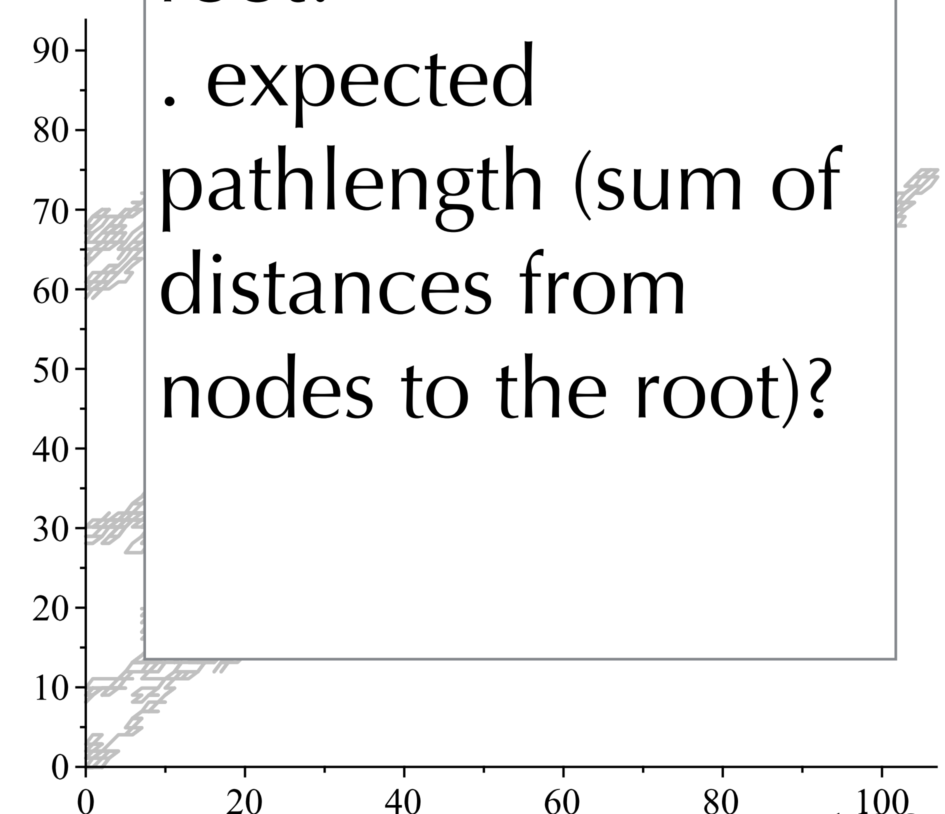

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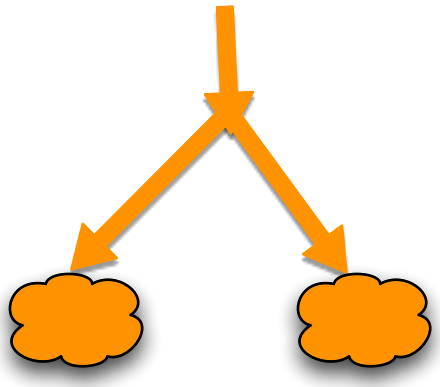
Combinatorics, Randomness and Analysis



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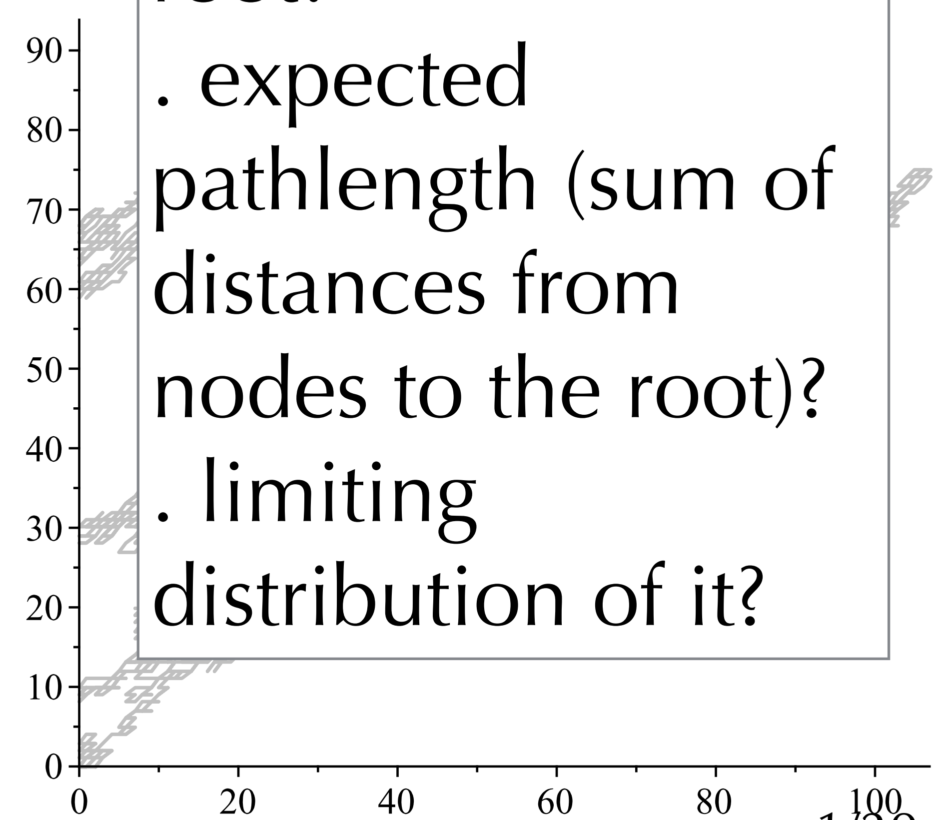
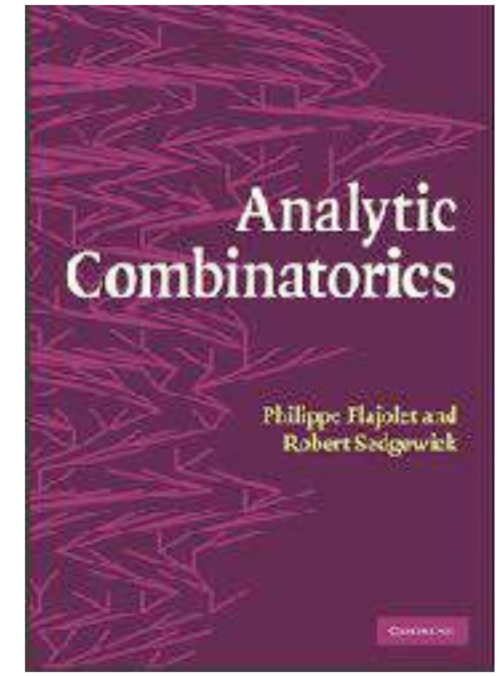
. limiting distribution of it?



Philippe Flajolet,
the Father of Analytic Combinatorics

1948-2011

Planned complete works span
7 volumes of approx 600 pp. each.



2009

Combinatorial specification → Generating functions →
- counting;
- random generation;
- asymptotic analysis.

If you can specify, you can analyze.

Constructible Structures

Language: $1, \mathcal{Z}, +, \times,$ SEQ, SET, CYC
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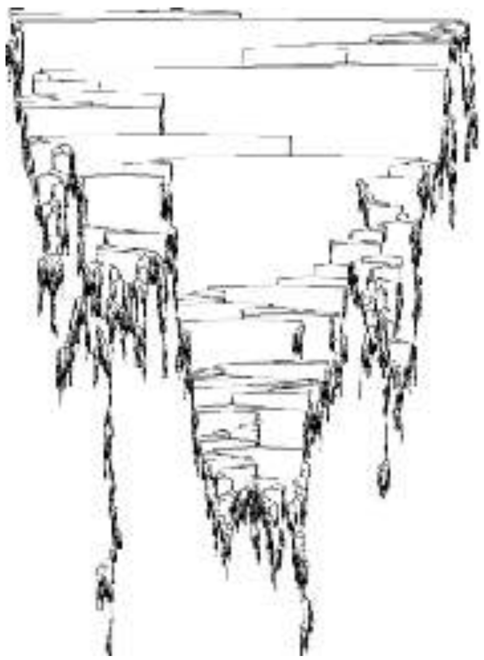

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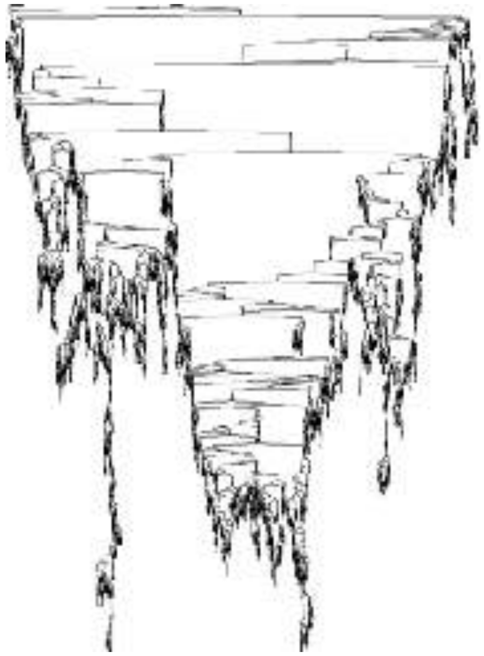

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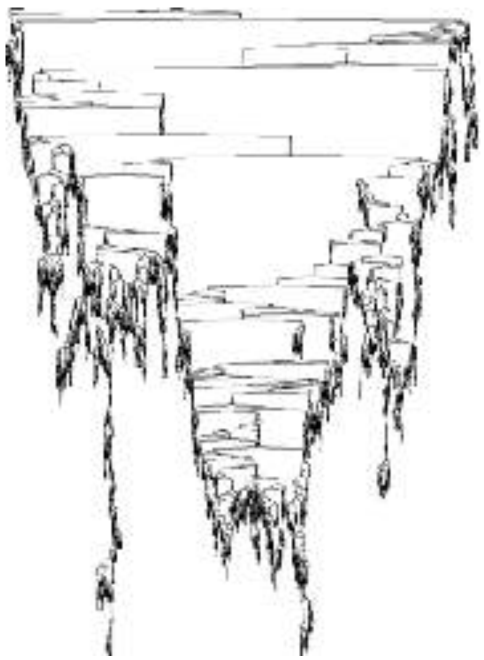

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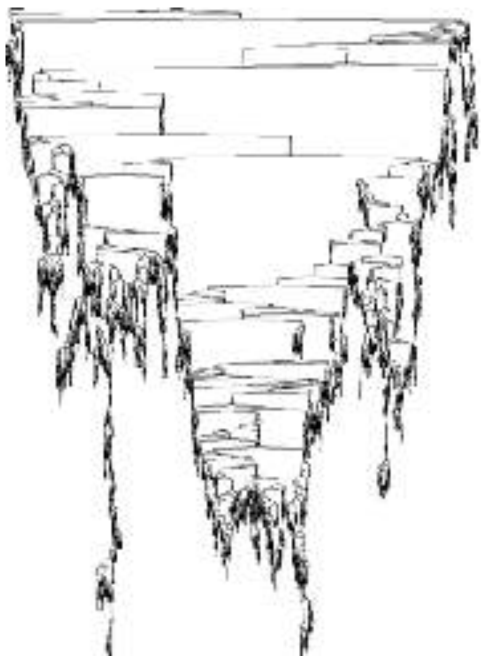

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- Functional graphs: $\mathcal{F}$=SET(CYC($\mathcal{T}$));

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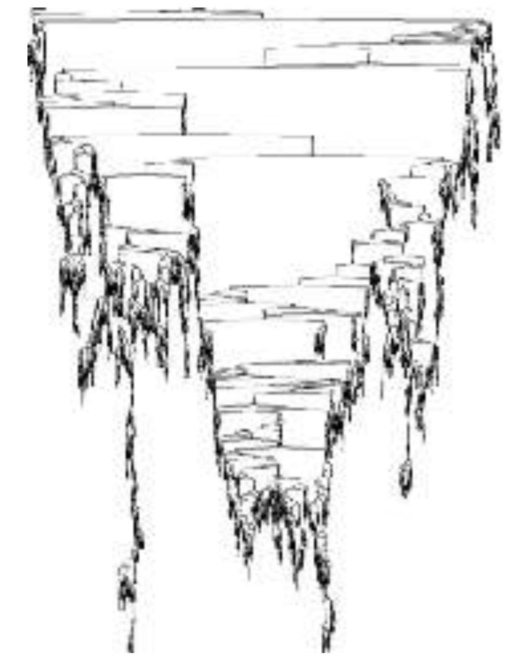

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- Series-parallel graphs:
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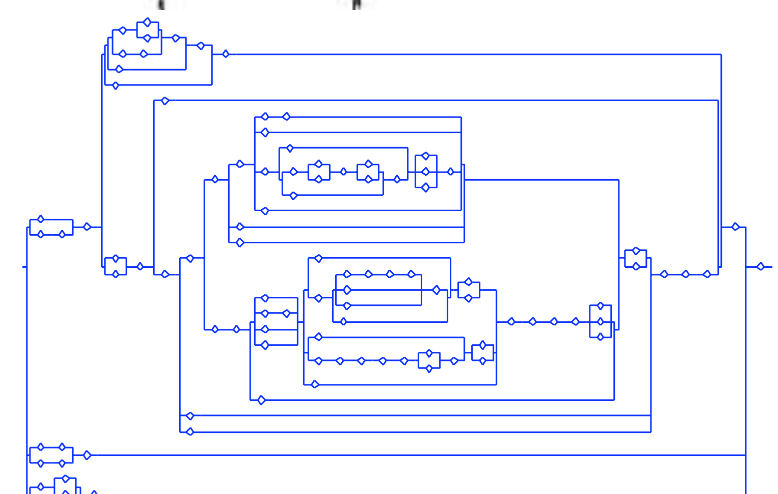

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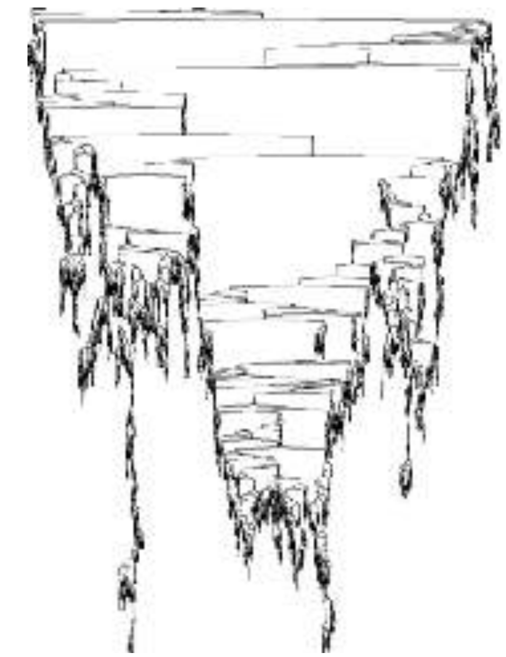

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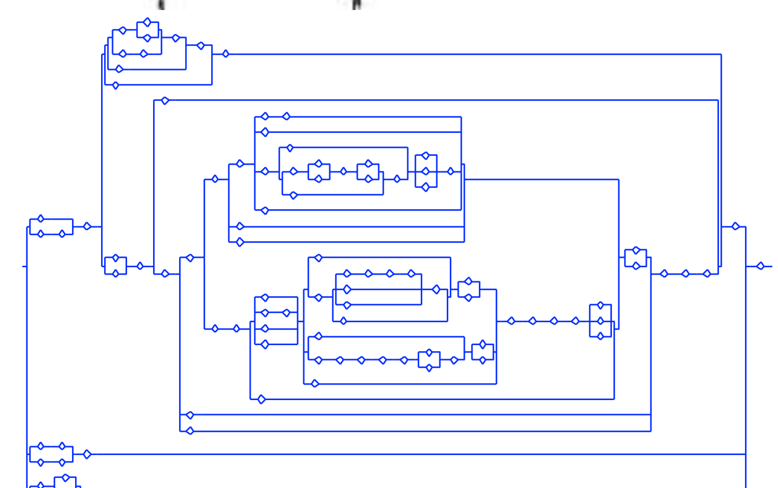
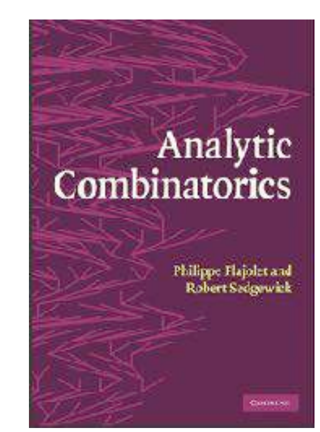
- ...hundreds of examples in "the purple book".





