1. Representation theory of finite groups

Notation: Let G be a finite group, $\mathfrak{C}$ the set of conjugacy classes of G, and $\{\pi_i\}_{i\in I}$ a full set of non-isomorphic irreducible representations of G. For $i\in I$ and $g\in C\in\mathfrak{C}$ we write $\chi_i(g)$ or $\chi_i(C)$ for the trace of g on π_i and set $n_i=\chi_i(1)=\dim\pi_i$.

1 (Schur's lemma). For any $i,\,j\in I$, $\mathrm{Hom}_G(\pi_j,\pi_i)$ is $\mathbb{C}\cdot\mathrm{Id}_{\pi_i}$ if $i=j$ and $\{0\}$ if $i\neq j$. This is obvious since any non-zero G-map $\pi_j\to\pi_i$ is an isomorphism and any linear map $\pi_i\to\pi_i$ has an eigenvalue.

2 (First orthogonality relation). Applying the general identity $|G|^{-1}\sum_{g\in G}\mathrm{Tr}(g,V)=\dim(V^G)$ to $V=\pi_i\otimes\pi_j^*$ $(i,j\in I)$ and using **1.** gives $\sum_{C\in\mathfrak{C}}|C|\,\chi_i(C)\overline{\chi}_j(C)=|G|\,\delta_{ij}$.

3 (Complete reducibility). Any finite-dimensional representation V of G is a direct sum of irreducible representaions. This follows by induction on the dimension, since if π is any subrepresentation of V then V splits as the direct sum of π and the orthogonal complement to π with respect to a non-degenerate G-invariant scalar product (which we can obtain by starting with any positive-definite Hermitian form on V and summing its translates under G).

4. For V as in **3.** we have canonically $V\cong\bigoplus_{i\in I}\mathrm{Hom}_{\mathbb{C}}(\mathrm{Hom}_G(V,\pi_i),\pi_i)$ (as G-modules), the map $V\to\mathrm{Hom}_{\mathbb{C}}(\mathrm{Hom}_G(V,\pi_i),\pi_i)$ being given by $v\mapsto(\phi\mapsto\phi(v))$. Indeed, this holds for $V=\pi_j$ by **1.** and in general by **3.**

5. For any representation V of G, $\mathrm{Hom}_G(\mathbb{C}[G],V)\cong V$ as G-representations, since $\phi\in\mathrm{Hom}_G(\mathbb{C}[G],V)$ is uniquely determined by $\phi(1)\in V$, which is arbitrary.

6. Applying **4.** to $\mathbb{C}[G]$ and using **5.** gives a canonical G-module isomorphism

$$\mathbb{C}[G]\;\cong\;\bigoplus_{i\in I}\mathrm{Hom}_{\mathbb{C}}(\pi_i,\pi_i)\;=\;\bigoplus_{i\in I}\mathrm{End}_{\mathbb{C}}(\pi_i)\tag{1}$$

which sends $[g]$ to $(\pi_i(g))_{i\in I}$. This equation is the central statement of the theory.

7. Comparing dimensions in (1), we find that $|G|=\sum_{i\in I}n_i^2$.

8. Since (1) is also an algebra homomorphism, $\mathbb{C}[G]\cong\prod_i M_{n_i}(\mathbb{C})$ as an algebra. Comparing the dimensions of the centers, we find that $|I|=\dim Z(\mathbb{C}[G])=|\mathfrak{C}|$, since clearly a basis for $Z(\mathbb{C}[G])$ is given by the elements $e_C=\sum_{g\in C}[g]\quad(C\in\mathfrak{C})$.

9 (Second orthogonality relation). Since a left inverse of a square matrix is also a right one, **2.** and **8.** imply $\sum_{i\in I}\chi_i(C_1)\overline{\chi}_i(C_2)=|G||C_1|^{-1}\delta_{C_1,C_2}\quad(C_1,\,C_2\in\mathfrak{C})$.

10. The isomorphism (1) is right G-equivariant, so $\mathbb{C}[G]=\sum_i\pi_i^*\otimes\pi_i$ as a $G\times G$-representation. Computing the trace of $(g_1,g_2)\in C_1\times C_2$ on both sides of (1) gives another proof of **9.** (and hence also of **2.**), since (g_1,g_2) acts on $\mathbb{C}[G]$ by $[g]\mapsto[g_1gg_2^{-1}]$.

