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# Stream Graphs, Link Streams and

Related Algorithmic Challenges

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Link Streams

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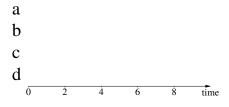
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#### interactions over time



• a, b, c, and d for 10 time units

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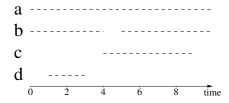
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#### interactions over time



- a, b, c, and d for 10 time units
- a always present, b leaves from 4 to 5, c present from 4 to 9, d from 1 to 3

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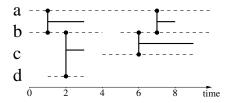
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#### interactions over time



- a, b, c, and d for 10 time units
- a always present, b leaves from 4 to 5, c present from 4 to 9, d from 1 to 3
- a and b interact from 1 to 3 and from 7 to 8; b and c from 6 to 9; b and d from 2 to 3.

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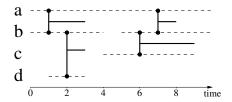
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e.g., social interactions, network traffic, money transfers, chemical reactions, etc.

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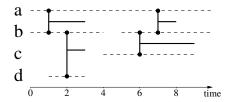
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e.g., social interactions, network traffic, money transfers, chemical reactions, etc.

how to describe such data?

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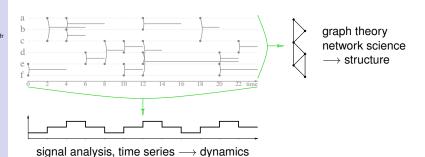
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# structure or dynamics



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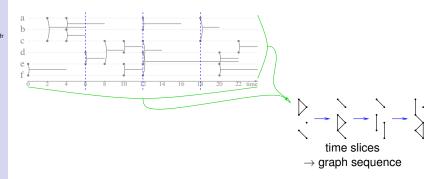
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# structure and dynamics?



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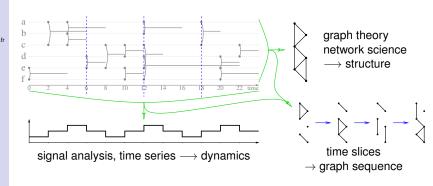
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#### structure and dynamics?



information loss what slices? graph sequences?

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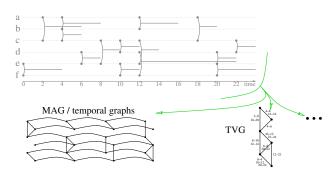
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### structure and dynamics



lossless but graph-oriented

+ ad-hoc properties (mostly path-related) + contact sequences + relational event models + ...

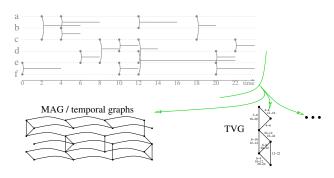
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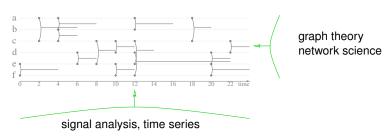
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#### Context

### what we propose

#### deal with the stream directly

#### stream graphs and link streams



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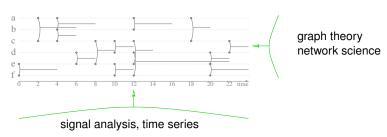
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#### what we propose

#### deal with the stream directly

#### stream graphs and link streams



wanted features: simple and intuitive, comprehensive, time-node consistent, generalizes graphs/signal

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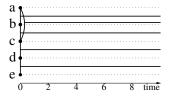
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# graph-equivalent streams

stream with no dynamics: nodes always present, either always or never linked









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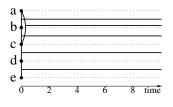
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# graph-equivalent streams

stream with no dynamics: nodes always present, either always or never linked







stream properties

graph properties

→ generalizes graph theory

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### our approach

very careful generalization of the most basic concepts
stream graphs and link streams
numbers of nodes and links
clusters and induced sub-streams
density and paths

→ buliding blocks for higher-level concepts

 neighborhood and degrees
 clustering coefficient
 betweenness centrality
 many others

+ ensure consistency with graph theory+ ensure classical relations are preserved

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**Basics** 

# definition of stream graphs

Graph G = (V, E) with  $E \subseteq V \otimes V$  $uv \in E \Leftrightarrow u$  and v are linked

$$(t,v) \in W \Leftrightarrow v \text{ is present at time } t$$

$$T_v = \{t, (t,v) \in W\}$$

$$(t, uv) \in E \Leftrightarrow u \text{ and } v \text{ are linked at time } t$$
  
 $T_{uv} = \{t, (t, uv) \in E\}$ 

$$(t, uv) \in E$$
 requires  $(t, u) \in W$  and  $(t, v) \in W$   
i.e.  $T_{uv} \subseteq T_u \cap T_v$ 

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#### definition of stream graphs

Graph G = (V, E) with  $E \subseteq V \otimes V$  $uv \in E \Leftrightarrow u$  and v are linked

