

Cours au Collège de France - Février 2016

Towards the ultimate precision limits in parameter estimation: An introduction to quantum metrology

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Troisième Leçon: Métrologie quantique et
décohérence

But de cette leçon

Dans cette leçon, on introduit l'extension pour les systèmes ouverts de la théorie de Cramér-Rao-Fisher. Le calcul de limites de précision est faite par moyen d'une theory qui offre un cadre générale pour l'estimation de paramètres de systèmes ouverts. Cette theory est appliquée au problème d'estimation de forces faibles, agissant sur an oscillateur harmonique amorti, et aussi à l'estimation de phase avec un interféromètre optique qui subit des pertes de photons ou la diffusion de la phase.

Rappel sur l'Information de Fisher Quantique

In the first lecture, we defined, for a given measurement corresponding to the POVM $\{\hat{E}(\xi)\}$, the Fisher information,

$$F[X; \{\hat{E}(\xi)\}] = \int d\xi p(\xi|X) \left[\frac{\partial \ln p(\xi|X)}{\partial X} \right]^2 = \int d\xi \frac{1}{p(\xi|X)} \left[\frac{\partial p(\xi|X)}{\partial X} \right]^2$$

and we have also defined the "Quantum Fisher information," which is obtained by maximizing the above expression with respect to all quantum measurements:

$$\mathcal{F}_Q(X) = \max_{\{\hat{E}(\xi)\}} F[X; \{\hat{E}(\xi)\}]$$

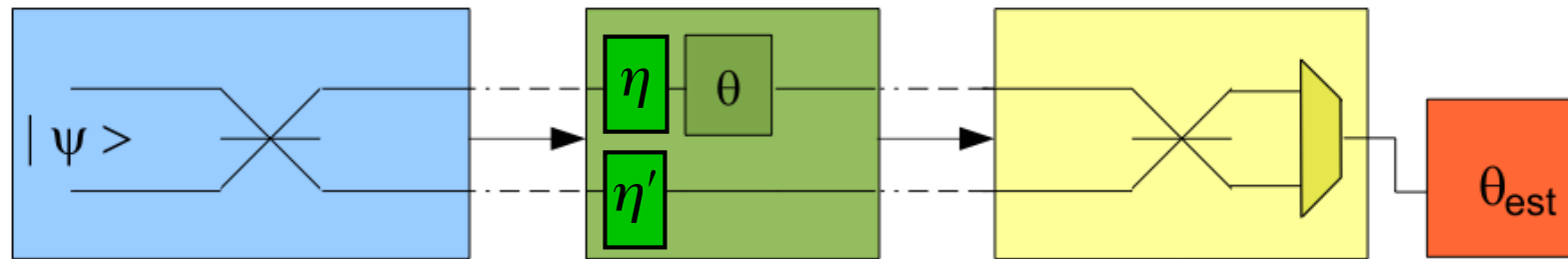
The lower bound for the precision in the measurement of the parameter X is then $\sqrt{\langle (\Delta X_{\text{est}})^2 \rangle} \geq 1/\sqrt{N\mathcal{F}_Q(X)}$, where N is the number of repetitions of the experiment.

We showed that the quantum Fisher information for pure states that evolve according to $|\psi(X)\rangle = \hat{U}(X)|\psi(0)\rangle$, where X is the parameter to be estimated and $\hat{U}(X)$ is a unitary operator, is

$$\mathcal{F}_Q(X) = 4\langle (\Delta \hat{H})^2 \rangle_0, \quad \langle (\Delta \hat{H})^2 \rangle_0 \equiv \langle \psi(0) | \left[\hat{H}(X) - \langle \hat{H}(X) \rangle_0 \right]^2 | \psi(0) \rangle$$

where $\hat{H}(X) \equiv i \frac{d\hat{U}^\dagger(X)}{dX} \hat{U}(X) = -i \hat{U}^\dagger(X) \frac{d\hat{U}(X)}{dX}$

Parameter estimation with decoherence



Loss of a single photon transforms NOON state into a separable state!

$$|\psi(N)\rangle = \frac{|N, 0\rangle + |0, N\rangle}{\sqrt{2}} \rightarrow |N-1, 0\rangle \text{ or } |0, N-1\rangle$$

No simple analytical expression for Fisher information!

For small N, more robust states can be numerically calculated

Experimental test with more robust states (for N=2):