11. Comparing traces on each π_i, we find that the image of e_C under the isomorphism $Z(\mathbb{C}[G])\cong\mathbb{C}^I$ of **8.** is $\{n_i^{-1}|C|\,\chi_i(C)\}_{i\in I}$. On the other hand, if A and B are two conjugacy classes then clearly $e_Ae_B=\sum_{C\in\mathfrak{C}}|C|^{-1}N(A,B,C^{-1})\,e_C$, where $N(A,B,C)$ denotes the number of triples $(a,b,c)\in A\times B\times C$ with $abc=1$. Multiplying this out and using **9.** we find **Frobenius's formula**

$$\frac{N(A,B,C)}{|A\times B\times C|}\;=\;\frac{1}{|G|}\sum_{i\in I}\frac{\chi_i(A)\chi_i(B)\chi_i(C)}{\chi_i(1)}\qquad(A,\,B,\,C\in\mathfrak{C})\,.\tag{2}$$

2. Explicit construction of the irreducible representations of $\mathfrak{S}_n$

A *Young diagram* is a finite union of sets of the form $\{0,1,\dots,a\}\times\{0,-1,\dots,-b\}\subset\mathbb{Z}^2$. We systematically identify the set $\mathcal{Y}_n$ of Young diagrams of cardinality n with the set $\mathcal{P}_n$ of partitions $\lambda=(\lambda_1\ge\lambda_2\ge\dots)$ of n by $\lambda\mapsto Y_\lambda$ = Young diagram with row-lengths λ_i. We will construct pairwise distinct isomorphism classes of representations V_λ of $\mathfrak{S}_n$ indexed by $\lambda\in\mathcal{P}_n$; since $|\mathcal{P}_n|$ is equal to the number of conjugacy classes of $\mathfrak{S}_n$, this solves the problem. (Actually, the space V_λ will be a specific representation of the group $\mathfrak{S}_{Y_\lambda}$ of permutations of the elements of Y_λ. Since Y_λ has cardinality n, this group is isomorphic to $\mathfrak{S}_n$, but the isomorphism, and hence the representation of the *fixed* group $\mathfrak{S}_n$ on V_{Y_λ}, is unique only up to conjugacy.) The idea of the construction we describe goes back to van der Waerden and von Neumann. Our presentation is a slight simplification of the one in the very nice book *Invariant Theory, Old and New* by J. Dieudonné and J. Carrell.

Denote by $\mathcal{A}_\lambda$ (resp. $\mathcal{B}_\lambda$) the subgroup of $\mathfrak{S}_{Y_\lambda}$ leaving invariant the rows (resp. columns) of Y_λ. Clearly $\mathcal{A}_\lambda\cap\mathcal{B}_\lambda=\{e\}$. Define three elements A_λ, B_λ, X_λ of the group algebra $\mathcal{R}_\lambda=\mathbb{C}[\mathfrak{S}_{Y_\lambda}]$ by

$$A_\lambda \;=\; \sum_{a\in\mathcal{A}_\lambda}[a]\,,\qquad B_\lambda \;=\; \sum_{b\in\mathcal{B}_\lambda}\varepsilon(b)[b]\,,\qquad X_\lambda \;=\; A_\lambda B_\lambda \;=\; \sum_{(a,b)\in\mathcal{A}_\lambda\times\mathcal{B}_\lambda}\varepsilon(b)\,[ab] \tag{1}$$

($\varepsilon(b)$ = sign of the permutation b), and set $V_Y=\mathcal{R}_\lambda X_\lambda\subseteq\mathcal{R}_\lambda$, a representation of $\mathfrak{S}_{Y_\lambda}$.

Theorem. *The representations V_λ ($\lambda\in\mathcal{P}_n$) are irreducible and pairwise non-isomorphic.*

The key to the proof is the following lemma, in which the elements of $\mathcal{P}_n$ have been ordered lexicographically (i.e. $\lambda>\mu$ if $\lambda_1=\mu_1,\ \dots\ ,\lambda_{i-1}=\mu_{i-1}$ and $\lambda_i>\mu_i$ for some i).

