Fast Fencing

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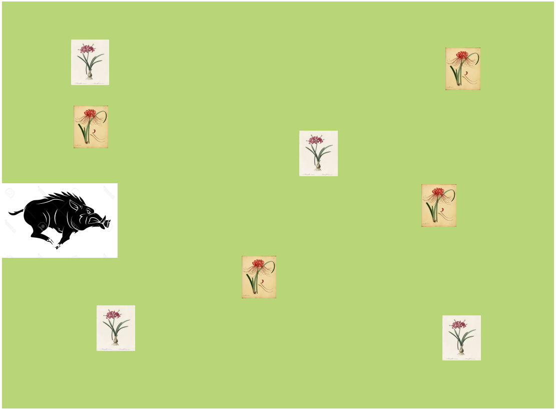
J. Gwyn Delin. — Hall Sculp.

Demi volte sur les coups forcés au dehors des armes. Plate 32.

Publish'd according to Act of Parliament, Feb. 1763.

How to protect these flowers from wild animals?



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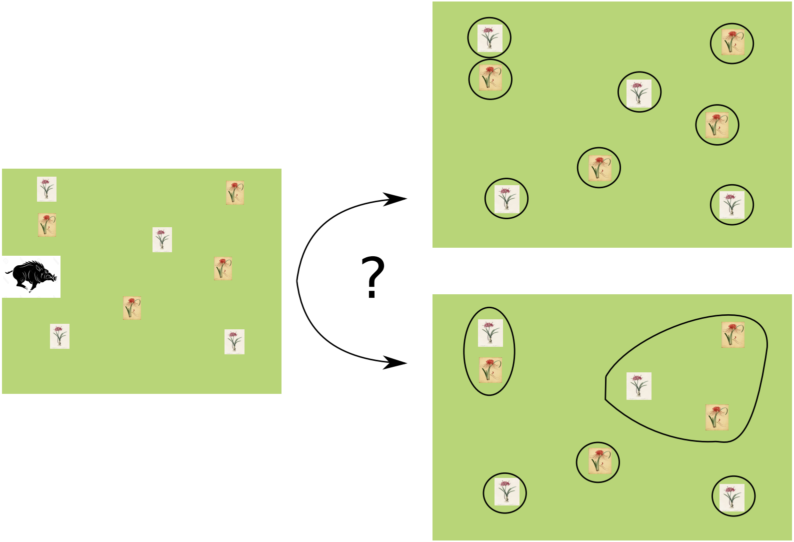

Which one of the two is the least expensive in fences?

More formally

Unit-disk fencing problem

Input: A set D of unit-disks in $\mathbb{R}^2$.
Output: A partition $\{D_1, \ldots, D_\ell\}$ of D that minimizes

$$\sum_{i=1}^{\ell} |\mathrm{ConvexHull}(D_i)|.$$

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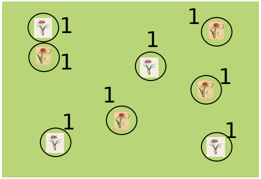
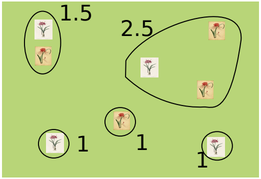


total cost = 8



total cost = 7

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Other variant:

k-cluster fencing problem

Input: A set P of points in $\mathbb{R}^2$.
Output: A partition $\{P_1, \ldots, P_k\}$ of P into k clusters that minimizes

$$\sum_{i=1}^{k} |\text{ConvexHull}(P_i)|.$$

A.k.a. min perimeter sum problem.

History

k-cluster fencing	$f(k)n^{O(k)}$	Capoyleas, Rote, Woeginger'91
k-cluster fencing	$f(k)n^{O(k)}$	Arkin, Khuller, Mitchell'93
2-cluster fencing	n^3	Mitchell and Wynters'91
2-cluster fencing	$n \log^4 n$	Abrahamsen, de Berg, Buchin, Mehr, Mehrabi'17
Unit-disk fencing	$\exp(n \log n)$	Arkin, Khuller, Mitchell'93: Run the algorithm for k-cluster, for each $1 \leq k \leq n$.

Conjecture by Arkin, Khuller, Mitchell'93:

k-cluster and unit-disk fencing are NP-Hard.

Our Results

Algorithms:

Unit-disk fencing	$n \operatorname{poly} \log n$
k-cluster fencing	n^{27}

Unit-disk algorithm extends to polygons.

Take-home message

Unit-disk fencing: Nice structure, simple polynomial time algorithms, more complicated near-linear time algorithm.

k-cluster fencing: Nice separator properties, not NP-Hard!