nature
photonics

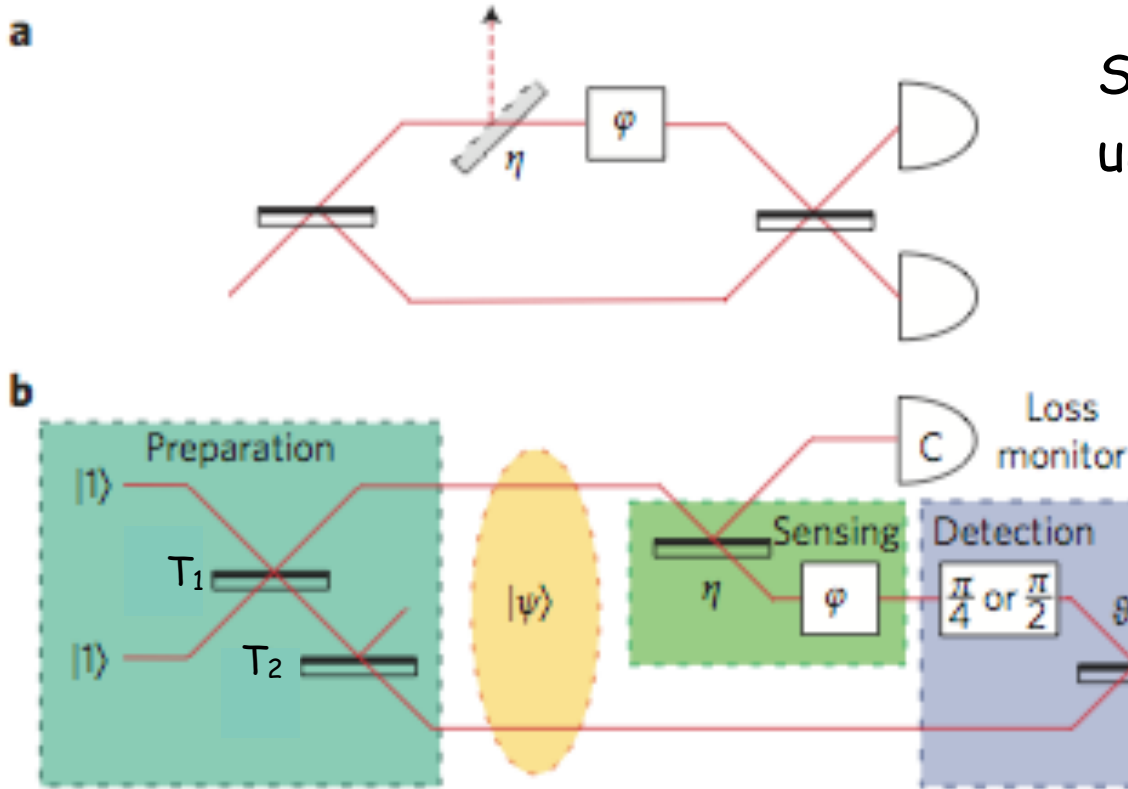
LETTERS

PUBLISHED ONLINE: 4 APRIL 2010 | DOI: 10.1038/NPHOTON.2010.39

Experimental quantum-enhanced estimation of a lossy phase shift

M. Kacprowicz¹, R. Demkowicz-Dobrzański^{1,2*}, W. Wasilewski², K. Banaszek^{1,2} and I. A. Walmsley³

Parameter estimation with losses - experiment



States leading to minimum uncertainty in the presence of noise:

$$|\psi\rangle = \sqrt{x_2}|20\rangle + \sqrt{x_1}|11\rangle - \sqrt{x_0}|02\rangle$$

Losses are simulated by a beam splitter in the upper arm, with transmissivity η .

States are prepared by two beam splitters, with transmissivities T_1 and T_2 .

First beam splitter produces the state $\sqrt{2T_1(1-T_1)}(|20\rangle - |02\rangle) + (2T_1 - 1)|11\rangle$.
 Second beam splitter allows to change the relative weights in this state.

When zero or one photon is lost, the conditional states are

$$\begin{aligned} \sqrt{p_0}|\psi_0\rangle &= \eta\sqrt{x_2}|20\rangle + \sqrt{\eta x_1}|11\rangle - \sqrt{x_0}|02\rangle && \text{(States with different total} \\ \sqrt{p_1}|\psi_1\rangle &= \sqrt{2\eta(1-\eta)x_2}|10\rangle + \sqrt{(1-\eta)x_1}|01\rangle && \text{photon number)} \end{aligned}$$

Maximization of Fisher information $p_0F(|\psi_0\rangle) + p_1F(|\psi_1\rangle)$ yields optimal values of x_0 , x_1 , and x_2 , for each value of η . (No information if two photons are lost).