I. Enumeration and Generating Functions

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$$F(z) = \sum_{n=0}^{\infty} f_n z^n$$

Example: Binary Trees

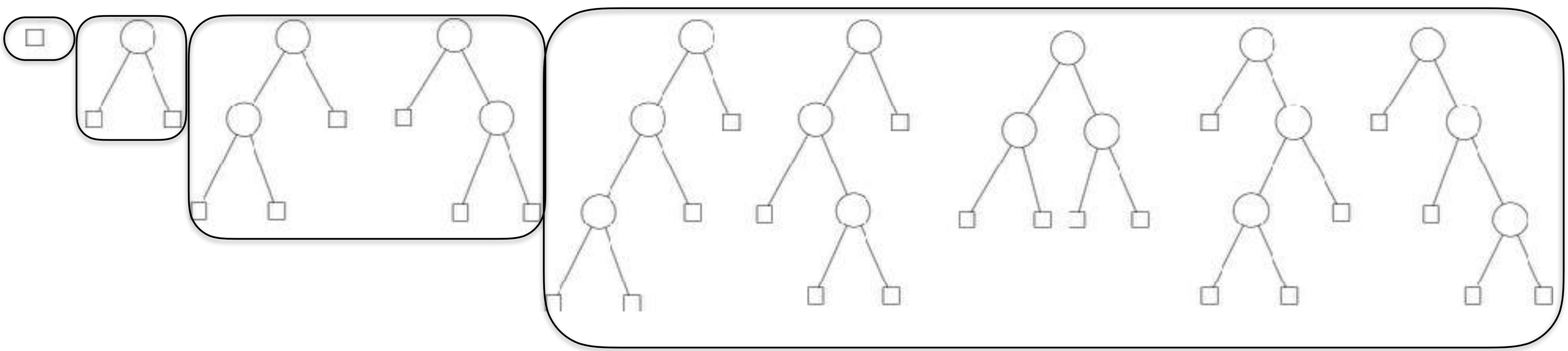
$$\mathscr{B} = 1 + \mathscr{Z} \times \mathscr{B} \times \mathscr{B}$$

$$B(z) = \sum_{n=0}^{\infty} B_n z^n$$

number of binary trees with *n* nodes

(Catalan)

$B_0 = B_1 = 1$ $B_2 = 2$ $B_3 = 5$



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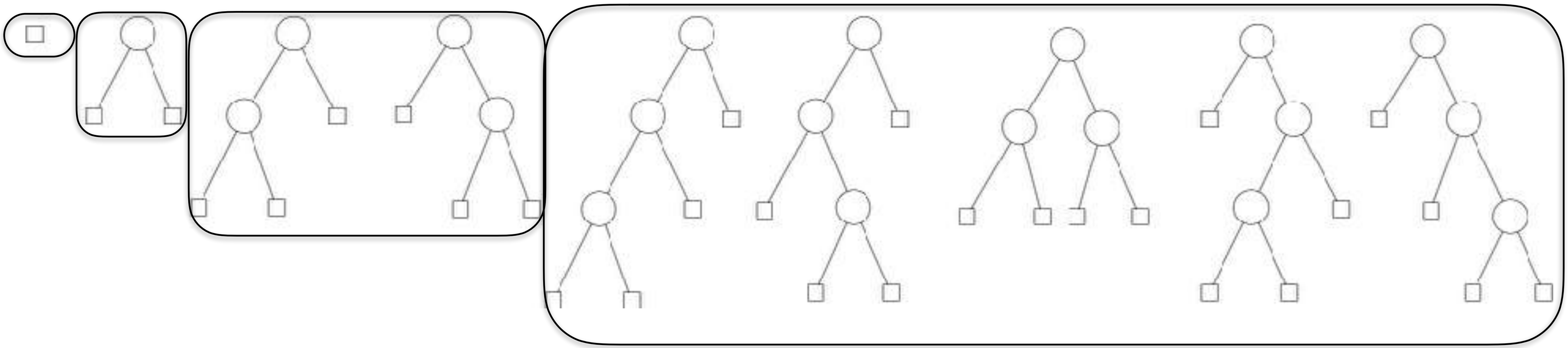

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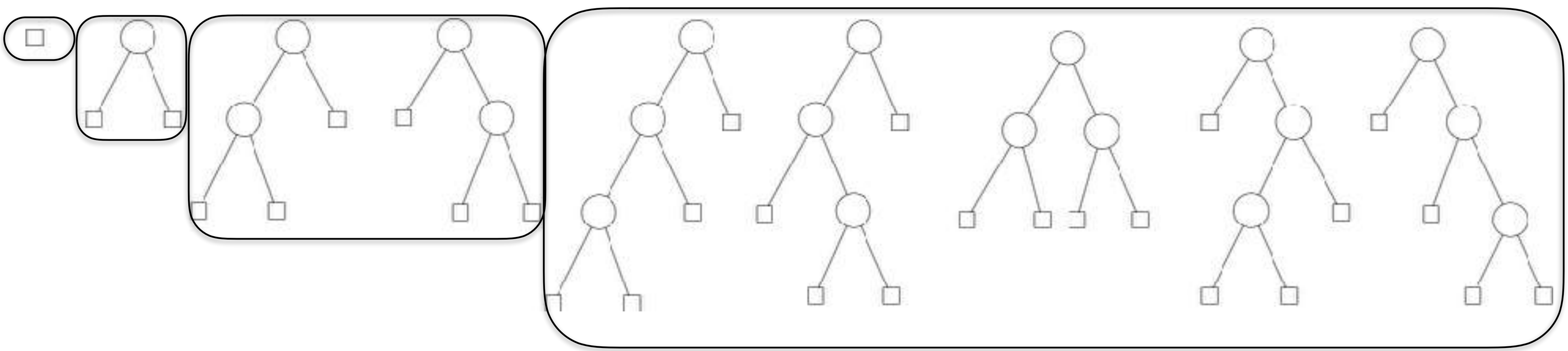

$$\text{For } n \geq 1, \quad B_n = \sum_{k=0}^{n-1} B_k B_{n-k-1} \Rightarrow B_n z^n = \sum_{k=0}^{n-1} z(B_k z^k)(B_{n-k-1} z^{n-k-1})$$

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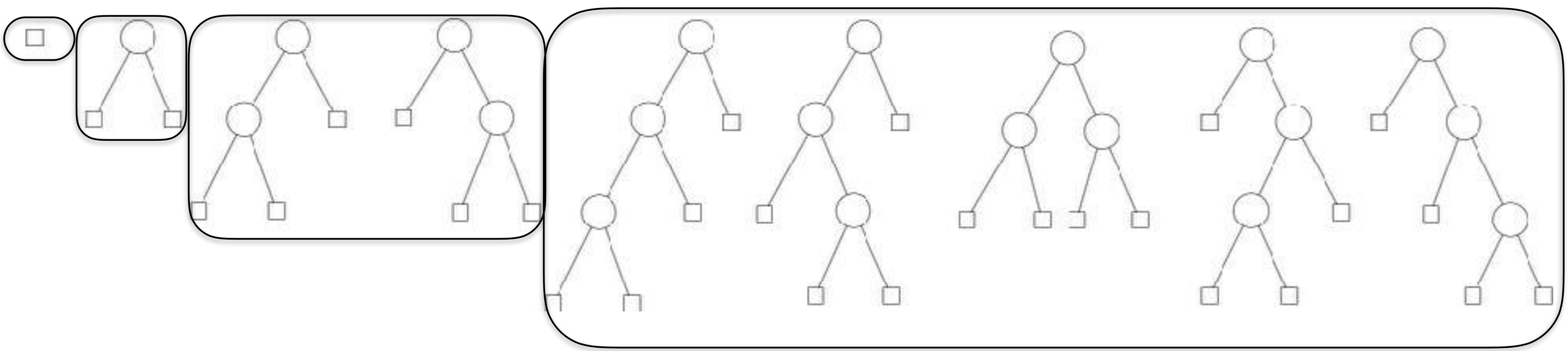

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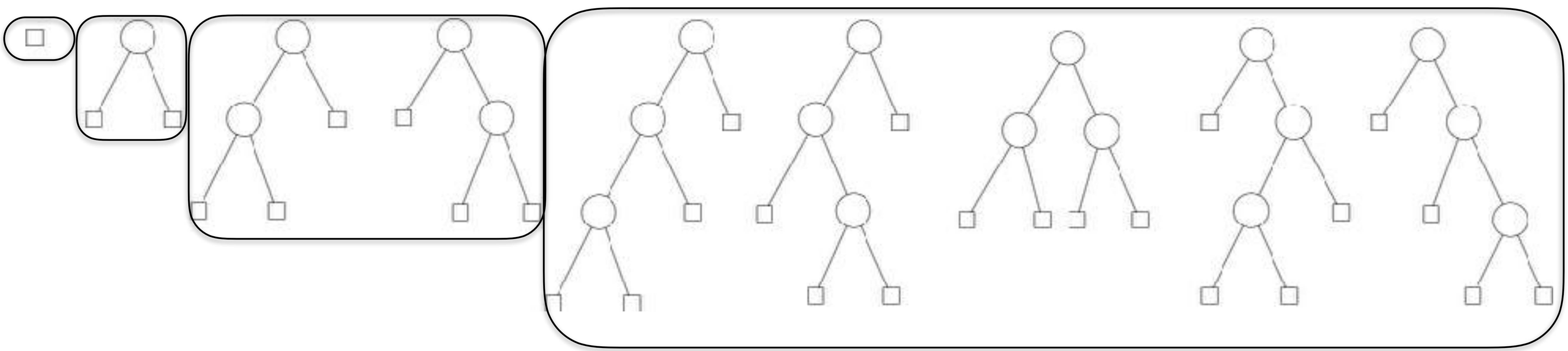

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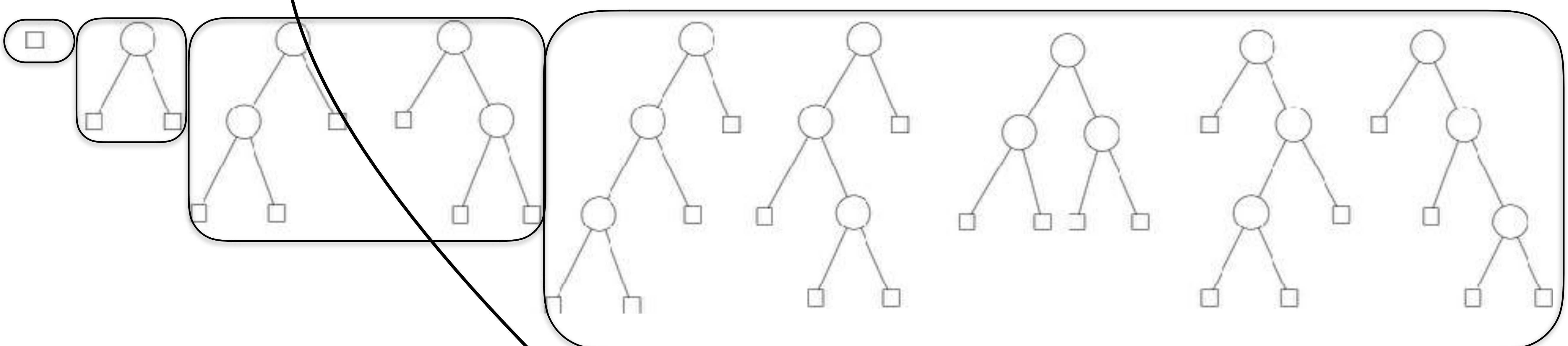

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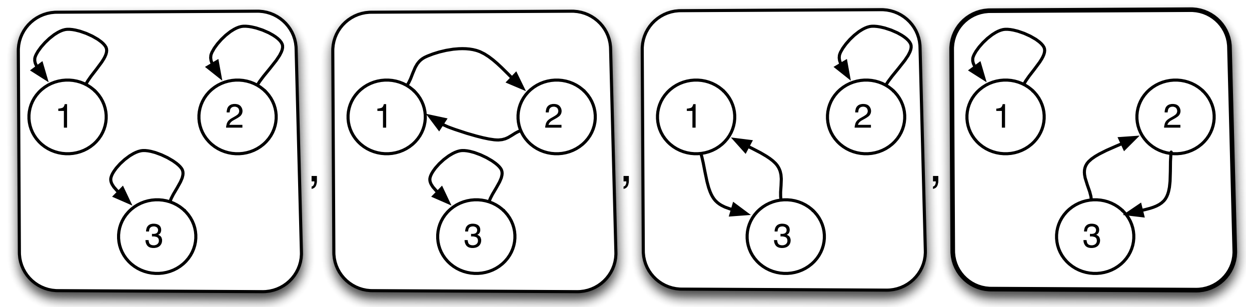
What do we count?

$\mathrm{Inv}[\{1,2,3\}] = \{$  $\}$

4 involutions;

3 of them permuted by $\mathfrak{S}_3 \rightarrow$ 2 *unlabelled structures*.

Exponential generating series (EGF):

$$F(z) = \sum_{n=0}^{\infty} f_n \frac{z^n}{n!}, \quad f_n = \text{nb. labelled structs of size } n. \qquad \mathrm{Inv}_3(z) = \frac{2}{3} z^3$$

Ordinary generating series (OGF):

$$\tilde{F}(z) = \sum_{n=0}^{\infty} \tilde{f}_n z^n, \quad \tilde{f}_n = \text{nb. unlabelled structs of size } n. \qquad \widetilde{\mathrm{Inv}}_3(z) = 2z^3$$

A Dictionary for Generating Series

Language: $1, \mathcal{Z}, +, \times, \textsc{Seq}, \textsc{Set}, \textsc{Cyc}$ and recursion.

describes ***constructible structures***

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Structure	EGF	OGF
$\mathcal{A}+\mathcal{B}$	$A(z)+B(z)$	$\tilde{A}(z)+\tilde{B}(z)$
$\mathcal{A}\times\mathcal{B}$	$A(z)\times B(z)$	$\tilde{A}(z)\times\tilde{B}(z)$

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Seq($\mathcal{A}$)	$\frac{1}{1-A(z)}$	$\frac{1}{1-\tilde{A}(z)}$

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SEQ($\mathcal{A}$)	$\frac{1}{1-A(z)}$	$\frac{1}{1-\tilde{A}(z)}$
SET($\mathcal{A}$)	$\exp(A(z))$	$\exp(\sum \tilde{A}(z^i)/i)$

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SET($\mathcal{A}$)	$\exp(A(z))$	$\exp(\sum\tilde{A}(z^i)/i)$
CYC($\mathcal{A}$)	$\log\frac{1}{1-A(z)}$	$\sum\frac{\phi(i)}{i}\log\frac{1}{1-\tilde{A}(z^i)}$

Examples

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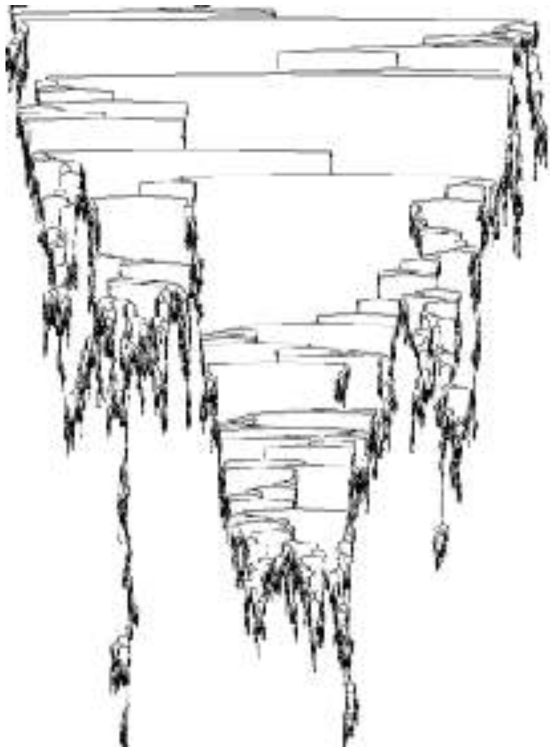
Binary trees: $\mathscr{B}$=1+$\mathscr{Z}$x$\mathscr{B}$x$\mathscr{B}$

$$\longrightarrow B(z) = 1 + zB(z)^2 = \tilde{B}(z)$$



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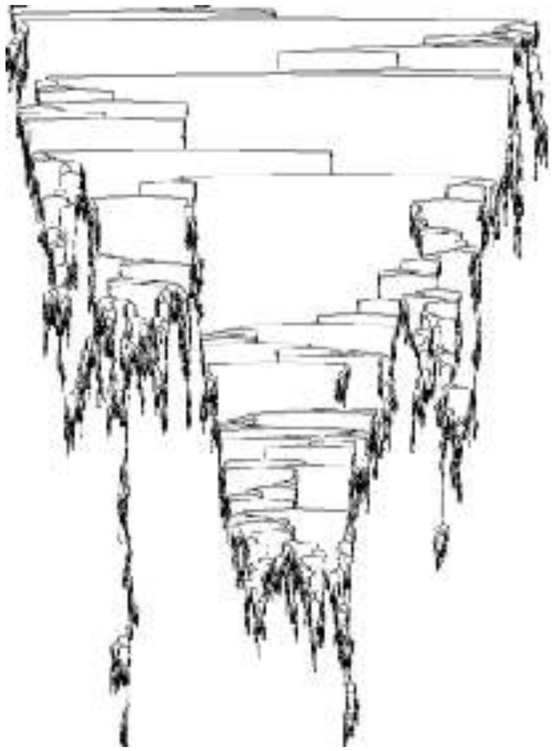