This talk: A polynomial time algorithm for unit-disk fencing

1. Simplify the problem
2. Structural property
3. Algorithm

Our problem for the rest of the talk

Input: A set P of points in $\mathbb{R}^2$. An opening cost c.
Output: A partition $\{P_1, \ldots, P_\ell\}$ of P that minimizes

$$\ell \cdot c + \sum_{i=1}^{\ell} \text{ConvexHull}(P_i).$$

Claim: Solving this problem helps us solve the unit-disk fencing problem.

Formally: For any instance of the unit-disk fencing problem D, define an instance of the new problem:

- c = perimeter of unit-disk
- $P = \{d \mid d$ center of a disk in $D\}$.

Any optimal solution for this problem is also an optimal solution for the unit-disk fencing problem.

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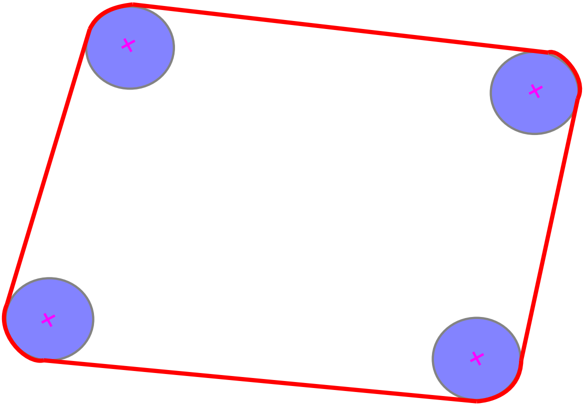
Solution for unit-disk: Cost is length of red lines.



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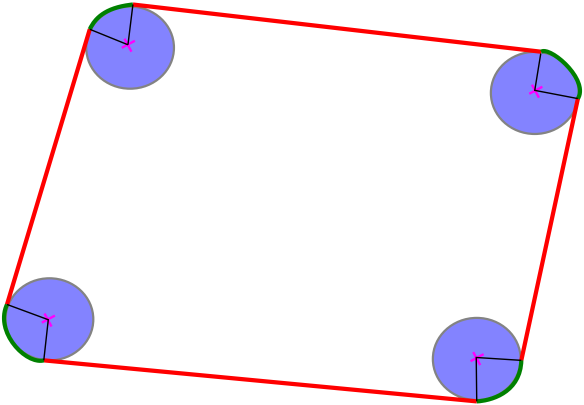

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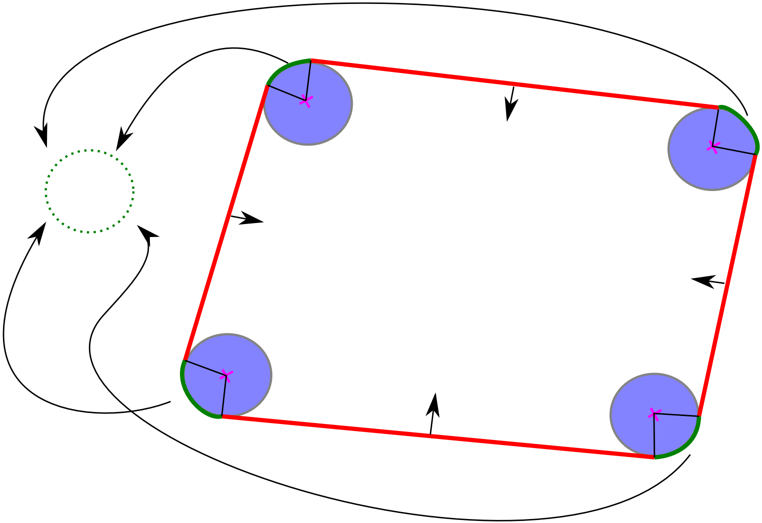

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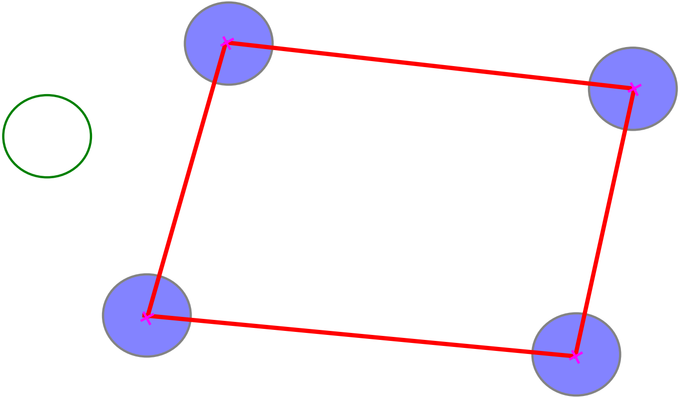
Solution for the new problem: cost is c + convexhull of centers of disks