Parameter estimation with losses - experiment

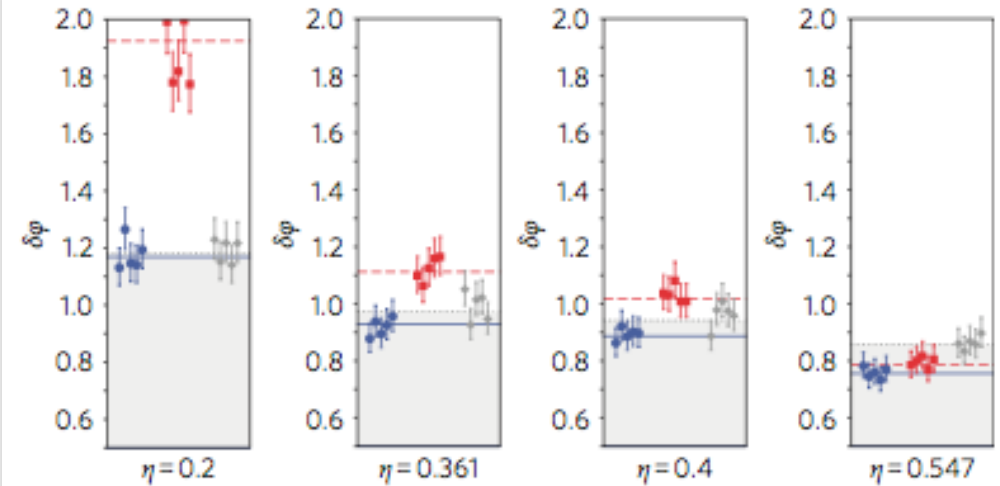
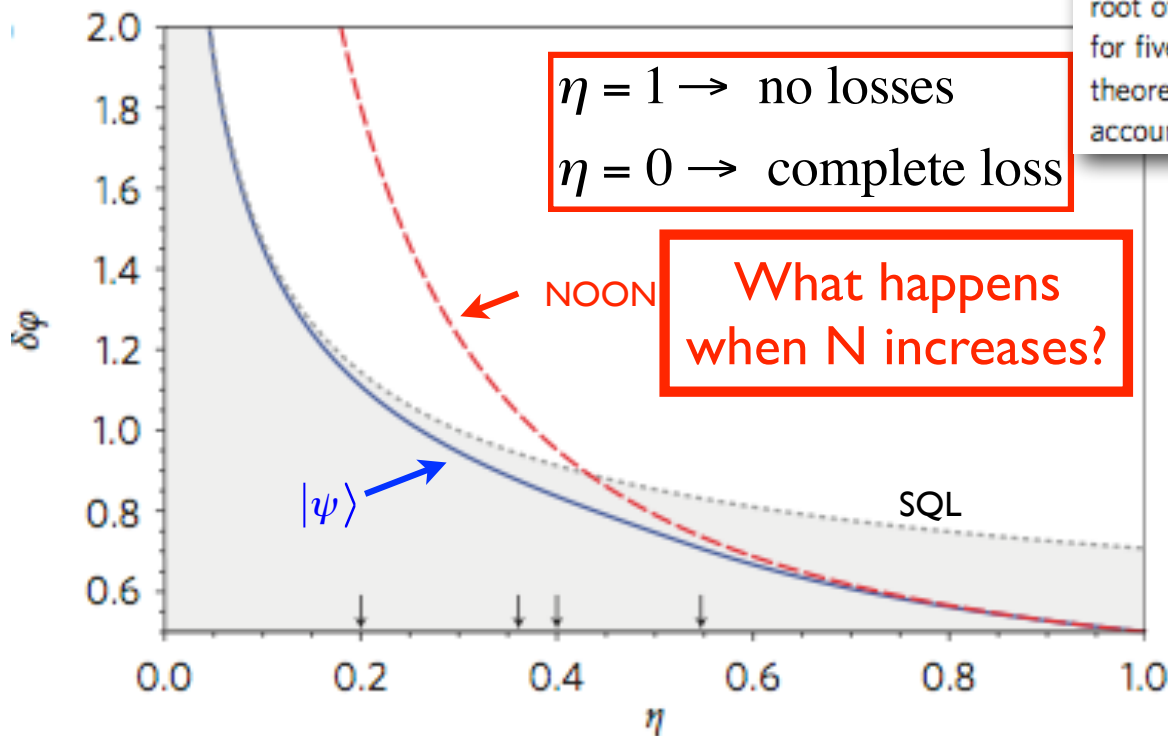


Figure 5 | Uncertainty of phase estimates. Uncertainties obtained using two-photon optimal (circles) and NOON (squares) states, as well as attenuated laser pulses in the SIL regime (diamonds), rescaled by the square root of the number of coincidences. For each transmission η , data are shown for five phases $\varphi = 0, \pm 0.2, \pm 0.4$ rad. Horizontal lines represent the theoretical Cramér-Rao bounds for given classes of input states, taking into account imperfections of the interferometer.



Parameter estimation with losses - theory

C. W. Helstrom, Quantum detection and estimation theory (Academic Press, New York, 1976); A. S. Holevo, Probabilistic and statistical aspects of quantum theory (North-Holland, Amsterdam, 1982); S. L. Braunstein and C. M. Caves, PRL 72, 3439 (1994).

We have now (Asymptotically attainable when $N \rightarrow \infty$)

$$\delta X \geq 1 / \sqrt{N \mathcal{F}_Q[\hat{\rho}(X_{\text{real}})]}, \quad \mathcal{F}_Q(\hat{\rho}) \equiv \max_{\hat{E}_j} F(\hat{\rho}, \hat{E}_j)$$

$$F(\hat{\rho}, \hat{E}_j) \equiv \sum_j p_j(X) \left(\frac{d \ln[p_j(X)]}{dx} \right)^2, \quad p_j(X) = \text{Tr}[\hat{\rho}(X) \hat{E}_j]$$

$$\hat{L}_{ij} = \frac{2}{p_i + p_j} [\partial \hat{\rho}(X) / \partial X]_{ij},$$

p_i, p_j eigenvalues of $\hat{\rho}$

General expression for the quantum Fisher information:

$\mathcal{F}_Q[\hat{\rho}(X)] = \text{Tr}[\hat{\rho}(X) \hat{L}^2(X)]$ where the operator \hat{L} ("symmetric logarithmic derivative") is defined by the equation $\frac{d\hat{\rho}(X)}{dX} = \frac{\hat{\rho}(X) \hat{L}(X) + \hat{L}(X) \hat{\rho}(X)}{2}$

For pure states:

$$\hat{\rho}^2 = \hat{\rho}, \quad \frac{d\hat{\rho}(X)}{dX} = \frac{d\hat{\rho}^2(X)}{dX} = \hat{\rho}(X) \frac{d\hat{\rho}(X)}{dX} + \frac{d\hat{\rho}(X)}{dX} \hat{\rho}(X) \Rightarrow \hat{L}(X) = 2 \frac{d\hat{\rho}(X)}{dX}$$

so that, from $\hat{\rho}(X) = \hat{U}(X) \hat{\rho}(0) \hat{U}^\dagger(X)$, one gets the previous result $\mathcal{F}_Q(X) = 4 \langle (\Delta \hat{H})^2 \rangle_0$, with $\hat{H}(X) \equiv i \frac{d\hat{U}^\dagger(X)}{dX} \hat{U}(X)$.

General case: \hat{L} difficult to evaluate - analytic expression not known.

General framework for estimating the ultimate precision limit in noisy quantum-enhanced metrology

B. M. Escher[★], R. L. de Matos Filho and L. Davidovich

news & views

QUANTUM METROLOGY

Beauty and the noisy beast

Elegant but extremely delicate quantum procedures can increase the precision of measurements. Characterizing how they cope with the detrimental effects of noise is essential for deployment to the real world.