Lemma (J. von Neumann). *Let λ, $\mu\in\mathcal{P}_n$ with $\lambda\ge\mu$, and let ϕ be any bijection from Y_λ to Y_μ. Then either* (i) *$\mathcal{A}_\lambda\cap\phi^{-1}\mathcal{B}_\mu\phi$ contains a transposition, or else* (ii) *$\lambda=\mu$ and $\phi^{-1}\in\mathcal{A}_\lambda\mathcal{B}_\lambda$.*

Proof. Alternative (i) says that there are two distinct elements (the ones interchanged by the transposition) belonging to the same row of Y_λ with images belonging to the same column of Y_μ. Assume this is not the case. Then in particular the images under ϕ of the elements of the first row of Y_λ belong to different columns of Y_μ. Since Y_μ has μ_1 columns and $\lambda_1\ge\mu_1$, this implies that $\lambda_1=\mu_1$ and that we can compose ϕ with an element $b_1\in\mathcal{B}_\mu$ (bringing these images up to the first row of Y_μ) and then $a_1\in\mathcal{A}_\mu$ (permuting the elements of the first row of Y_μ) so that the composite $a_1b_1\phi: Y_\lambda\to Y_\mu$ is the identity on the first row. Now the same argument applied to the remaining part of the diagrams shows that $\lambda_2=\mu_2$ and that there exist $a_2\in\mathcal{A}_\mu$ and $b_2\in\mathcal{B}_\mu$ such that $a_2b_2\phi$ is the identity on the first two rows of Y_λ. Continuing in the same way we finally obtain (ii). $\square$

Corollary. *The elements A_λ, B_λ, X_λ defined in* (1) *satisfy $A_\lambda\,\mathcal{R}_\lambda\,B_\lambda\;=\;\mathbb{C}\cdot X_\lambda\;\subseteq\;\mathcal{R}_\lambda$.*

Proof. If $x=\sum x_\sigma[\sigma]\in A_\lambda\mathcal{R}_\lambda B_\lambda$, then $axb=\varepsilon(b)x$ for all $a\in\mathcal{A}_\lambda$, $b\in\mathcal{B}_\lambda$, so $x_{a\sigma b}=\varepsilon(b)x_\sigma$ for all σ. Thus $x_\sigma=\varepsilon(b)x_e$ for $\sigma=ab\in\mathcal{A}_\lambda\mathcal{B}_\lambda$. But $x_\sigma=0$ for $\sigma\notin\mathcal{A}_\lambda\mathcal{B}_\lambda$, because the lemma (with $\lambda=\mu$, $\phi=\sigma^{-1}$) gives us transpositions $a\in\mathcal{A}_\lambda$ and $b\in\mathcal{B}_\lambda$ with $a\sigma b=\sigma$, so that $x_\sigma=-x_\sigma$. $\square$

Proof of the theorem. If $V\subseteq V_\lambda$ is an irreducible subrepresentation, then $X_\lambda V\subseteq X_\lambda\mathcal{R}_\lambda X_\lambda\subseteq\mathbb{C}X_\lambda$. Also $X_\lambda V\ne\{0\}$ since $\mathcal{R}_\lambda X_\lambda V=V_\lambda V\supseteq V^2=V$. Hence $\mathbb{C}X_\lambda=X_\lambda V\subseteq V$, so $V_\lambda\subseteq V$.

Now suppose that $\lambda>\mu$ and that there is a bijection $\psi: Y_\mu\to Y_\lambda$ such that V_λ and $V'_\mu=\psi V_\mu\psi^{-1}$ are isomorphic subrepresentations of $\mathcal{R}_\lambda$. The lemma applied to $\phi=\sigma\psi^{-1}\tau$ with $\sigma\in\mathfrak{S}_{Y_\lambda}$, $\tau\in\mathfrak{S}_{Y_\mu}$ gives transpositions $s\in\mathcal{A}_\lambda$ and $s'\in\mathcal{B}_\mu$ with $s'=\phi s\phi^{-1}$. Then $A_\lambda s=A_\lambda$ and $s'B_\mu=-B_\mu$, so $A_\lambda\phi^{-1}B_\mu=0$. Hence $A_\lambda\mathcal{R}_\lambda\psi\mathcal{R}_\mu B_\mu=0$, so $V_\lambda V'_\mu=0$ and Schur's lemma implies $V_\lambda\not\cong V'_\mu$. $\square$

Remark. Note that V_λ has a natural integral structure: $V_\lambda=L_\lambda\otimes_{\mathbb{Z}}\mathbb{C}$, where $L_\lambda=\mathbb{Z}[\mathfrak{S}_{Y_\lambda}]\,X_\lambda$. This gives another proof of the fact—otherwise proved by noting that any two elements of $\mathfrak{S}_n$ generating the same subgroup are conjugate—that the irreducible characters of $\mathfrak{S}_n$ are $\mathbb{Z}$-valued.