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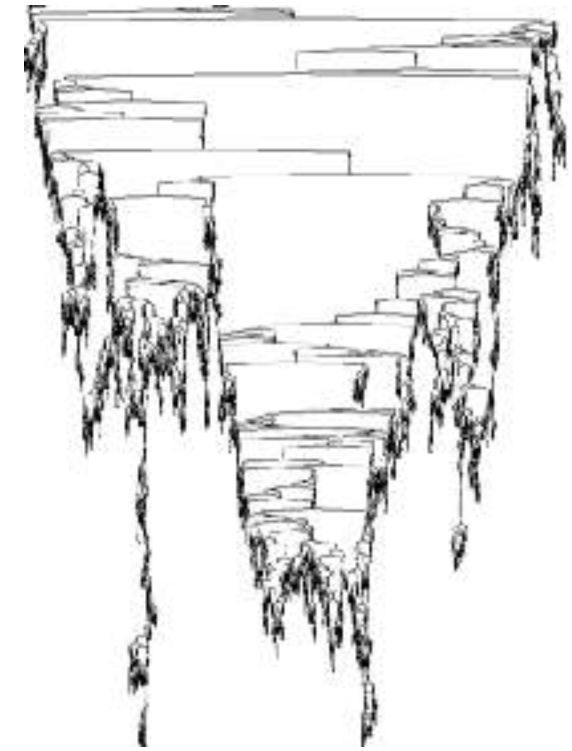

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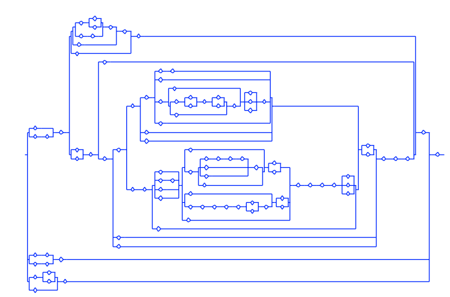
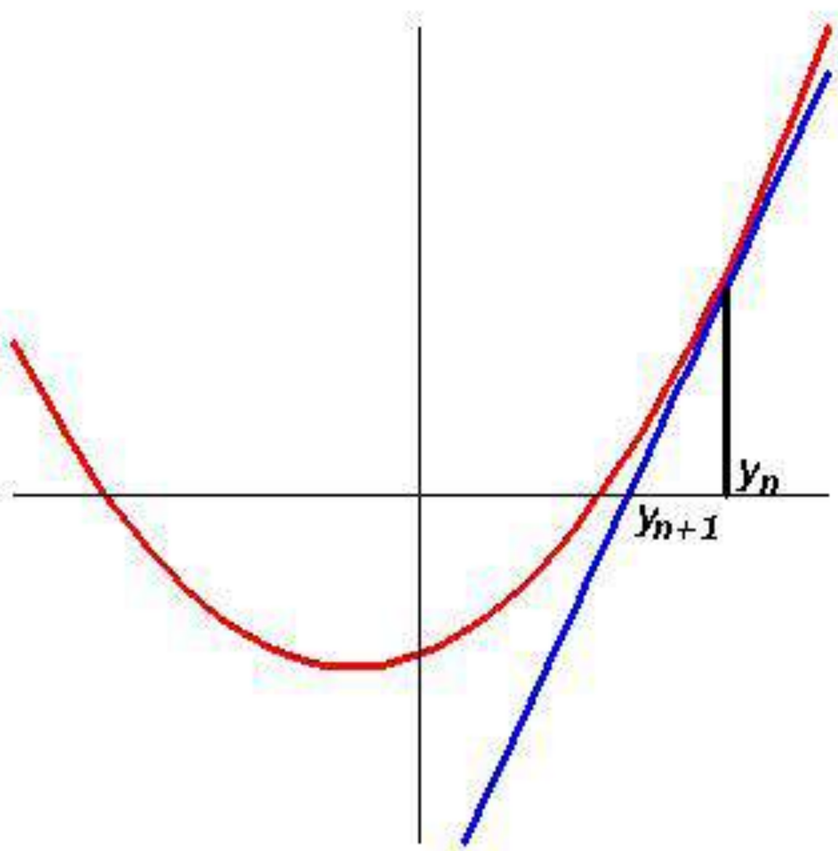


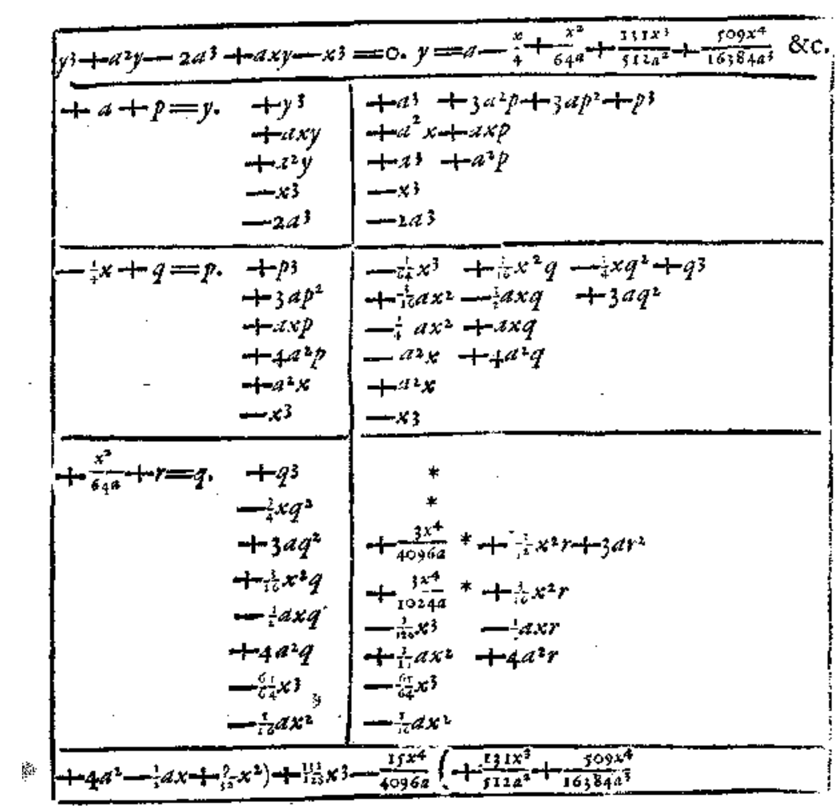
Series-parallel graphs:

$$\mathcal{G} = \mathcal{Z} + \mathcal{S} + \mathcal{P}, \mathcal{S} = \textsc{Seq}_{>0}(\mathcal{Z}+\mathcal{P}), \mathcal{P} = \textsc{Set}_{>0}(\mathcal{Z}+\mathcal{S})$$

$$\longrightarrow \left\{ G(z) = z + S(z) + P(z), S(z) = \frac{1}{1-z-P(z)} - 1, P(z) = e^{z+S(z)} - 1 \right\}$$

II. Newton Iteration and Fast Enumeration





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Numerical Newton Iteration

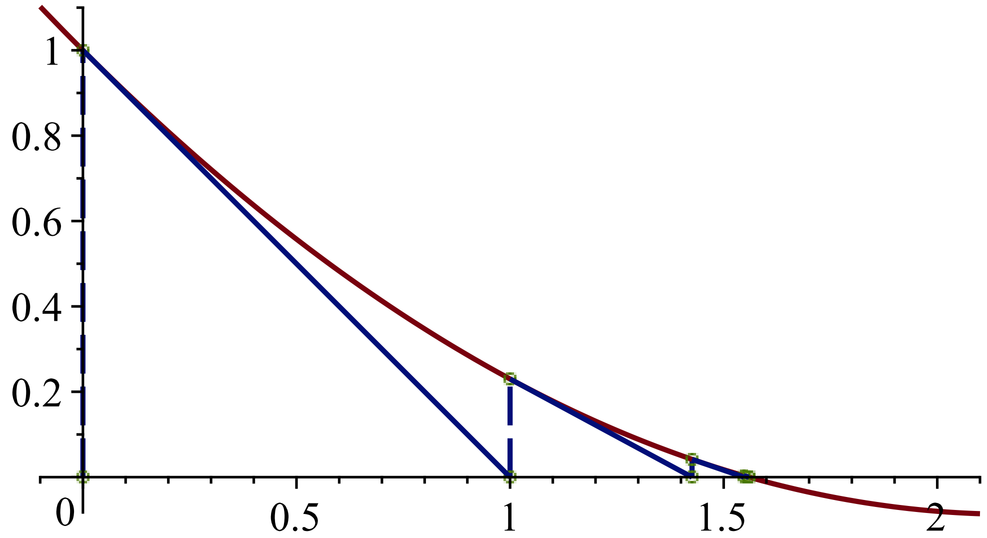
To solve $\phi(y) = 0$, iterate

$$y^{[n+1]} = y^{[n]} + u^{[n]},$$

with $\phi(y^{[n]}) + \phi'(y^{[n]})u^{[n]} = 0.$

$$\phi(y) = 1 + zy^2 - y$$

$$z = 0.23$$



Numerical Newton Iteration

To solve $\phi(y) = 0$, iterate

$$y^{[n+1]} = y^{[n]} + u^{[n]},$$

with $\phi(y^{[n]}) + \phi'(y^{[n]})u^{[n]} = 0.$

$$\phi(y) = 1 + zy^2 - y$$

$$y^{[n+1]} = \mathcal{N}(y^{[n]}) = y^{[n]} + \frac{1 + z{y^{[n]}}^2 - y^{[n]}}{1 - 2zy^{[n]}}$$

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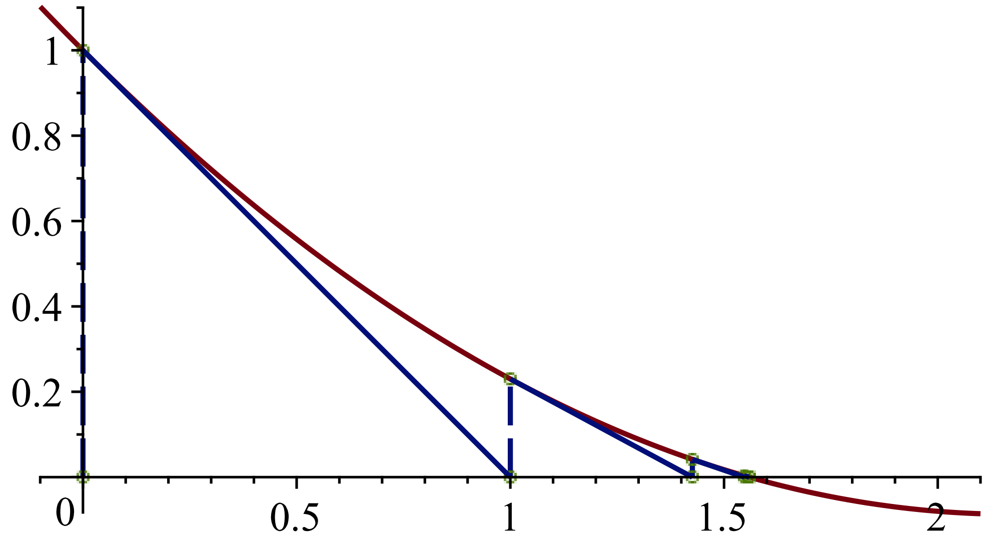

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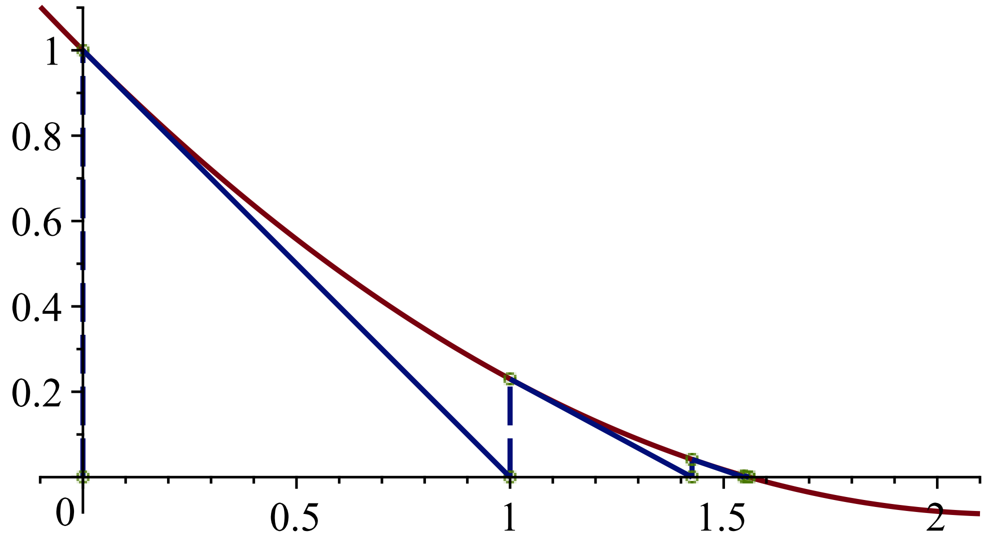

$$y^{[0]} = 0,$$
$$y^{[1]} = 1.0000000000000000,$$
$$y^{[2]} \simeq 1.4259259259259259,$$
$$y^{[3]} \simeq 1.5471933181836303,$$
$$y^{[4]} \simeq 1.5589256602748822,$$
$$y^{[5]} \simeq 1.5590375713926592,$$
$$y^{[6]} \simeq 1.5590375815769151$$

Quadratic convergence

Newton Iteration for Power Series

Same Newton Iteration

$$y^{[n+1]} = \mathcal{N}(y^{[n]}) = y^{[n]} + \frac{1 + z{y^{[n]}}^2 - y^{[n]}}{1 - 2zy^{[n]}}$$

Newton Iteration for Power Series

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$$y^{[0]} = 0$$

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$$y^{[2]} = 1 + z + 2z^2 + 4z^3 + 8z^4 + 16z^5 + 32z^6 + 64z^7 + \cdots$$

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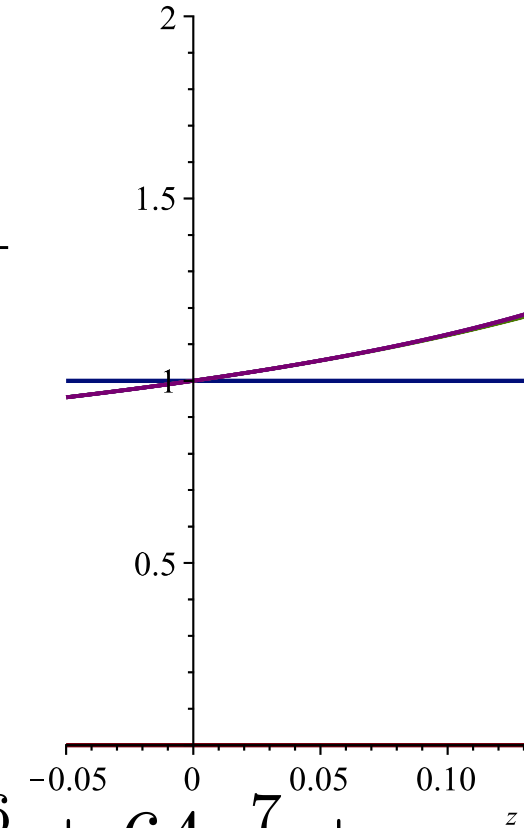