Lorenzo Maccone and Vittorio Giovannetti

Open-system evolution and quantum channels

The evolution of an open system can be described by the Hamiltonian

$$H = H_S \otimes \mathbf{1}_E + \mathbf{1}_S \otimes H_E + V_{SE}$$

where H_S and H_E stand for the free-evolution Hamiltonians of the system and environment, respectively, and V_{SE} is the interaction between the two parties. We aim to describe the effective time evolution of S: $\rho_S(t) = \text{Tr}_E [\rho_{SE}(t)] = \$[\rho_S(0)]$ where $\$$ is a linear map. Assuming that initially S and E are not correlated, and that the initial state of the environment is $|0\rangle_E$, then $\rho_{SE}(0) = \rho_S^{\text{in}} \otimes |0\rangle_E \langle 0|$ and

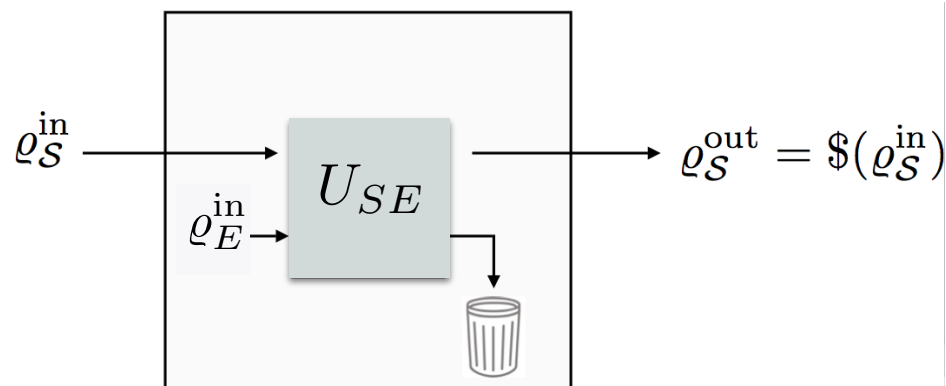
$$\rho_{SE}(t) = U_{SE} (\rho_S^{\text{in}} \otimes |0\rangle_E \langle 0|) U_{SE}^\dagger$$

where U_{SE} is the evolution operator corresponding to Hamiltonian H.

Then
$$\rho_S^{\text{out}} = \text{Tr}_E \left[U_{SE} (\rho_S^{\text{in}} \otimes |0\rangle_E \langle 0|) U_{SE}^\dagger \right]$$

$$= \sum_{\mu} {}_E \langle \mu | U_{SE} | 0 \rangle_E \rho_S^{\text{in}} {}_E \langle 0 | U_{SE}^\dagger | \mu \rangle_E$$

$$= \sum_{\mu} K_{\mu} \rho_S^{\text{in}} K_{\mu}^\dagger = \$(\rho_S^{\text{in}})$$



where $\{|\mu\rangle\}$ is a basis of E, and $K_{\mu} \equiv {}_E \langle \mu | U_{SE} | 0 \rangle_E$ are the **Kraus operators**, which define the **quantum channel** $\$$ (this is the Kraus decomposition of a quantum channel). The differential form of this evolution leads to the master equation for the reduced density matrix of the system.

Example of quantum channels: amplitude damping

• Amplitude damping channel. $U_{SE}^{AD} |0\rangle_S |0\rangle_E = |0\rangle_S |0\rangle_E$

$$U_{SE}^{AD} |1\rangle_S |0\rangle_E = \sqrt{1-p} |1\rangle_S |0\rangle_E + \sqrt{p} |0\rangle_S |1\rangle_E$$

$p = 1 - \exp(-\Gamma t)$: corresponds to Weisskopf-Wigner approach to the spontaneous emission of an atom into a zero-temperature environment. In this case, the states $|0\rangle_E$ and $|1\rangle_E$ correspond to the vacuum and one-photon states of the reservoir, while $|0\rangle_S$ and $|1\rangle_S$ correspond to the ground and excited states of a two-level atom.

The reduced density matrix obtained by tracing out the reservoir degrees of freedom coincides with the solution, within the Markovian approximation, of the master equation corresponding to the Hamiltonian of a two-level atom interacting with a continuum of electromagnetic field modes.

The same map also corresponds to the Jaynes-Cummings model, which describes the Rabi oscillations of a two-level atom interacting with a single mode of the electromagnetic field, if one sets $p = \sin^2(\Omega t/2)$. So it is advantageous to analyze this map in terms of \mathbf{p} rather than \mathbf{t} , since the same analysis covers both cases. Along the same line, we consider that the system interacts with an environment, not necessarily a reservoir with many degrees of freedom.

Example of quantum channels: amplitude damping (2)

• **Kraus operators.** From $U_{SE}^{AD} |0\rangle_S |0\rangle_E = |0\rangle_S |0\rangle_E$
 $U_{SE}^{AD} |1\rangle_S |0\rangle_E = \sqrt{1-p} |1\rangle_S |0\rangle_E + \sqrt{p} |0\rangle_S |1\rangle_E$
 and from the definition $K_\mu \equiv {}_E \langle \mu | U_{SE} | 0 \rangle_E$ one gets

$$K_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \quad K_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}$$

The evolution of the reduced density matrix of S can be obtained either by tracing out the states of the "reservoir", or directly by applying the Kraus operators:

$$\begin{aligned} \rho &= K_0 \rho K_0^\dagger + K_1 \rho K_1^\dagger \\ &= \begin{pmatrix} \rho_{00} & \sqrt{1-p} \rho_{01} \\ \sqrt{1-p} \rho_{10} & (1-p) \rho_{11} \end{pmatrix} + \begin{pmatrix} p \rho_{11} & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \rho_{00} + p \rho_{11} & \sqrt{1-p} \rho_{01} \\ \sqrt{1-p} \rho_{10} & (1-p) \rho_{11} \end{pmatrix} \end{aligned}$$

For $p = 1 - \exp(-\Gamma t)$, the population of the excited state decays exponentially (rate Γ), feeding the population of the ground state. The coherences (non-diagonal elements) also decay exponentially, with rate $\Gamma/2$.