Stream graph 
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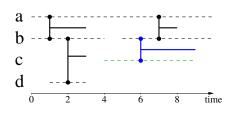
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**Basics** 

#### an example



$$T = [0, 10]$$
  $V = \{a, b, c, d\}$ 

$$W = T \times \{a\} \cup ([0,4] \cup [5,10]) \times \{b\} \cup [4,9] \times \{c\} \cup [1,3] \times \{d\}$$
 $T_a = T$   $T_b = [0,4] \cup [5,10]$   $T_c = [4,9]$   $T_d = [1,3]$ 

$$E = ([1,3] \cup [7,8]) \times \{ab\} \cup [6,9] \times \{bc\} \cup [2,3] \times \{bd\}$$
  
 $T_{ab} = [1,3] \cup [7,8]$   $T_{bc} = [6,9]$   $T_{bd} = [2,3]$   $T_{ad} = \emptyset$ 

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Basics

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Algorithm

#### a few remarks

works with... discrete time, continuous time, instantaneous interactions or with durations, directed, weighted, bipartite...

if  $\forall v, T_v = T$  then  $S \sim L = (T, V, E)$  is a **link stream** 

if  $\forall u, v, T_{uv} \in \{T, \emptyset\}$  then  $S \sim G = (V, E)$  is a graph-equivalent stream

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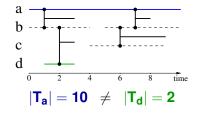
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# size of a stream graph

How many nodes? How many links?



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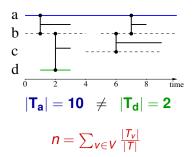
Degree

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### size of a stream graph

How many nodes? How many links?



$$n = \frac{|\mathbf{T_a}|}{10} + \frac{|T_b|}{10} + \frac{|T_c|}{10} + \frac{|\mathbf{T_d}|}{10} = 1 + 0.9 + 0.5 + 0.2 = 2.6$$
 nodes

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# size of a stream graph

How many nodes? How many links?

a
b
c
d
$$|\mathbf{T_a}| = \mathbf{10} \neq |\mathbf{T_d}| = \mathbf{2}$$

$$n = \sum_{v \in V} \frac{|T_v|}{|T|}$$

$$m = \sum_{uv \in V \otimes V} \frac{|T_{uv}|}{|T|}$$

$$n = \frac{|T_a|}{10} + \frac{|T_b|}{10} + \frac{|T_c|}{10} + \frac{|T_d|}{10} = 1 + 0.9 + 0.5 + 0.2 = 2.6 \text{ nodes}$$

$$m = \frac{|T_{ab}|}{10} + \frac{|T_{bc}|}{10} + \frac{|T_{bd}|}{10} = 0.3 + 0.3 + 0.1 = 0.7 \text{ links}$$

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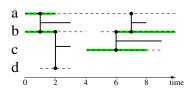
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#### clusters, sub-streams

Cluster in G = (V, E): a subset of V. Cluster in S = (T, V, W, E): a subset of  $W \subseteq T \times V$ .



$$C = [0,2] \times \{a\} \cup ([0,2] \cup [6,10]) \times \{b\} \cup [4,8] \times \{c\}$$

$$S(C)$$
 sub-stream induced by  $C$   
 $S(C) = (T, V, C, E_C)$ 

 $\hookrightarrow$  properties of (sub-streams induced by) clusters

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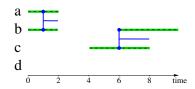
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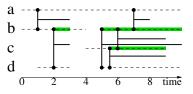
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# neighborhood

in 
$$G = (V, E)$$
:  $N(v) = \{u, uv \in E\}$   
in  $S = (T, V, W, E)$ :  $N(v) = \{(t, u), (t, uv) \in E\}$ 



$$N(d) = ([2,3] \cup [5,10]) \times \{b\} \cup [5.5,9] \times \{c\}$$

N(v) is a cluster

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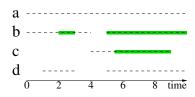
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Degrees

# degree

in G and in S:

$$d(v)$$
 is the size of  $N(v)$ 



$$N(d) = ([2,3] \cup [5,10]) \times \{b\} \cup [5.5,9] \times \{c\}$$
$$d(d) = \frac{|[2,3] \cup [5,10]|}{10} + \frac{|[5.5,9]|}{10} = 0.6 + 0.35 = 0.95$$

- degree distribution, average degree, etc
- if graph-equivalent stream then graph degree
- relation with n and m

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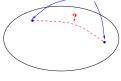
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#### density

#### in G:

proba two random nodes are linked

$$\delta(G) = \frac{\text{nb links}}{\text{nb possible links}}$$
$$= \frac{2 \cdot m}{n \cdot (n-1)}$$



random

#### in S:

proba two random nodes are linked at a random time instant

$$\delta(S) = \frac{\text{nb links}}{\text{nb possible links}}$$
$$= \frac{\sum_{uv \in V \otimes V} |T_{uv}|}{\sum_{uv \in V \otimes V} |T_{u} \cap T_{v}|}$$