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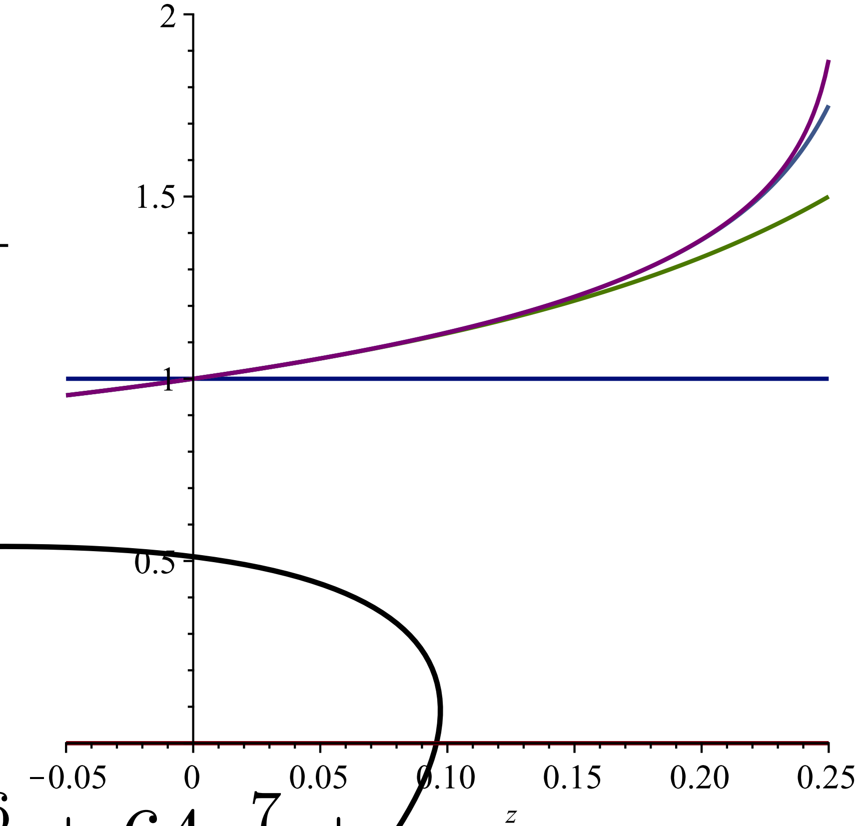

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Proving numerical convergence requires control over the tails

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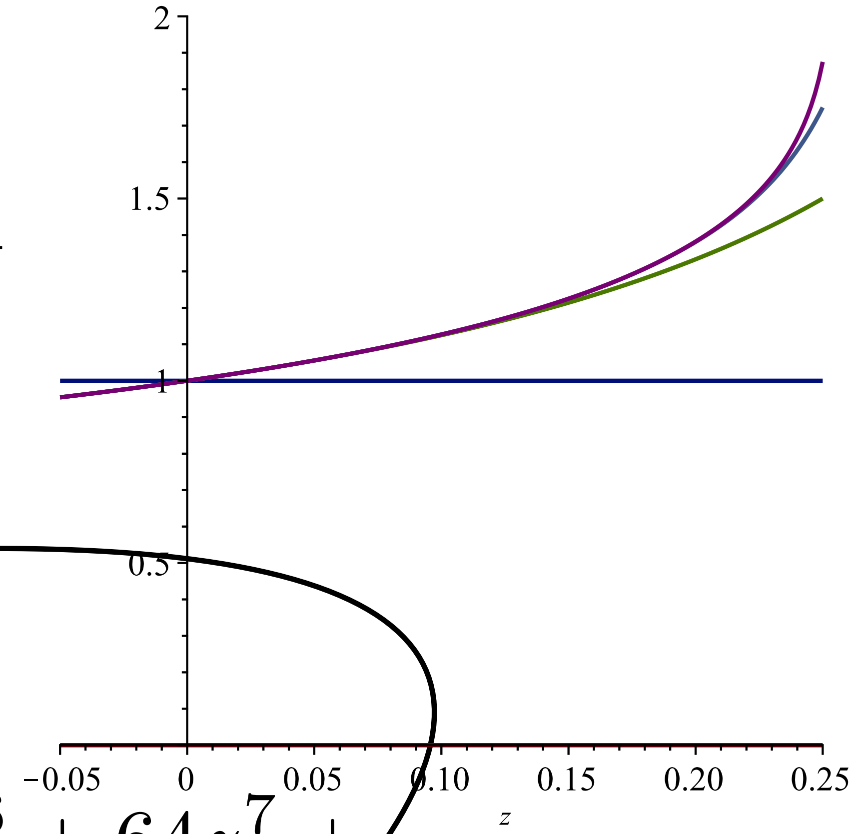

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On power series: $y - y^{[\infty]} = O(z^m) \Longrightarrow \mathcal{N}(y) - y^{[\infty]} = O(z^{2m(+1)})$

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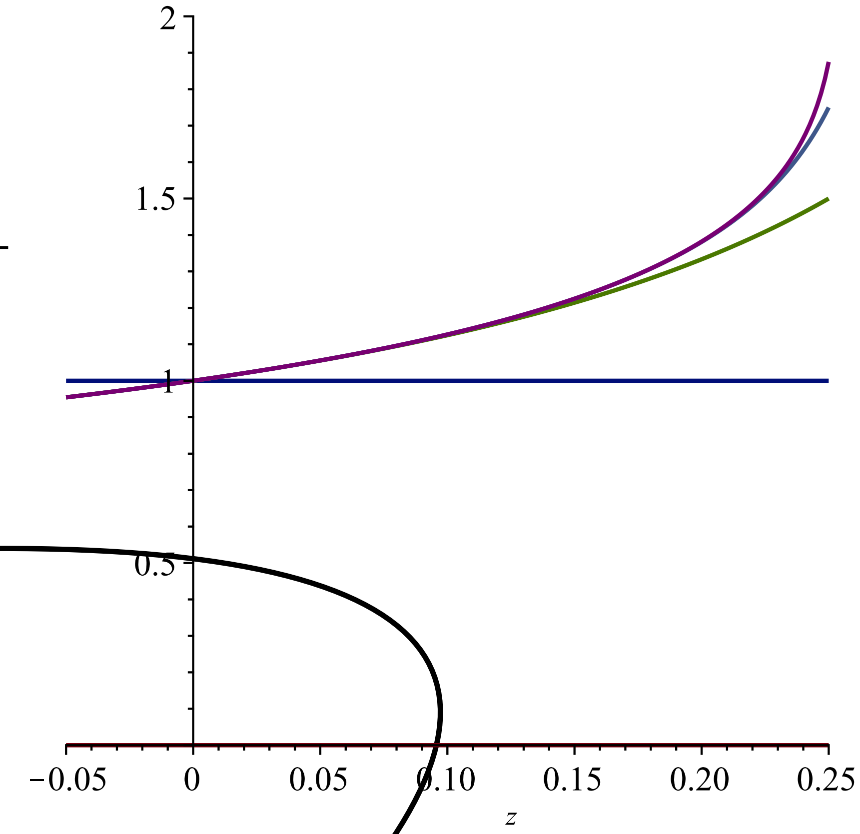

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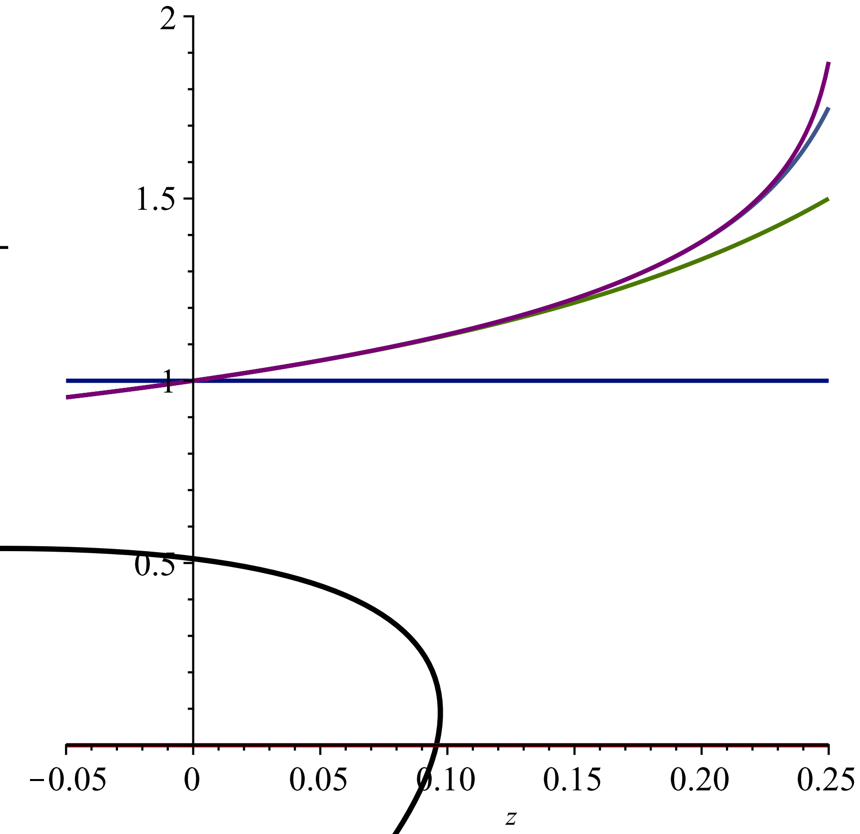

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```
Expand(N) = {
  res=Expand(N/2);
  a=ϕ(res); b=ϕ'(res);
  u=Solve(a+bx,x);
  return res+u; }
```

Newton Iteration for Power Series

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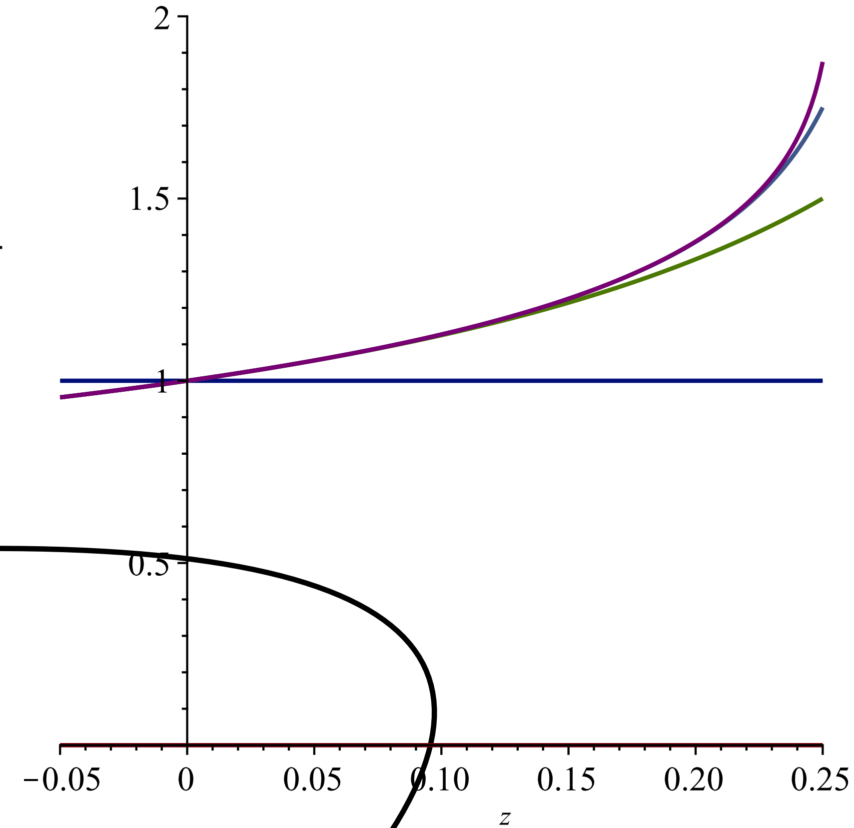

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```

Cost(N) ≤ ct × Cost(last step)

Fast Enumeration

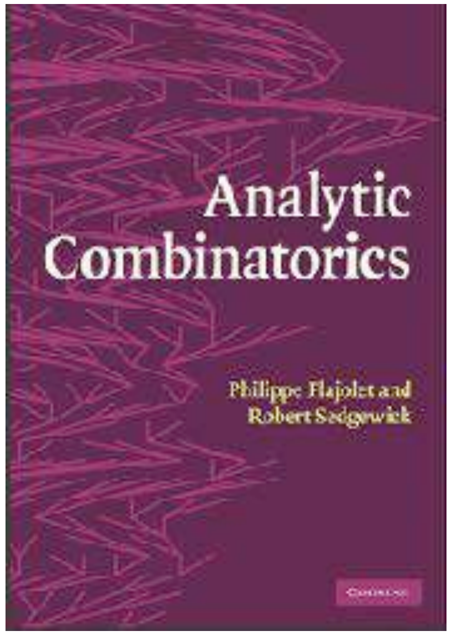
Thm. First N coefficients of GFs of all *constructible* structures:

1. arithmetic complexity $O(N \log N)$ (both ogf & egf);

2. bit complexity

$O(N^2 \log^2 N \log\log N)$ (ogf); $O(N^2 \log^3 N \log\log N)$ (egf).

Ingredients: Newton iteration & FFT.



[Pivoteau-S-Soria 2012]

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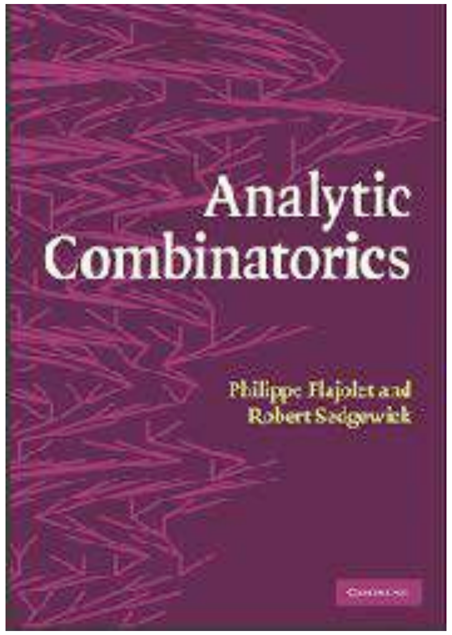
Ingredients: Newton iteration & FFT.