Purification of an evolution

Given the Kraus decomposition of a quantum channel, it is possible to find a correspondent unitary evolution of the system plus an environment.

That is done by picking up a basis in S , $\{|\phi_i\rangle_S\}$, and as many orthonormal vectors in E' , $\{|i\rangle_{E'}\}$, as the number of Kraus operators. We define then:

$$U_{SE'}|\phi_j\rangle_S|0\rangle_{E'} = \sum_i K_i|\phi_j\rangle_S|i\rangle_{E'}$$

It is easy to see that this evolution preserves the norm, which implies that it can be extended to a unitary operator acting on $\mathcal{H}_{SE'}$.

This unitary evolution is not necessarily the same as the one derived from the original Hamiltonian: the vectors $|i\rangle_{S'}$ may span an "effective" environment with smaller (finite) dimension than the real environment E , which leads however to the same dynamics for all the states in S .

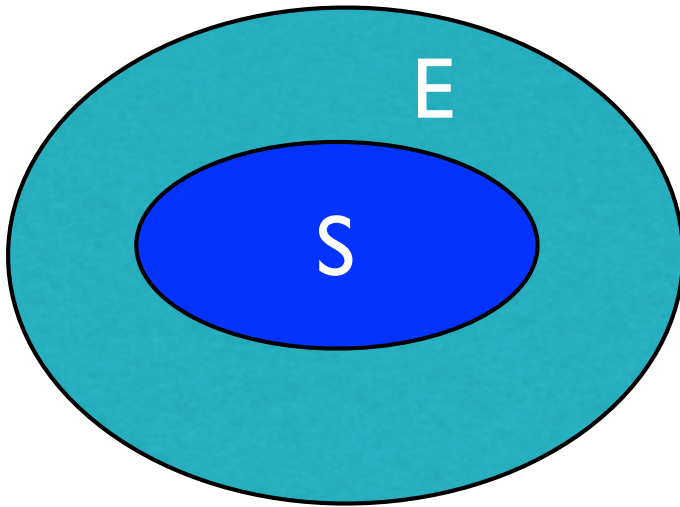
We shall use this purification strategy in order to develop a general framework for the estimation of parameters in noisy quantum-enhanced metrology.

Parameter estimation in open systems:

Extended space approach

B. M. Escher, R. L. Matos Filho, and L. D., *Nature Physics* 7, 406 (2011);
Braz. J. Phys. 41, 229 (2011)

Given initial state and non-unitary evolution, define in $S+E$



$$|\Phi_{S,E}(x)\rangle = \hat{U}_{S,E}(x) |\psi\rangle_S |0\rangle_E \text{ (Purification)}$$

Then

$$\mathcal{F}_Q \equiv \max_{\hat{E}_j^{(S)} \otimes \hat{1}} F\left(\hat{E}_j^{(S)} \otimes \hat{1}\right) \leq \max_{\hat{E}_j^{(S,E)}} F\left(\hat{E}_j^{(S,E)}\right) \equiv \mathcal{C}_Q$$

since measurements on $S+E$ should yield more information than measurements on S alone.

Least upper bound: Minimization over all unitary evolutions in $S+E$ - difficult problem

Physical meaning of this bound: information obtained about parameter when $S+E$ is monitored

Bound is attainable - there is always a purification such that $\mathcal{C}_Q = \mathcal{F}_Q$

Then, monitoring $S+E$ yields same information as monitoring S

Minimization procedure

PRL 109, 190404 (2012)

PHYSICAL REVIEW LETTERS

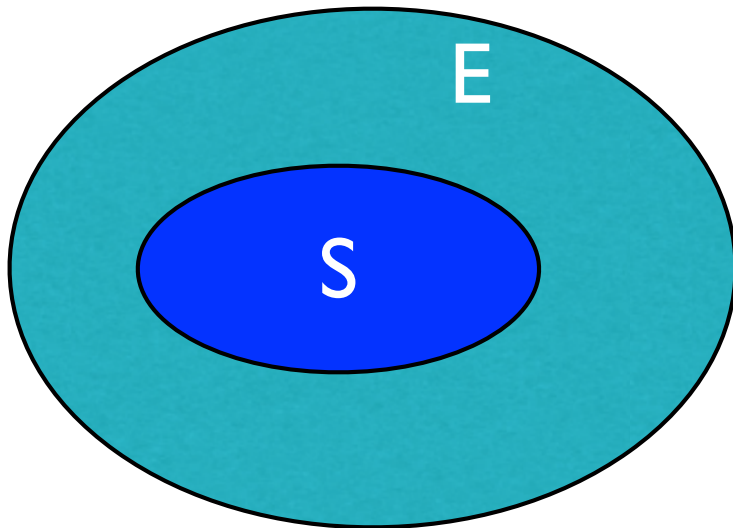
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Quantum Metrological Limits via a Variational Approach

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There is always a unitary operator acting only on E that connects two different purifications of ρ_S

Given $|\Phi_{S,E}(x)\rangle = \hat{U}_{S,E}(x)|\psi\rangle_S|0\rangle_E$,

$$i\frac{d|\Phi_{S,E}(x)\rangle}{dx} = \hat{H}_{S,E}(x)|\Phi_{S,E}(x)\rangle,$$

then any other purification can be written as:

$$|\Psi_{S,E}(x)\rangle = u_E(x)|\Phi_{S,E}(x)\rangle$$

Define $\hat{h}_E(x) = i\frac{d\hat{u}_E^\dagger(x)}{dx}\hat{u}_E(x)$

Minimize now C_Q over all Hermitian operators $h_E(x)$ that act on E . Above paper proposes iterative procedure for doing this.