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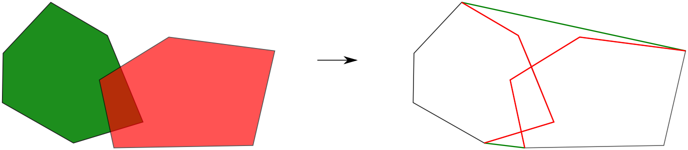
Obvious observation:

In an optimal solution, the convex hulls of the clusters do not intersect.



Key Definition:

A set of points P is indivisible if the optimal solution for P is $\{P\}$.

Observation:

The clusters of OPT are indivisible.

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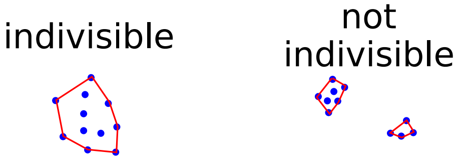


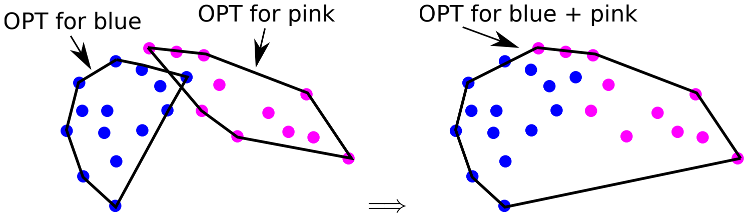
Observation:

The clusters of OPT are indivisible.

Structural Result

Lemma

Consider two indivisible sets of points A and B.
If their convex hulls intersect then $A \cup B$ is an indivisibile set of points.



Proof of Structural Result

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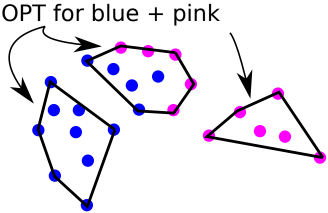

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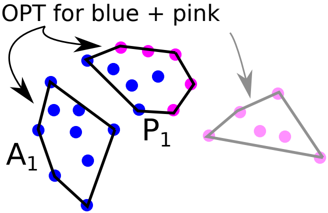

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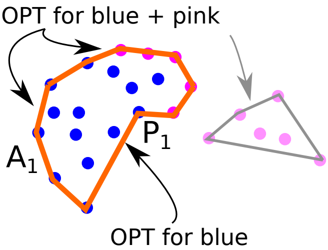

$A_1 \cup P_1$ is not indivisible.

$$\text{Conv.Hull}(A_1 \cup P_1) \leq$$
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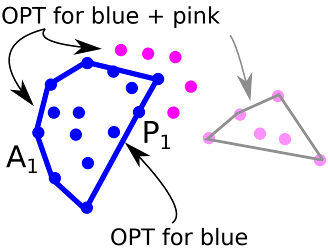

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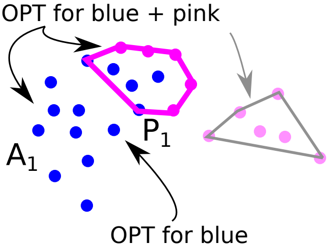

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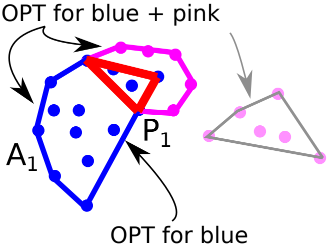

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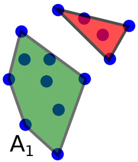

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By indivisibility of blue:
$\text{OPT}(\text{blue}) = \text{Conv.Hull}(\text{blue}) + c \leq$
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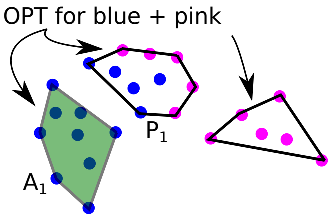

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Summing up the two equations
$\text{Conv.Hull}(A_1 \cup P_1) + c \leq \text{Conv.Hull}(A_1) + \text{Conv.Hull}(P_1) + 2c$
Thus $A_1 \cup P_1$ is indivisible, a contradiction.