```
> with(NewtonGF):
> BinTrees:={B=Union(Epsilon,Prod(Z,B,B))}:
> GFSeries(BinTrees,labelled,z,21)[2];
```

$$B = 1 + z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + 132z^6 + 429z^7 + 1430z^8 + 4862z^9 + 16796z^{10} + 58786z^{11} + 208012z^{12} + 742900z^{13} + 2674440z^{14} + 9694845z^{15} + 35357670z^{16} + 129644790z^{17} + 477638700z^{18} + 1767263190z^{19} + 6564120420z^{20} + O(z^{21})$$

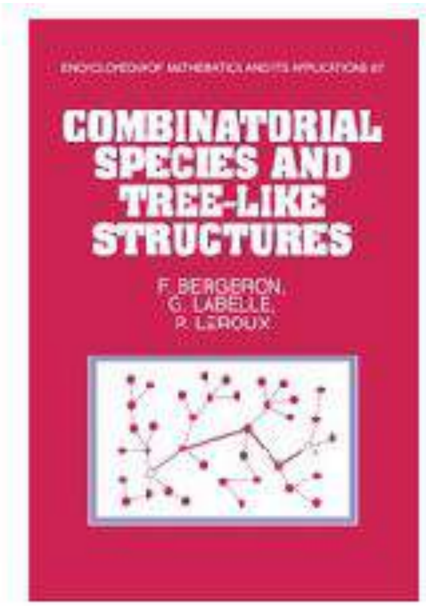


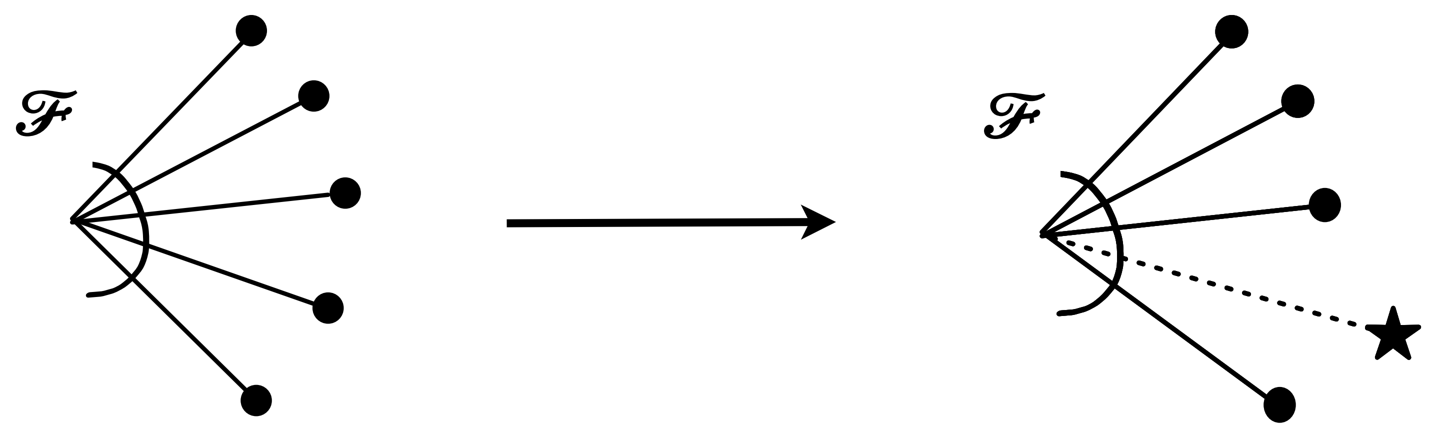
Demo
NewtonGF

[Pivoteau-S-Soria 2012]

Derivative of Combinatorial Structures

(origin: species theory)





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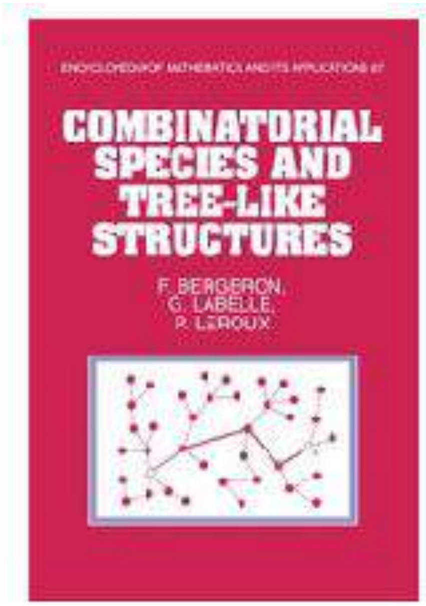
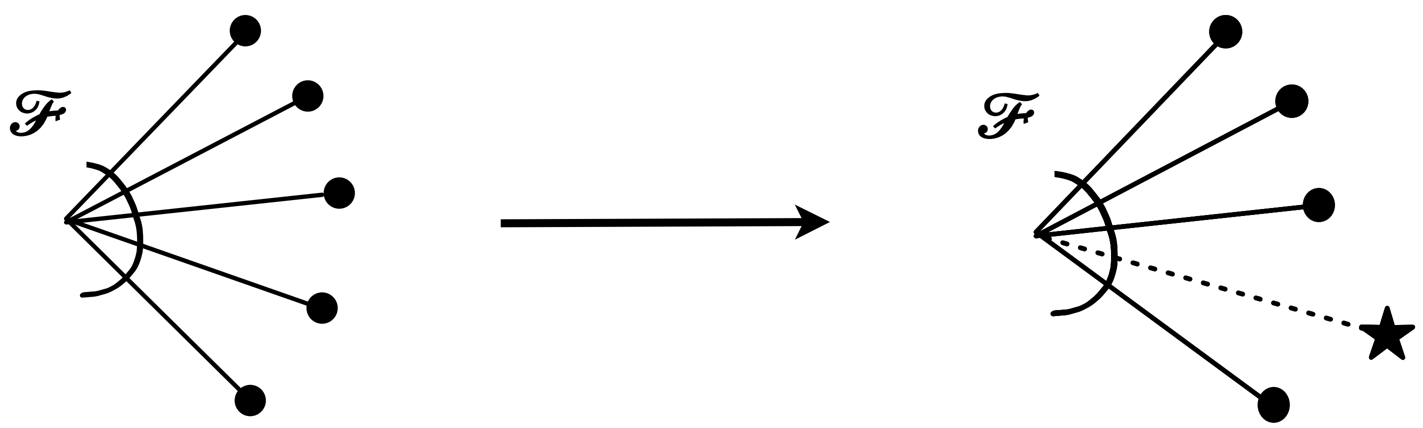
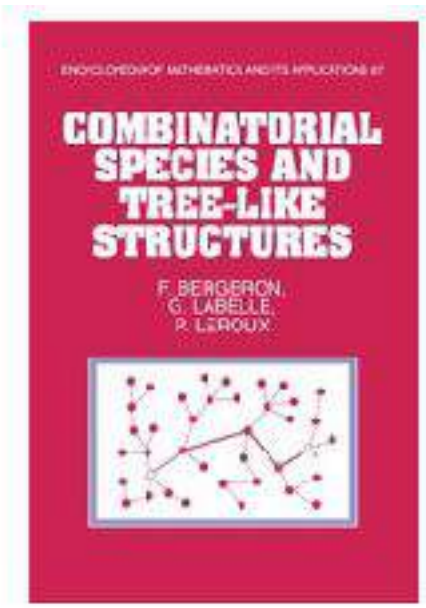




- $(\mathscr{F}+\mathscr{G})'=\mathscr{F}'+\mathscr{G}'$; $(\mathscr{F}\times\mathscr{G})'=\mathscr{F}'\times\mathscr{G}+\mathscr{F}\times\mathscr{G}'$;

Derivative of Combinatorial Structures



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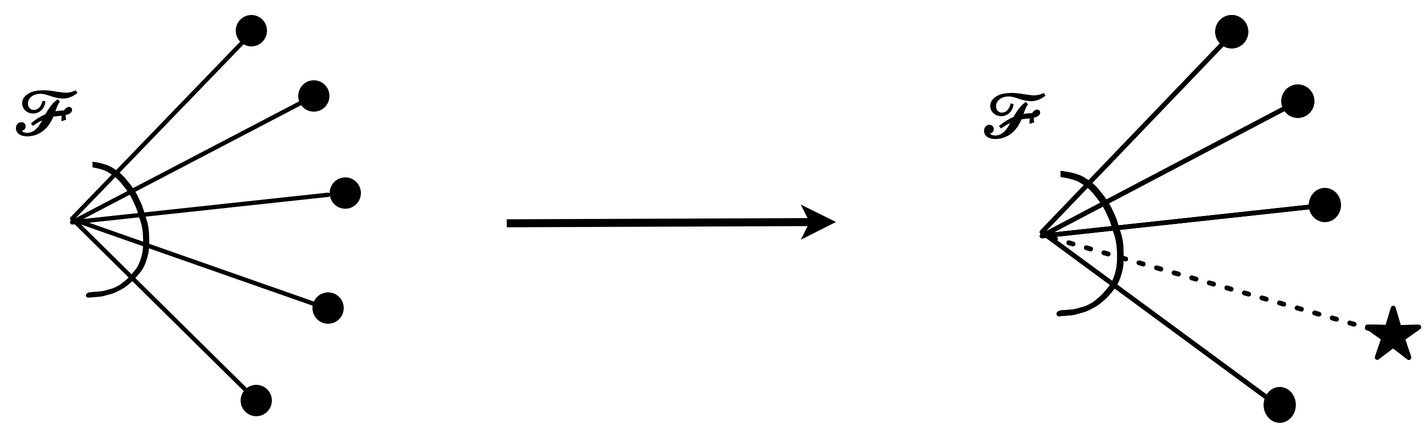

- $(\mathscr{F}+\mathscr{G})'=\mathscr{F}'+\mathscr{G}'$;$(\mathscr{F}\times\mathscr{G})'=\mathscr{F}'\times\mathscr{G}+\mathscr{F}\times\mathscr{G}'$;
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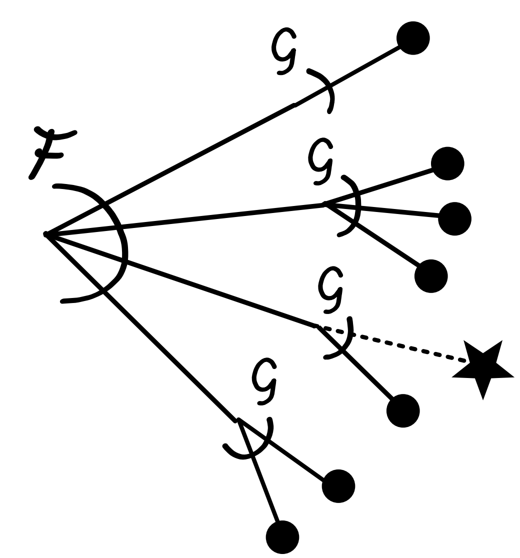

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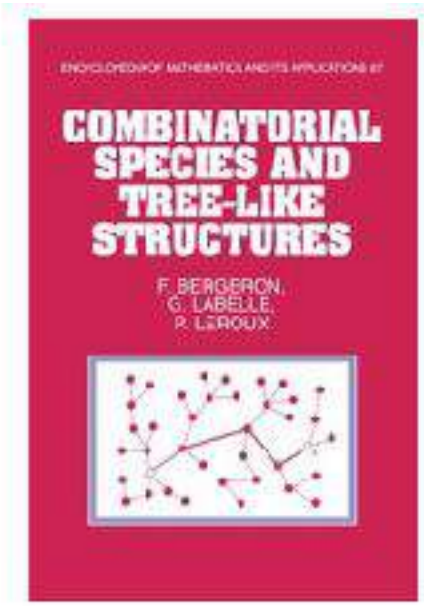

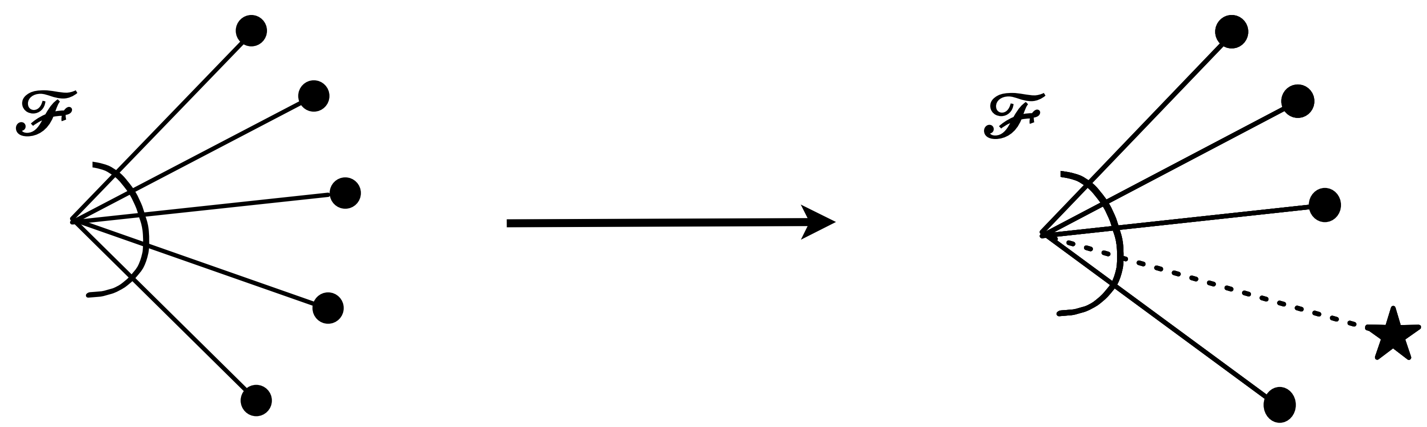

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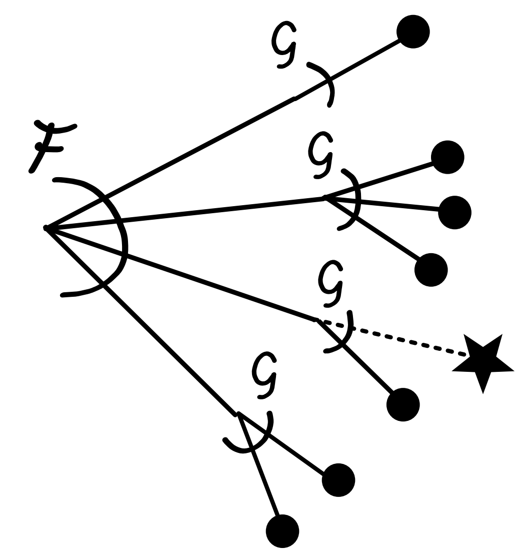
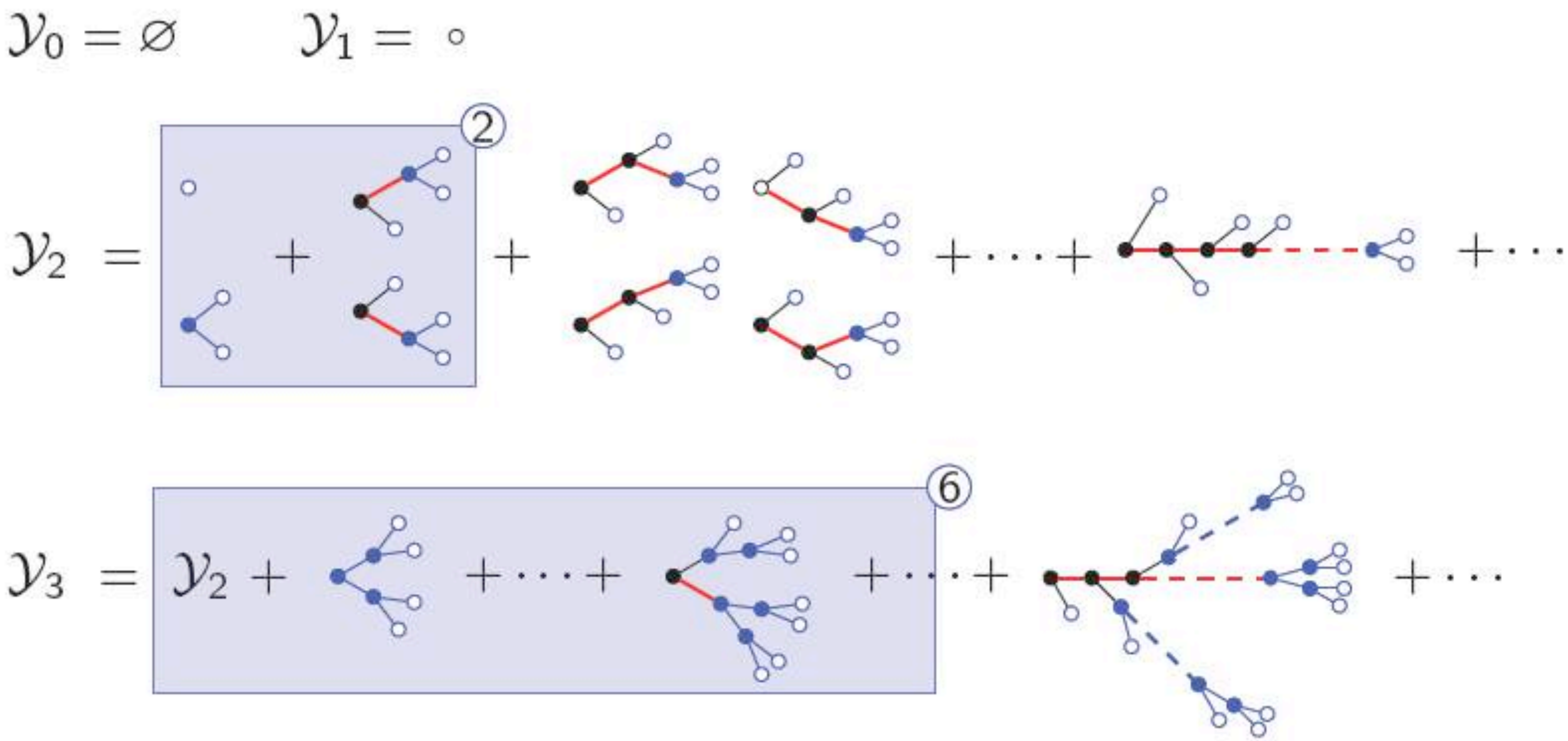
- $\mathscr{F}(\mathscr{G})'=\mathscr{F}'(\mathscr{G})\times\mathscr{G}'$;
- $0'=1'=0$; $\mathscr{Z}'=1$;