Analytical solution for the forced noisy harmonic oscillator

PHYSICAL REVIEW A 88, 042112 (2013)

Quantum limit for the measurement of a classical force coupled to a noisy quantum-mechanical oscillator

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Let us consider first the noiseless oscillator:

$$\hat{H}_S / \hbar\omega = \frac{1}{2}(\hat{P}^2 + \hat{X}^2) + F \cos(\omega t) \hat{X}$$

$$\hat{X} = (\hat{a} + \hat{a}^\dagger) / \sqrt{2}$$

$$\hat{P} = -i(\hat{a} - \hat{a}^\dagger) / \sqrt{2}$$

$$F = f \sqrt{\hbar / (m\omega)^3}$$

Interaction Hamiltonian in the interaction picture

$$\begin{aligned} \hat{H}_I &= \hbar\omega F \cos(\omega t) (\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}) / \sqrt{2} \approx \hbar\omega (F/2) (\hat{a} + \hat{a}^\dagger) / \sqrt{2} \\ &\approx \hbar\omega F \hat{X} / 2 \quad (\text{Rotating-wave approximation}) \end{aligned}$$

Unitary evolution: $\hat{U}_I(t, 0) = \exp \{-i\omega t F \hat{X} / 2\}$

This is a momentum displacement operator.

The displacement p is given by

$$p = \omega F t / 2 \Rightarrow \delta F = \frac{2}{\omega t} \delta p$$

$\delta F \rightarrow$ uncertainty in the estimation of F (proportional to the uncertainty in the estimation of the momentum displacement)

Uncertainty in the estimation of the force (noiseless oscillator)

$$\hat{U}_I(t, 0) = \exp(-i\omega t F \hat{X}/2)$$

Quantum Fisher information: $\mathcal{F}_Q(F) = (\omega t)^2 \Delta^2 \hat{X}_0$

Uncertainty in the estimation of the Force (and of the momentum displacement!):

$$\delta F = \frac{2}{\omega t} \delta p \geq \frac{1}{\omega t \sqrt{\nu \Delta^2 \hat{X}_0}}$$

Standard quantum limit:

$$\Delta^2 \hat{X}_0 = \frac{1}{2} \Rightarrow \delta F_{\text{std}} = \frac{\sqrt{2}}{\omega t \sqrt{\nu}}$$

Ultimate precision limit: maximize variance under the conditions

$$\langle \psi | \psi \rangle = 1 \text{ and } \langle \psi | (\hat{H}_0 / \hbar \omega) | \psi \rangle = E \Rightarrow \text{Squeezed state}$$

$$\Delta^2 \hat{X}_0 = (E + \sqrt{4E^2 - 1}) \Rightarrow \delta F \geq \frac{1}{\omega t \sqrt{\nu(E + \sqrt{4E^2 - 1})}} \xrightarrow{E \rightarrow \infty} \frac{1}{\omega t \sqrt{3\nu E}}$$

"Heisenberg limit"

Minimum energy: $E=1/2$ (ground state). Notice that the precision increases with the measurement time (this will not longer be true when noise is present).

Noisy forced oscillator

The Langevin approach allows in this case a simple physical picture:

$$\frac{d\hat{P}}{dt} = \omega F/2 - \gamma\hat{P} + \hat{f}_\gamma(t) \quad \langle \hat{f}_\gamma(t) \rangle = 0 \quad \langle \hat{f}_\gamma(t)\hat{f}_\gamma(t') \rangle = 2\gamma\delta(t-t')$$

We consider for simplicity a zero-temperature reservoir — see PRA **88**, 042112 (2013) for generalization to temperatures different from zero.

Solution of the Langevin equation:

$$\hat{P}(t) = \hat{P}(0)e^{-\gamma t} + \frac{\omega F}{2\gamma}(1 - e^{-\gamma t}) + e^{-\gamma t} \int_0^t dt' \hat{f}_\gamma(t')e^{\gamma t'}$$

momentum displacement $p = FD(\eta)/2$

$$\eta = e^{-2\gamma t}$$
$$D(\eta) \equiv (\omega/\gamma)(1 - \sqrt{\eta})$$

Uncertainty in p is equal to standard deviation of $\hat{P}(t)$ in the initial state:

$$\delta p = \Delta\hat{P}(t)|_0 = \sqrt{\langle [\hat{P}(t) - \langle \hat{P} \rangle]^2 \rangle_0}$$

$$\text{Uncertainty in } F \text{ is then } \delta F = \frac{2}{D(\eta)}\delta p = \frac{2}{D(\eta)}\Delta\hat{P}(t)|_0$$

From the above equation for $\hat{P}(t)$, one gets: $\Delta^2\hat{P}(t) = \eta\Delta^2\hat{P}(0) + (1 - \eta)/2.$

Noisy forced oscillator (2)