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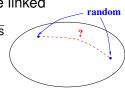
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random

#### in S:

proba two random nodes are linked at a random time instant

 $\delta(S) = \frac{\text{nb links}}{\text{nb possible links}} = \frac{\sum_{uv \in V \otimes V} |T_{uv}|}{\sum_{uv \in V \otimes V} |T_{u} \cap T_{v}|}$ 

- if graph-equivalent stream then graph density
- relation with n, m, and average degree

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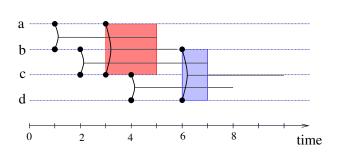
# cliques

in G: sub-graph of density 1 all nodes are linked together



#### in S: sub-stream of density 1

#### all nodes interact all the time



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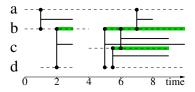
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Algorithn

### clustering coefficient

in G and in S: density of the neighborhood

$$cc(v) = \delta(N(v))$$



$$N(d) = ([2,3] \cup [5,10]) \times \{b\} \cup [5.5,9] \times \{c\}$$

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#### clustering coefficient

in G and in S: density of the neighborhood

$$cc(v) = \delta(N(v))$$

$$N(d) = ([2,3] \cup [5,10]) \times \{b\} \cup [5.5,9] \times \{c\}$$
$$cc(d) = \delta(N(d)) = \frac{|[6,9]|}{|[5.5,9]|} = \frac{6}{7}$$

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# paths

from a to d: (a, b), (b, c), (c, d) length: 3

 $\rightarrow \text{shortest paths}$ 

in S

from (1, d) to (9, c):

(2, d, b), (3, b, a), (7.5, a, b), (8, b, c)

length: 4

→ shortest paths

→ fastest paths

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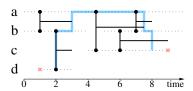
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# paths

from a to d: (a, b), (b, c), (c, d) length: 3

 $\rightarrow$  shortest paths

in S:



from (1, d) to (9, c):

$$(2, d, b), (3, b, a), (7.5, a, b), (8, b, c)$$

length: 4 duration: 6

 $\rightarrow \text{shortest paths}$ 

 $\rightarrow$  fastest paths

and Link Streams

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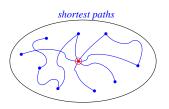
Paths

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# betweenness centrality

in G:

b(v) = fraction of  $shortest\ paths$ from any u to any w in Vthat involve v



in S:

b(t, v) = fraction of  $shortest \, \text{fastest } paths$ from any (i, u) to any (j, w) in Wthat involve (t, v)

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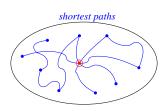
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### betweenness centrality

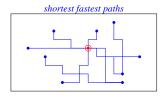
in G:

b(v) = fraction of  $shortest\ paths$ from any u to any w in Vthat involve v



in *S*:

b(t, v) = fraction of shortest fastest paths from any (i, u) to any (j, w) in Wthat involve (t, v)



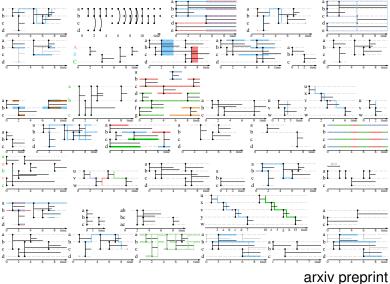
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### algorithmic concerns

extension of graph concepts...

...extension of graph algorithms?

some properties of S derive from properties of  $G_t$ neighborhood, degrees, k-cores, ...

some don't but algorithms may be adapted density, cliques (greedy, Bron-Kerbosch), ...

some still don't ⇒ new algorithms needed

(directed) paths, betweenness, patterns, ...

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Algorithms

# algorithmic concerns

extension of graph concepts...

...extension of graph algorithms?

some properties of S derive from properties of  $G_t$ neighborhood, degrees, k-cores, ...

some don't but algorithms may be adapted density, cliques (greedy, Bron-Kerbosch), ...

some still don't ⇒ new algorithms needed (directed) paths, betweenness, patterns, ...

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Algorithms

# algorithmic challenges

classical ones

streaming/on-line fully dynamic approximation space complexity

new ones

cliques, paths, betweenness unbounded number of links prediction?

good news

time-induced locality knowledge of dynamics better than induced graph?

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#### conclusion

#### we provide a language (set of concepts) that:

- makes it easy to deal with interaction traces,
- combines structure and dynamics in a consistent way,
- generalizes graphs / networks ; signals / time series ?
- meets classical and new algorithmic challenges,
- opens new perspectives for data analysis,
- clarifies the interplay interactions ←→ relations.

**studies in progress:** internet traffic, financial transactions, mobility/contacts, mailing-lists, sales, etc.

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# calls for papers

special issues of international journals

**Theoretical Computer Science (TCS)** 

Link Streams: models and algorithms

**Computer Networks** 

Link Streams: methods and case studies

deadline: July 1st http://link-streams.com