  SET′=SET; CYC′=SEQ; SEQ′=SEQ×SEQ.



Combinatorial Newton Iteration

$$\mathcal{Y} = 1 + \mathcal{Z} \times \mathcal{Y} \times \mathcal{Y} =: \mathcal{H}(\mathcal{Z}, \mathcal{Y})$$



[Décoste, Labelle, Leroux 1982]

Combinatorial Newton Iteration

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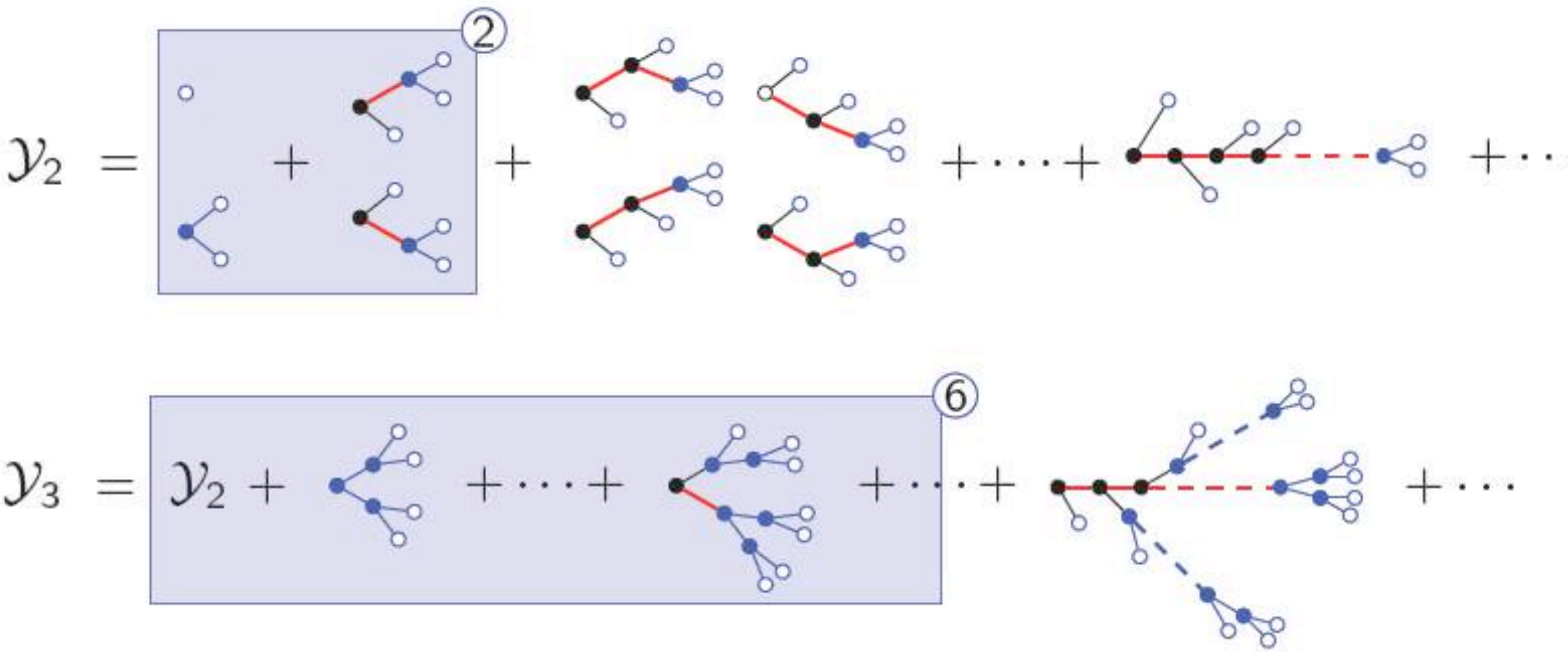
$\mathcal{Y}_0 = \varnothing \qquad \mathcal{Y}_1 = \circ$



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$$\left(\mathrm{Id} - \frac{\partial \mathcal{H}}{\partial \mathcal{Y}}(\mathcal{Z}, \mathcal{Y}^{[n]})\right)^{-1}$$

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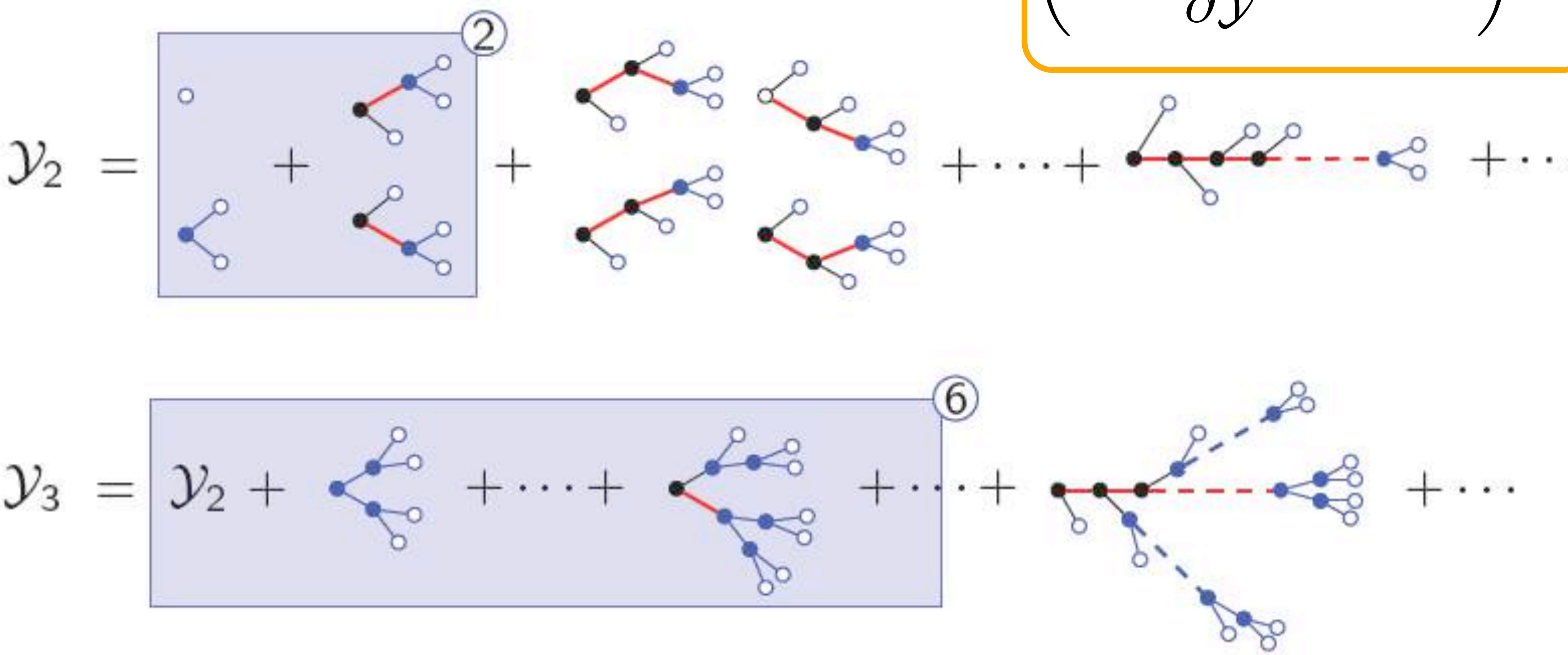

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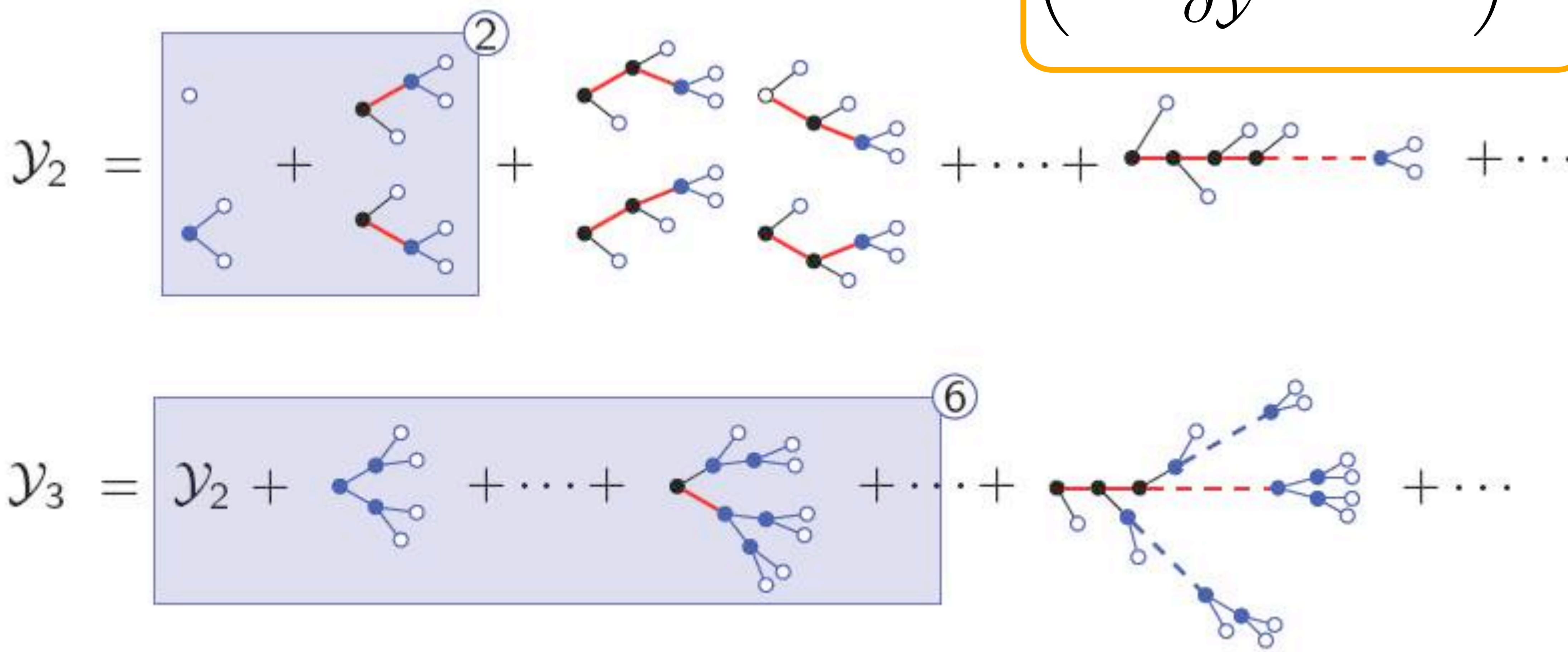
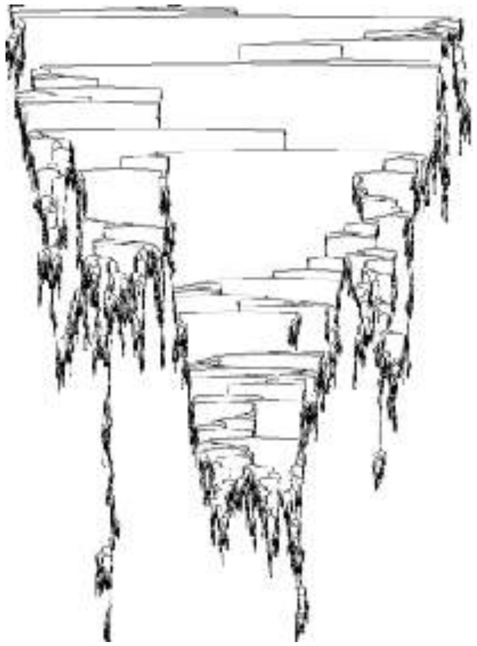


Ccl: numerical convergence of the Newton iteration starting from 0.

[Décoste, Labelle, Leroux 1982]

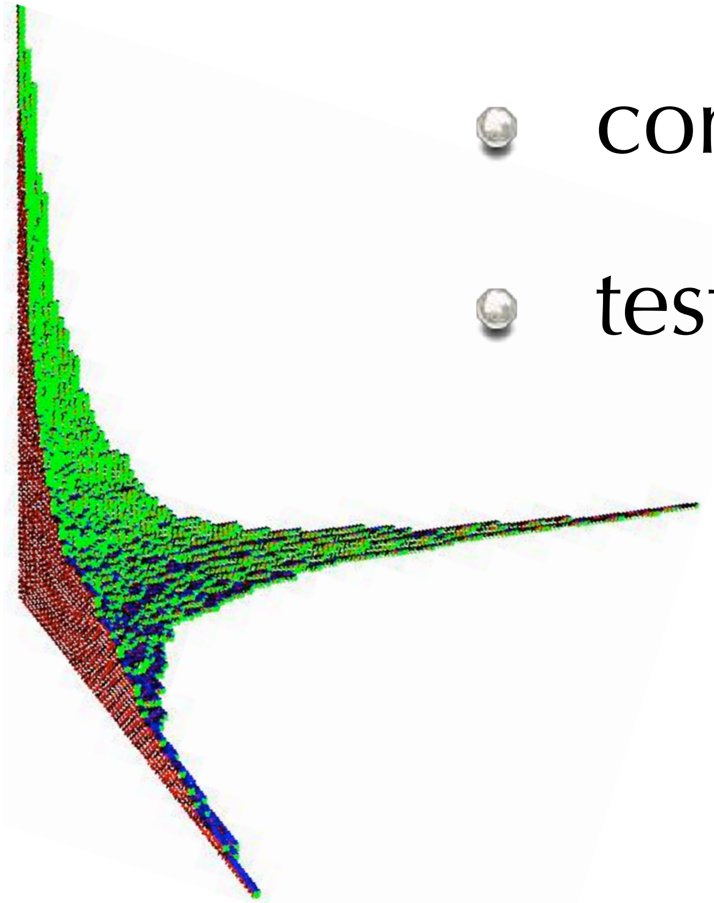
III. Random Generation

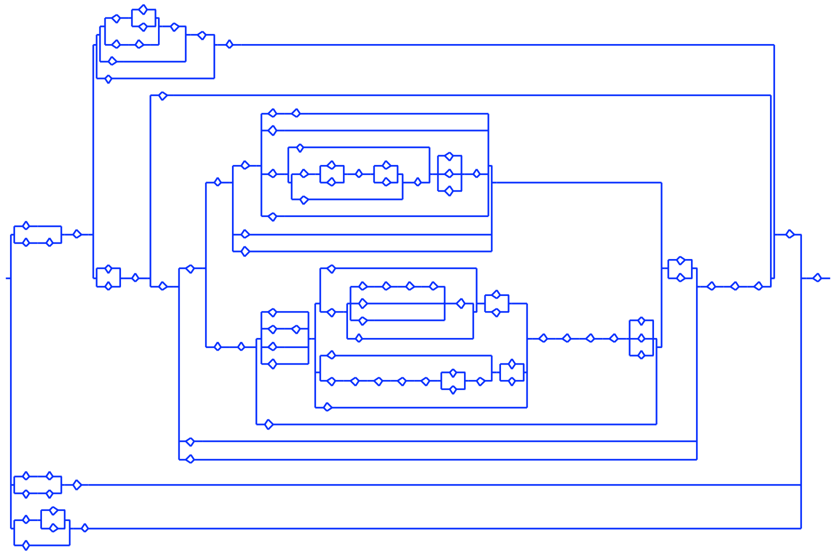
Why Random Generation?



Simulation in the discrete world; helps

- evaluate parameters;
- compare/refine models;
- test software.





Boltzmann Sampling

Proba(t)= $\dfrac{x^{|t|}}{T(x)}$ with $T(x) = \sum_{t \in \mathcal{T}} x^{|t|} = \sum_n T_n x^n$

x is a parameter of the algorithm

Boltzmann Sampling

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Expected size:

$$\frac{xT'(x)}{T(x)}$$

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Boltzmann Sampling

$$\text{Proba}(t) = \frac{x^{|t|}}{T(x)} \quad \text{with} \quad T(x) = \sum_{t \in \mathcal{T}} x^{|t|} = \sum_n T_n x^n$$

Expected size: $\dfrac{xT'(x)}{T(x)}$

x is a parameter of the algorithm