Therefore

$$\delta F = \frac{2}{D(\eta)} \Delta \hat{P}(t) \Big|_0 = \frac{2}{D(\eta)} \left[\eta \Delta^2 \hat{P}(0) \Big|_0 + (1 - \eta)/2 \right]^{1/2}$$

$$\geq \frac{1}{D(\eta)} \left[\frac{\eta}{\Delta^2 \hat{X}(0) \Big|_0} + 2(1 - \eta) \right]^{1/2}$$

where in the last step the Heisenberg uncertainty relation was used. Clearly the minimum uncertainty is reached for a minimum uncertainty state, then

$$\delta F_{\min} = \frac{1}{D(\eta)} \left[\frac{\eta}{\Delta^2 \hat{X}(0) \Big|_0} + 2(1 - \eta) \right]^{1/2}$$

$$\eta = e^{-2\gamma t}$$

$$D(\eta) \equiv (\omega/\gamma)(1 - \sqrt{\eta})$$

This bound was obtained through a measurement of the momentum displacement. One should still maximize $\Delta^2 \hat{X}(0) \Big|_0$, which for fixed average energy E is obtained with a squeezed state, as seen before. Then

$$\Delta^2 \hat{X}(0) \Big|_0 = E + \sqrt{4E^2 - 1} \Rightarrow \delta F_{\min} = \frac{1}{D(\eta)} \left[\frac{\eta}{E + \sqrt{4E^2 - 1}} + 2(1 - \eta) \right]^{1/2}$$

When $\gamma \rightarrow 0$, one recovers the result for the noiseless oscillator.

Noisy forced oscillator (3)

Another way of getting this result is through the Fisher information.

Take initial state as a minimum-uncertainty Gaussian state. Then final state is also Gaussian (since momentum displacement is a Gaussian operation). If the momentum is measured, the Fisher information is

$$\mathcal{F}_{\text{Gaussian}}(F; |P\rangle\langle P|) \equiv \int dP \langle P | \hat{\rho}(F) | P \rangle \left(\frac{d \ln[\langle P | \hat{\rho}(F) | P \rangle]}{dF} \right)^2$$

$$\Rightarrow \mathcal{F}_{\text{Gaussian}}(F) = [D(\eta)]^2 \frac{\Delta^2 \hat{X}|_0}{\eta + 2(1-\eta)\Delta^2 \hat{X}|_0}$$

$$\eta = e^{-2\gamma t}$$
$$D(\eta) \equiv (\omega/\gamma)(1 - \sqrt{\eta})$$

It is easy to check that

$$\delta F_{\min} = \frac{1}{\sqrt{\mathcal{F}_{\text{Gaussian}}(F)}}$$

coincides with the expression obtained before.

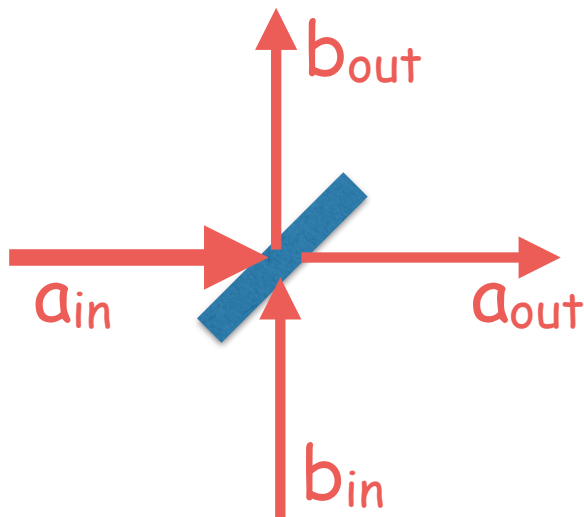
Important question:
Is this the maximum possible value?

Physical picture helps to find the quantum Fisher information

Consider that the harmonic oscillator corresponds to one mode of the electromagnetic field. Notice that

$$\Delta^2 \hat{P}(t) \Big|_0 = \eta \Delta^2 \hat{P}(0) \Big|_0 + (1 - \eta)/2 = \eta \Delta^2 \hat{P}(0) \Big|_0 + (1 - \eta) \Delta^2 \hat{P}(0) \Big|_{GS}$$