Proof of Structural Result

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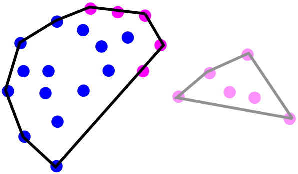
Solution better than
OPT for blue + pink



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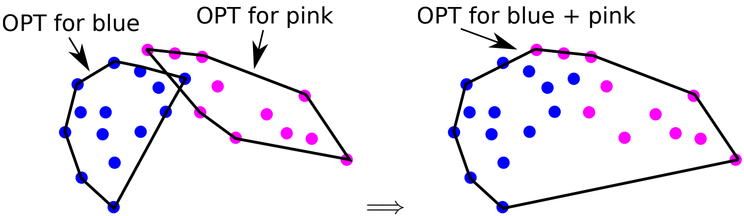
This suggests the following algorithmic approach



Algorithm

Algorithm: Step 1

Recursively divide $\mathbb{R}^2$ into cells.

Level 0 cells.

Goal: Solve for cells of level i using solution for cells of level $i + 1$.

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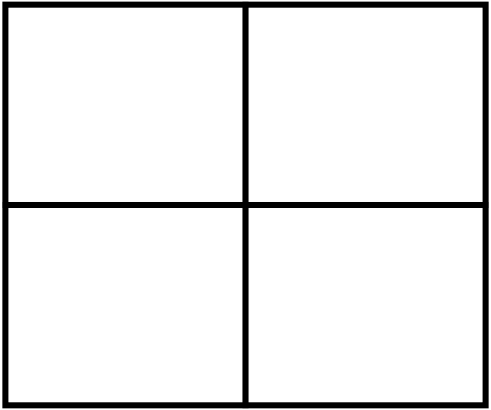
Level 1 cells.



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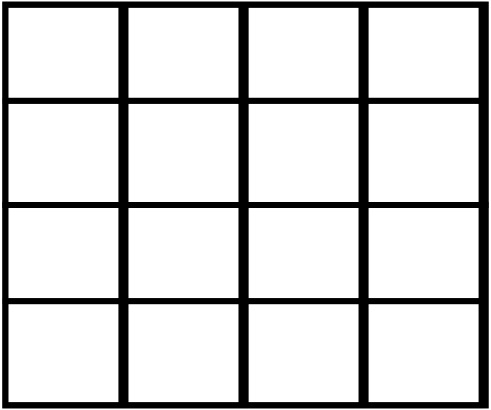
Level 2 cells.



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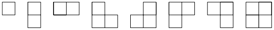
Define Shapes with Level-i Cells

Basic Polyominos: 

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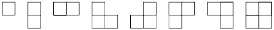
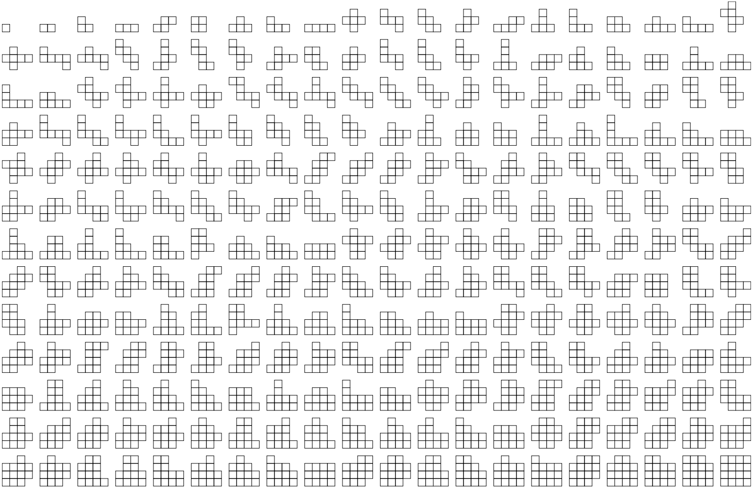
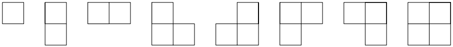
Basic Polyominos: 

Evolved Polyominos:



Property

Optimal solution for basic polyominoes made of level-i cells can be computed from optimal solutions for evolved polyominoes made of level-$i+1$ cells.



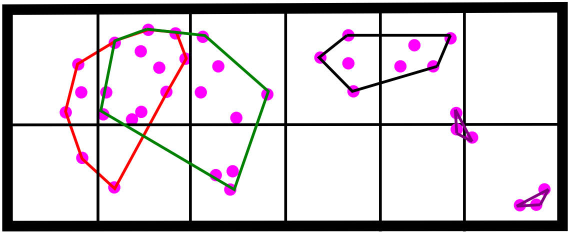
Property

Optimal solution for evolved polyominoes made of level-i cells computed:

- from optimal solution for basic polyominoes made of level-i cells, and
- Sweeping step to find out if there is a cluster intersecting all cells.