```
Generate(h∈ℋ) = {
   if ℋ=1 return 1;
   if ℋ=𝒵 return 𝒵;
   if ℋ=ℱxℊ {
       Generate(f∈ℱ);
       Generate(g∈𝒢);
       return (f,g); }
   if ℋ=ℱ+𝒢 {
       b:=Bernoulli(F(x)/H(x));
       if b=1 return Generate(f∈ℱ);
       else return Generate(g∈𝒢);
   if ℋ=Set(ℱ) {
       …
```

[Duchon, Flajolet, Louchard, Schaeffer 2002-04]

Boltzmann Sampling

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       Generate(f∈𝓕);
       Generate(g∈𝓖);
       return (f,g); }
   if 𝓗=𝓕+𝓖 {
       b:=Bernoulli(F(x)/H(x));
       if b=1 return Generate(f∈𝓕);
       else return Generate(g∈𝓖);
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```

F(x),*H(x)* by Newton iteration

[Duchon, Flajolet, Louchard, Schaeffer 2002-04]

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       ...
```

F(x),H(x) by Newton iteration

Example: binary trees

$$\mathcal{B} = 1 + \mathcal{Z} \times \mathcal{B} \times \mathcal{B}$$

with 1/B(.23)≃1/1.559≃.6414

Coin: 0,0,1,0,1,1,1,1,0,1,…

[Duchon, Flajolet, Louchard, Schaeffer 2002-04]

Boltzmann Sampling

Proba(t)= $\dfrac{x^{|t|}}{T(x)}$ with $T(x)=\sum_{t\in\mathcal{T}} x^{|t|}=\sum_n T_n x^n$

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      else return Generate(g∈𝒢);
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      …
```

F(x),H(x) by Newton iteration

Example: binary trees

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Coin: 0,0,1,0,1,1,1,1,0,1,…

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F(x),*H(x)* by Newton iteration

Example: binary trees

$\mathcal{B}=1+\mathcal{Z}\times\mathcal{B}\times\mathcal{B}$

with 1/B(.23)≃1/1.559≃.6414

Coin: 0,0,1,0,1,1,1,1,0,1,…

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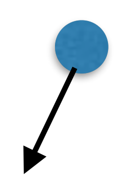

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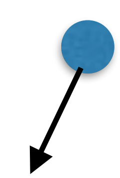

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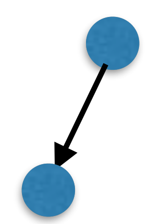

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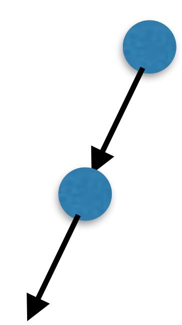

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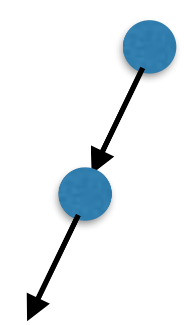

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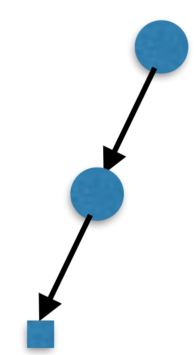

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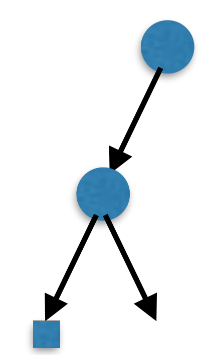

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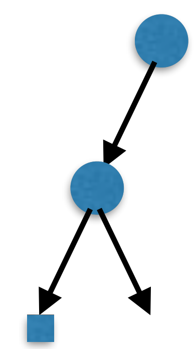

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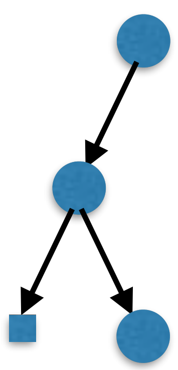

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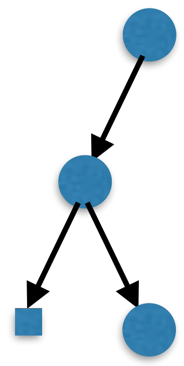

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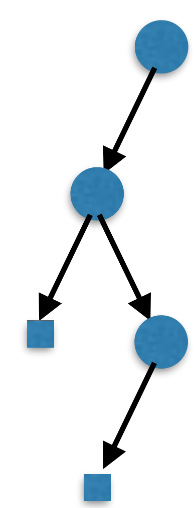

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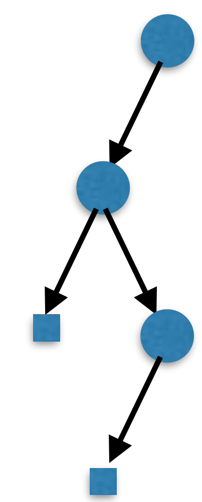

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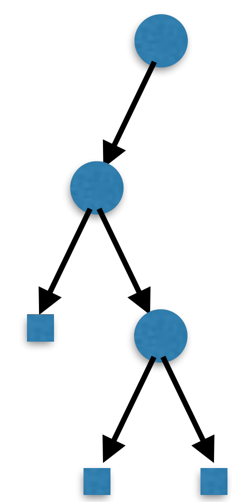

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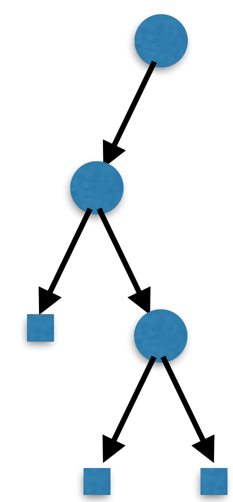

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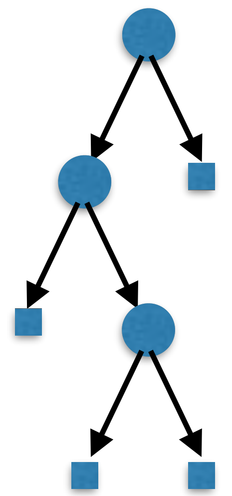

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   if 𝓗=Set(𝓕) {
      …
```

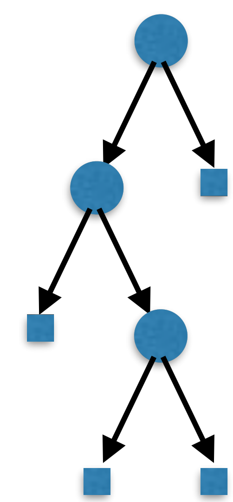
F(x),*H(x)* by Newton iteration

Example: binary trees

$\mathcal{B}=1+\mathcal{Z}\times\mathcal{B}\times\mathcal{B}$

with 1/B(.23)≃1/1.559≃.6414

Coin: 0,0,1,0,1,1,1,1,0,1,…



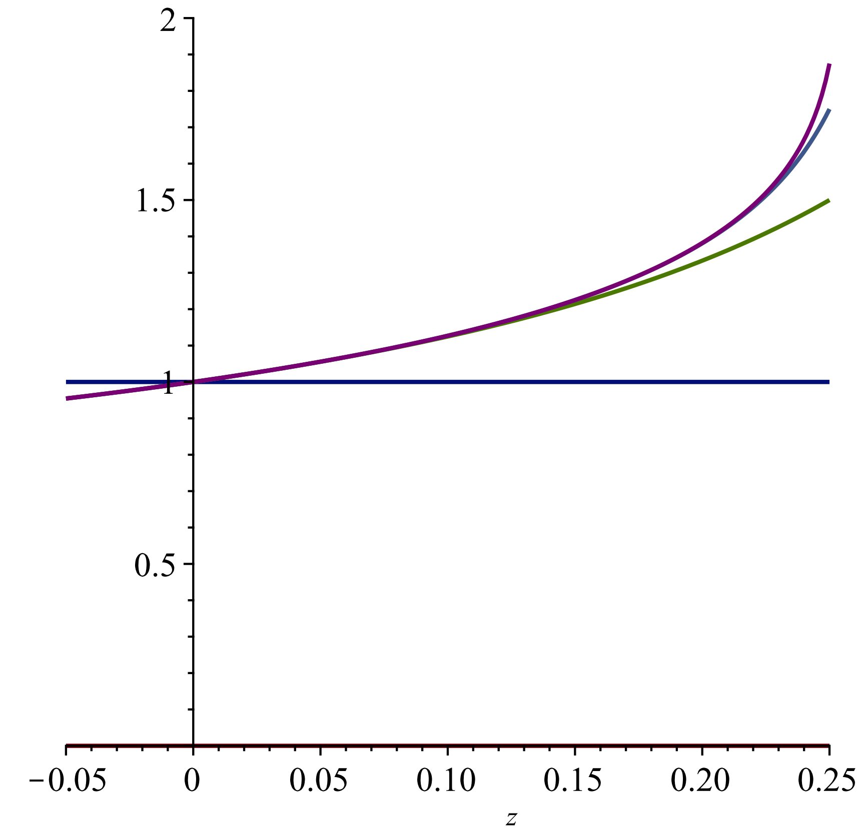
Exp. size

$\sim \dfrac{1}{2\sqrt{1-4x}}$

[Duchon, Flajolet, Louchard, Schaeffer 2002-04]

Example: XML-trees

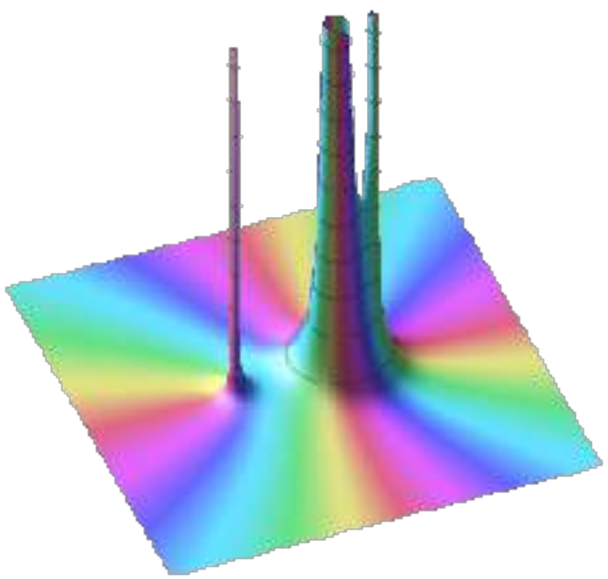
Grammar	File size	#eq.	Newton
rss	9.5k	16	0.02s.
Relax NG	124k	114	0.10s.
XSLT	168k	122	0.12s.
XHTML Basic	284k	96	0.32s.
SVG	6.3M	232	0.23s.
OpenDocument	2.8M	814	0.34s.
DocBook	11M	977	23s.



Time for *x* s.t. E(size)=10,000

[Darrasse2010]

IV. Asymptotic Analysis



$$F(z) = \sum_{n=0}^{\infty} f_n z^n \longrightarrow f_n \sim \cdots, \quad n \to \infty$$

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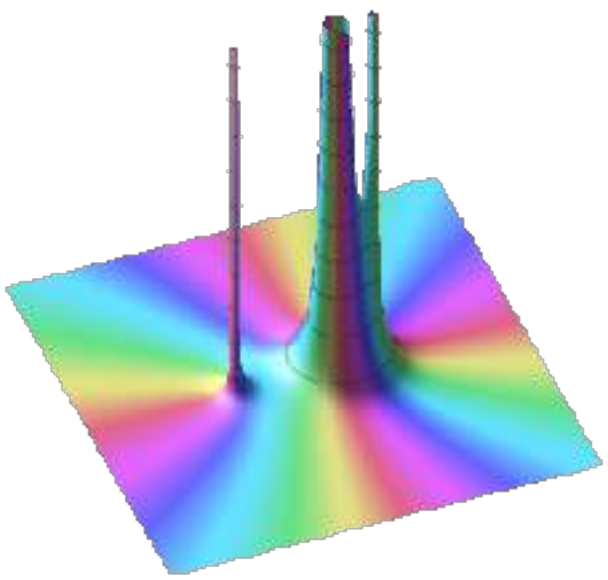

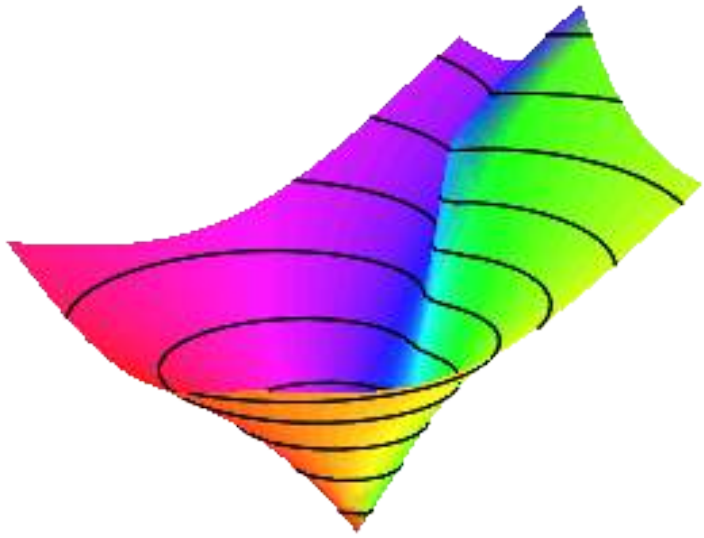

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Singularity Analysis

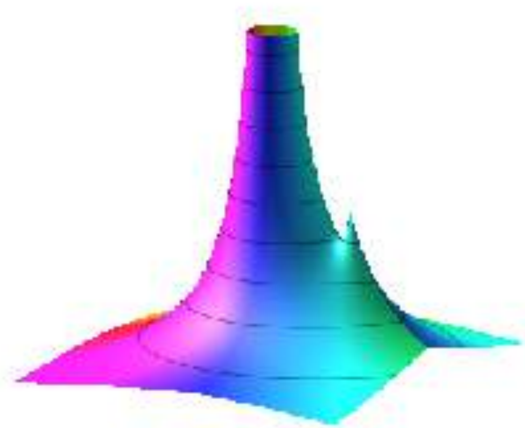
counts the number of objects of size *n*

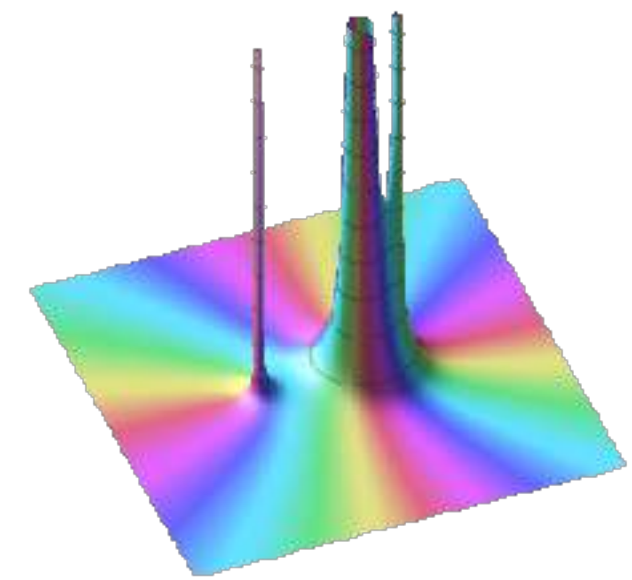
$$(a_n) \mapsto A(z) := \sum_{n \geq 0} a_n z^n$$

captures some structure

A 3-Step Method:

$$a_n = \frac{1}{2\pi i} \oint \frac{A(z)}{z^{n+1}} \, dz$$





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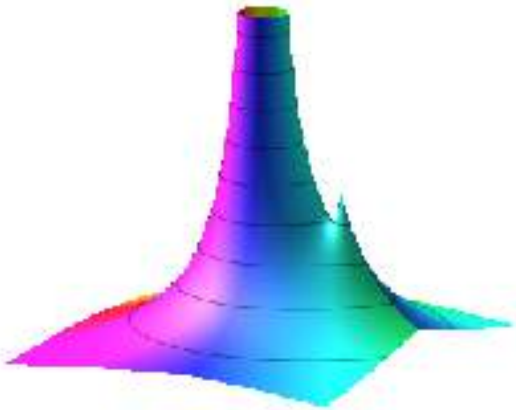

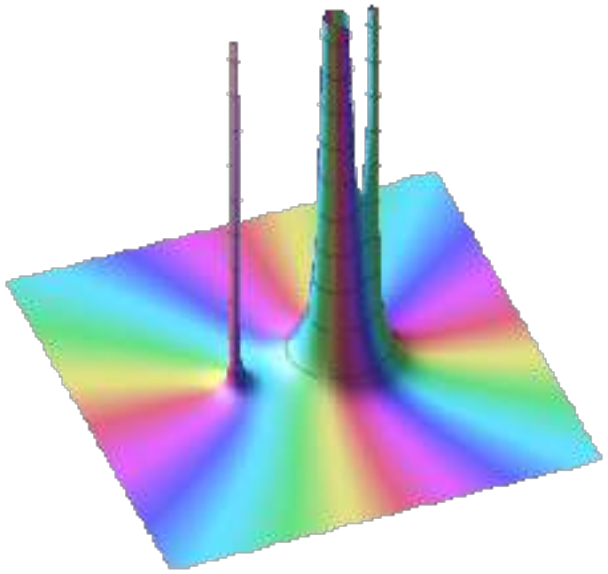

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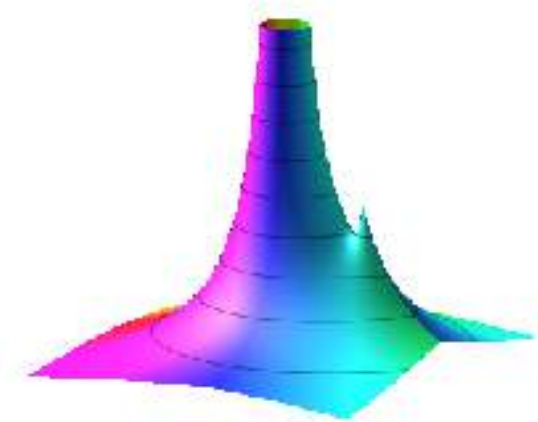

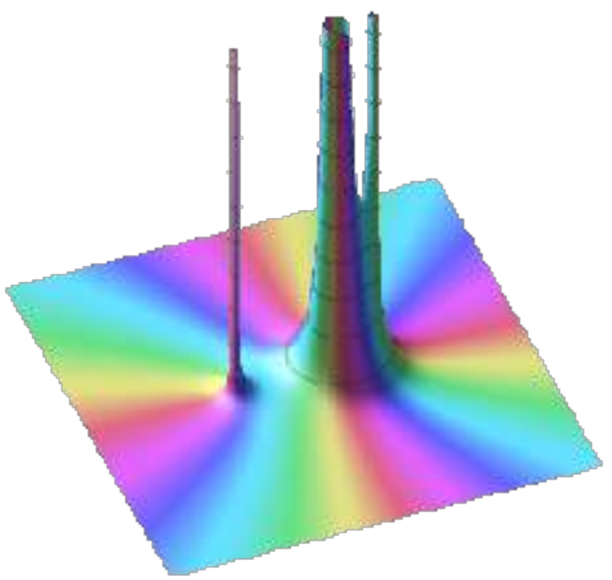

$$A(z) \underset{z\to\rho}{\sim} c\left(1-\frac{z}{\rho}\right)^{\alpha}\log^m\frac{1}{1-\frac{z}{\rho}}$$

$$a_n \underset{n\to\infty}{\sim} c\,\rho^{-n}\frac{n^{-\alpha-1}}{\Gamma(-\alpha)}\log^m n$$

$(\alpha \notin \mathbb{N})$

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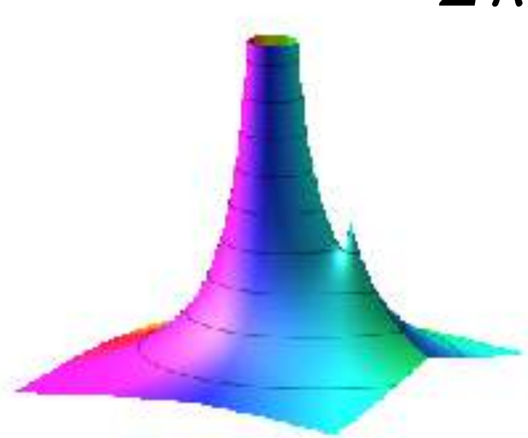

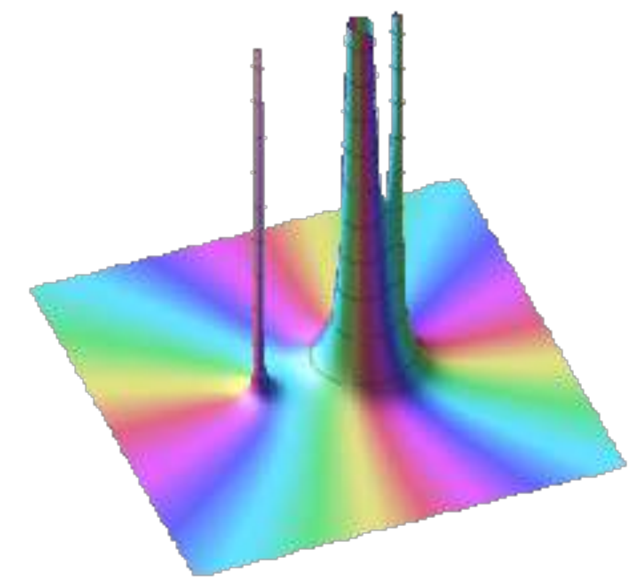

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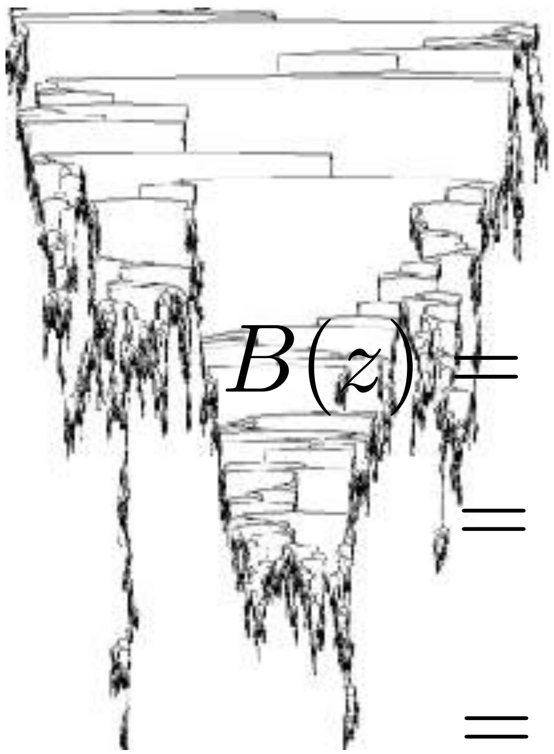
full asymptotic expansion available

Example: Binary Trees (Again!)



$$B(z) = 1 + zB(z)^2$$
$$= 1 + z + 2z^2 + 5z^3 + \cdots$$
$$= \frac{1-\sqrt{1-4z}}{2z}$$

A 3-Step Method:

1. Locate dominant singularities;
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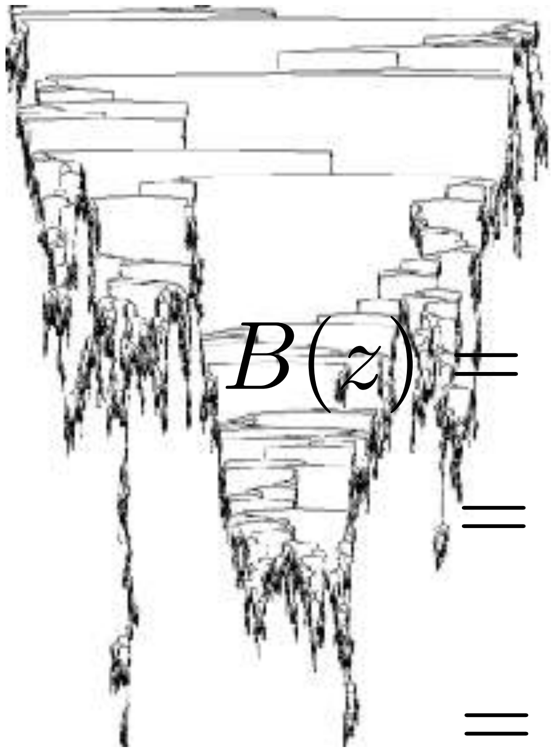

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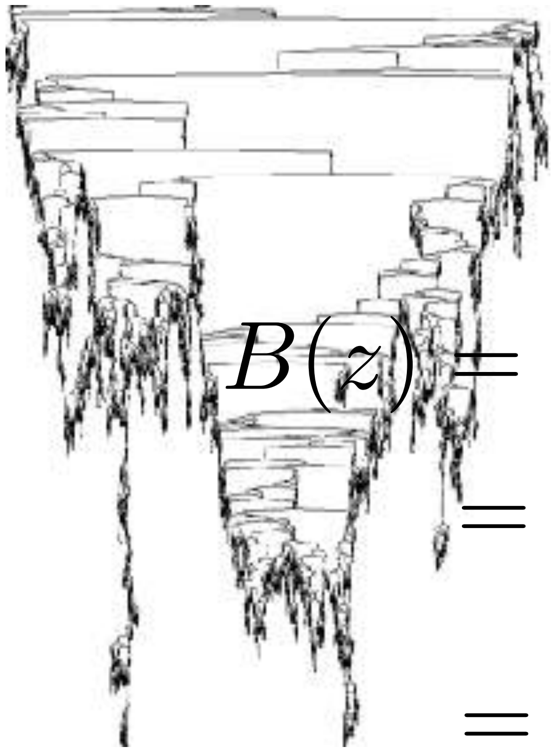

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- Equations over combinatorial structures + parameters

Ex.: path length in binary trees

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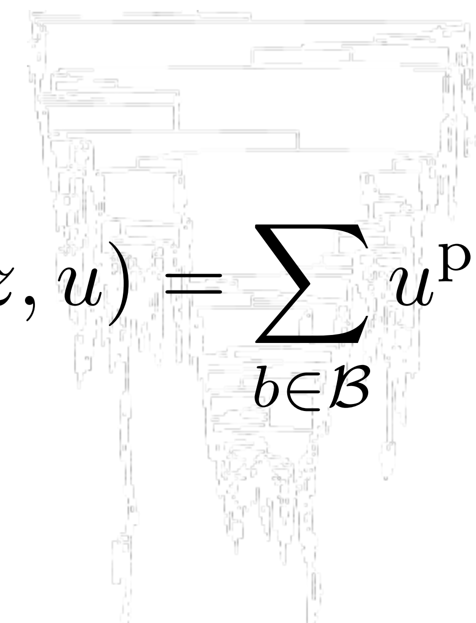

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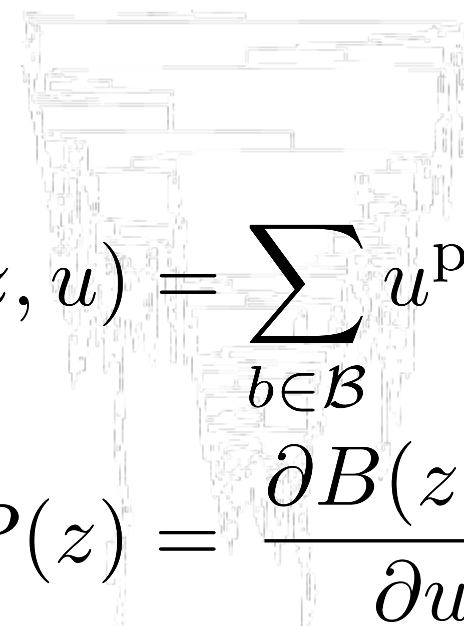

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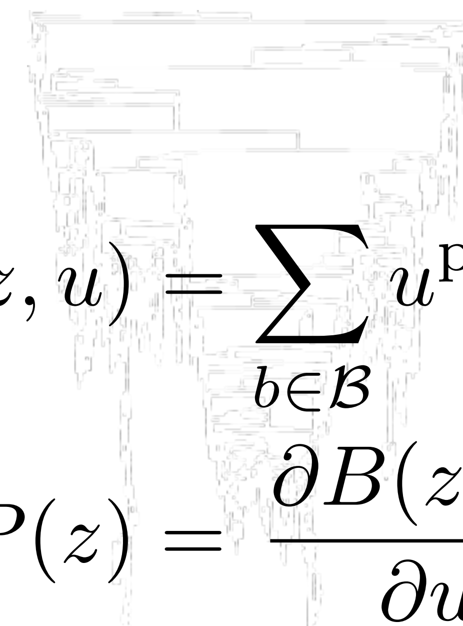

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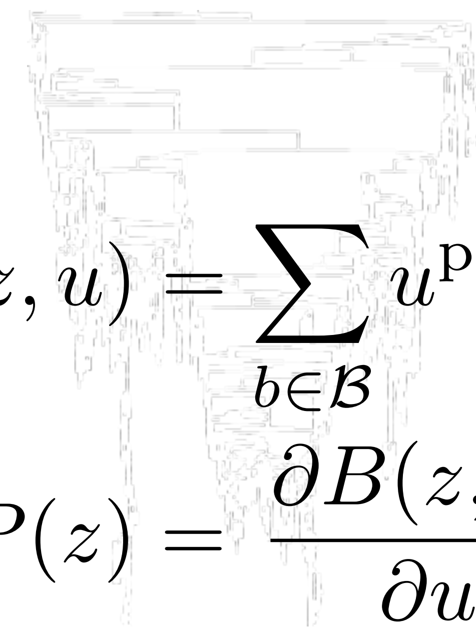

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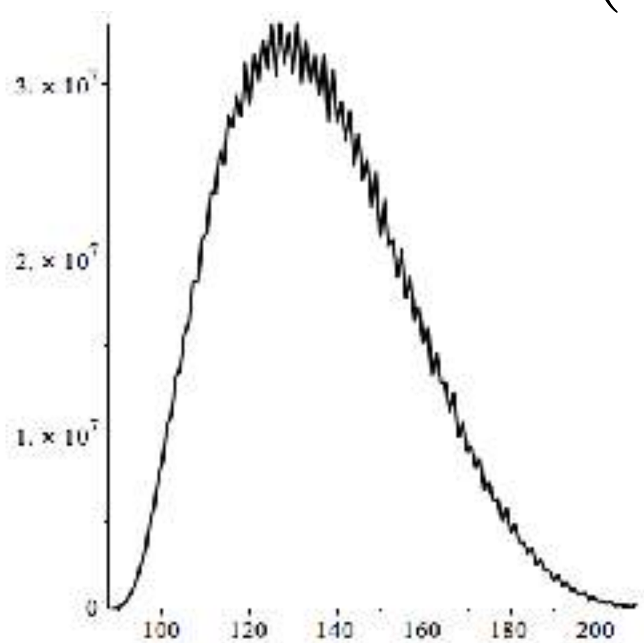

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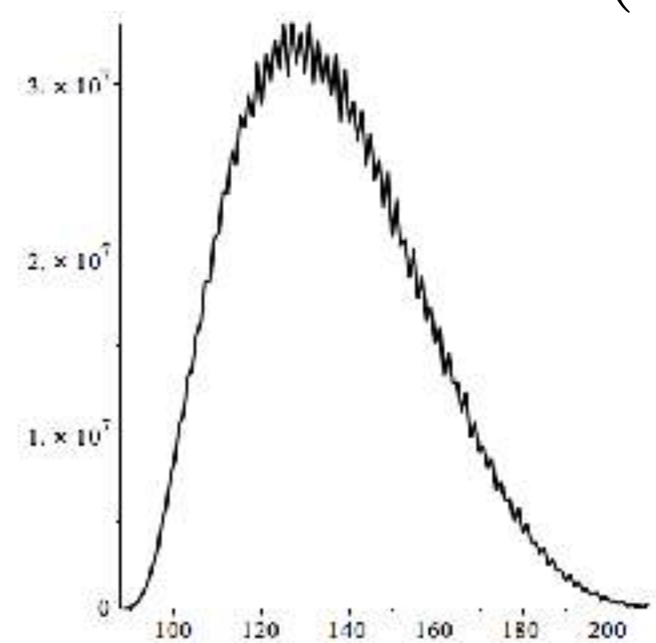

Demo.

If you can specify it, you can analyze it!

Permutations	Languages
Mappings	Integers
Words	Compositions
Strings	Partitions
Urns	...
Trees	

+ *parameters*

Universality phenomena:

Ex.: # trees of various types $\longrightarrow$ $K\rho^n n^{-3/2}$ $\longrightarrow$ pathlength in $n^{1/2}$

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Combinatorial specification →

	Automated?
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asymptotic analysis.	✗ (not fully)

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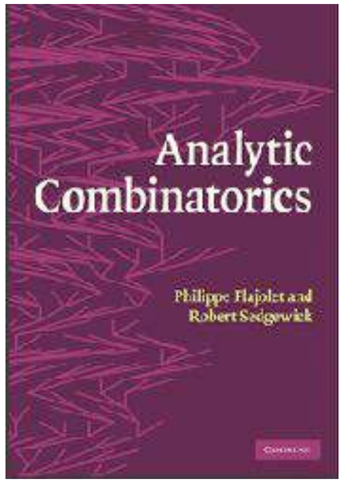

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