Alg:

1. Greedily merge optimal clusters of the basic polyominos that overlap.
2. Guess a point that belongs to the large cluster, compute the best cluster that intersects all cells, if it decreases the cost, keep it.



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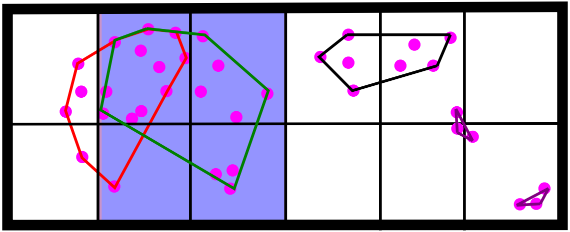

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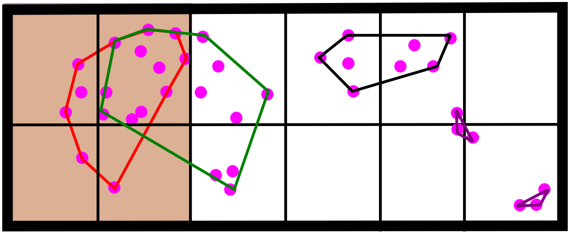

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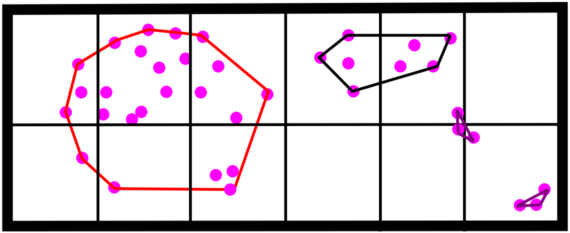

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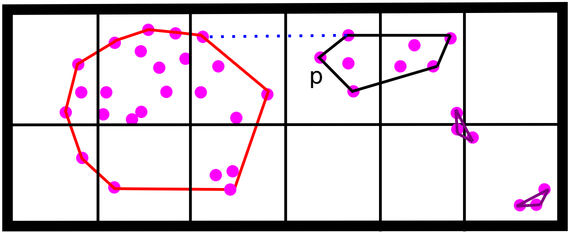

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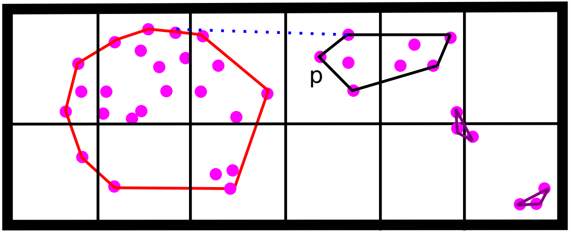

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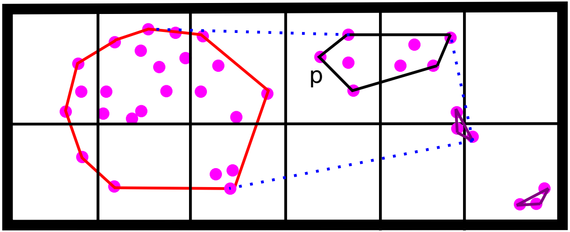

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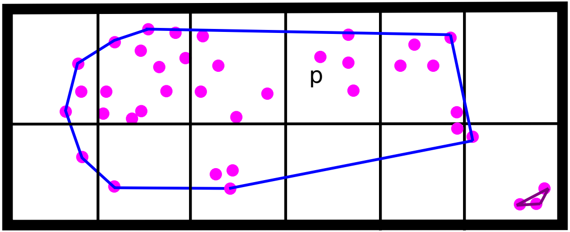

Overall Running Time

- $O(n^2)$ time per level
- $O(\log n)$ levels in total
- Overall running time : $O(n^2 \log n)$.

Faster

$O(n \text{ poly} \log n)$, using a more sophisticated algorithm for the last step.

k-Clustering Fencing Problem

Other structural result:

Separators with small complexity: Cells only intersect a few clusters of OPT.

Dynamic programming on cells.

Our Results

Algorithms:

Unit-disk fencing	$n \operatorname{poly} \log n$
k-cluster fencing	n^{27}

Unit-disk alg. extends to polygons w/ slightly worse complexity.

Take-home message

Unit-disk fencing: Nice structure, simple polynomial time algorithms, more complicated near-linear time algorithm.

k-cluster fencing: Nice separator properties, not NP-Hard!