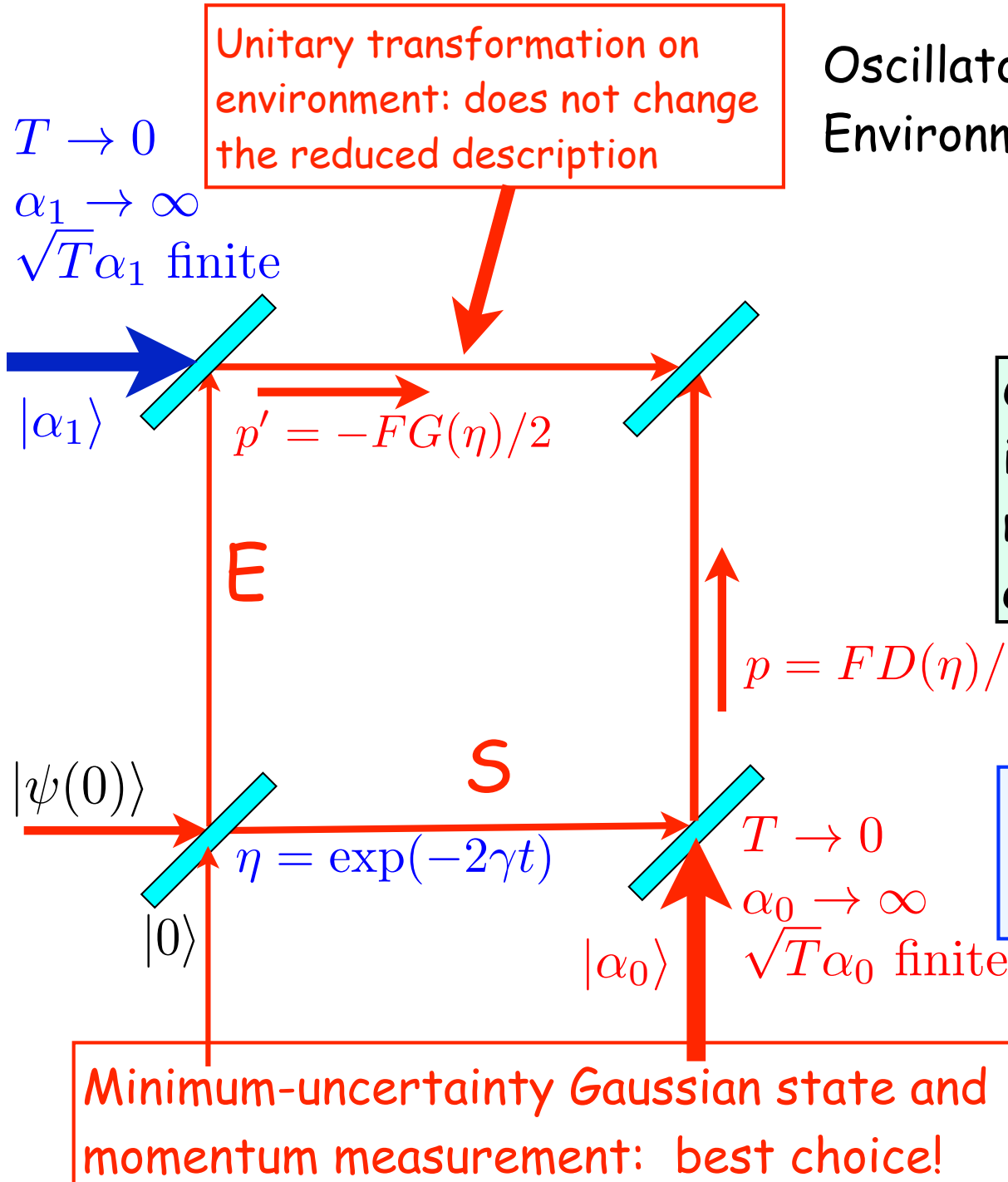
where $\Delta^2 \hat{P}(0) \Big|_{GS} = 1/2$ is the variance of $\hat{P}(0)$ in the ground state of the harmonic oscillator (a minimum-uncertainty state). This relation corresponds precisely to that for the variance of quadratures of a mode of the electromagnetic field going through a beam splitter with transmissivity η .



The incoming mode b_{in} has no photons (vacuum state), and the outgoing mode b_{out} can be considered a reservoir, which takes photons out of the incoming mode a_{in} .

This suggests a natural purification of mode a , which will be discussed now.

Noisy forced oscillator: Purification procedure



Oscillator = Field mode (S)
 Environment = Another field mode (E)

Purification!

Calculate quantum Fisher information for SE, choose G to minimize it \Rightarrow upper bound for quantum Fisher information of S

$$\mathcal{F}_{QSE}(F) = [D(\eta)]^2 \frac{\Delta^2 \hat{X}_0}{\eta + 2(1-\eta)\Delta^2 \hat{X}_0}$$

Same expression as before!

Exact quantum limit

$$\delta F \geq \frac{\gamma}{\omega(1 - \sqrt{\eta})\sqrt{\nu}} \left[2(1 - \eta) + \frac{\eta}{(E + \sqrt{E^2 - 1/2})} \right]^{1/2}$$

$$\eta = \exp(-2\gamma t)$$

Depending on which term dominates,
one gets standard or Heisenberg limit

$$f = F \sqrt{(m\omega)^3 / \hbar}$$

$$E + \sqrt{4E^2 - 1} \gg \frac{\eta}{1 - \eta} \Rightarrow \text{Standard scaling}$$

$$E + \sqrt{4E^2 - 1} \ll \frac{\eta}{1 - \eta} \Rightarrow \text{Heisenberg scaling}$$

Bound saturates as time grows...

$$\delta F \geq \frac{\gamma}{\omega\sqrt{\nu}} \sqrt{2(1 + 2\bar{n}_T)}$$

It does not pay to wait for a long
time, as in the noiseless case...

Minimization of
bound implies
maximization of
variance of
position: for fixed
average energy E ,
squeezed state!

Thermal reservoir:

$$2(1 - \eta) \rightarrow 2(1 - \eta)(2\bar{n}_T + 1)$$

Better strategy: Divide to conquer...

Force acts during a time t_{total} . Probe force during time τ , measure the probe system, reset this system and repeat this procedure ν times, with $\nu = t_{\text{total}}/\tau$. Minimize measurement uncertainty with respect to τ

Diffusive limit: $\gamma \rightarrow 0$, $n_T \rightarrow \infty$, with $\gamma n_T = \mathcal{D}$

[Maiwald, R. et al. Stylus ion trap for enhanced access and sensing. Nature Phys. 5, 551-554 (2009)]

$$\delta f \geq \sqrt{\frac{4m\hbar\omega\mathcal{D}}{t_{\text{tot}}}} \sqrt{1 + \frac{1}{4\mathcal{D}\langle(\Delta\hat{X})^2\rangle_0 t_{\text{tot}}}}$$

Correction to heuristic calculation

Further generalizations

PHYSICAL REVIEW A 88, 042112 (2013)

Quantum limit for the measurement of a classical force coupled to a noisy quantum-mechanical oscillator

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Estimation of amplitude f of force $f\zeta(t)$, with known $\zeta(t)$, such that $\text{Max} |\zeta(t)| = 1$. Also for temperature different from zero.

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PHYSICAL REVIEW LETTERS

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4 MARCH 2011

Fundamental Quantum Limit to Waveform Estimation

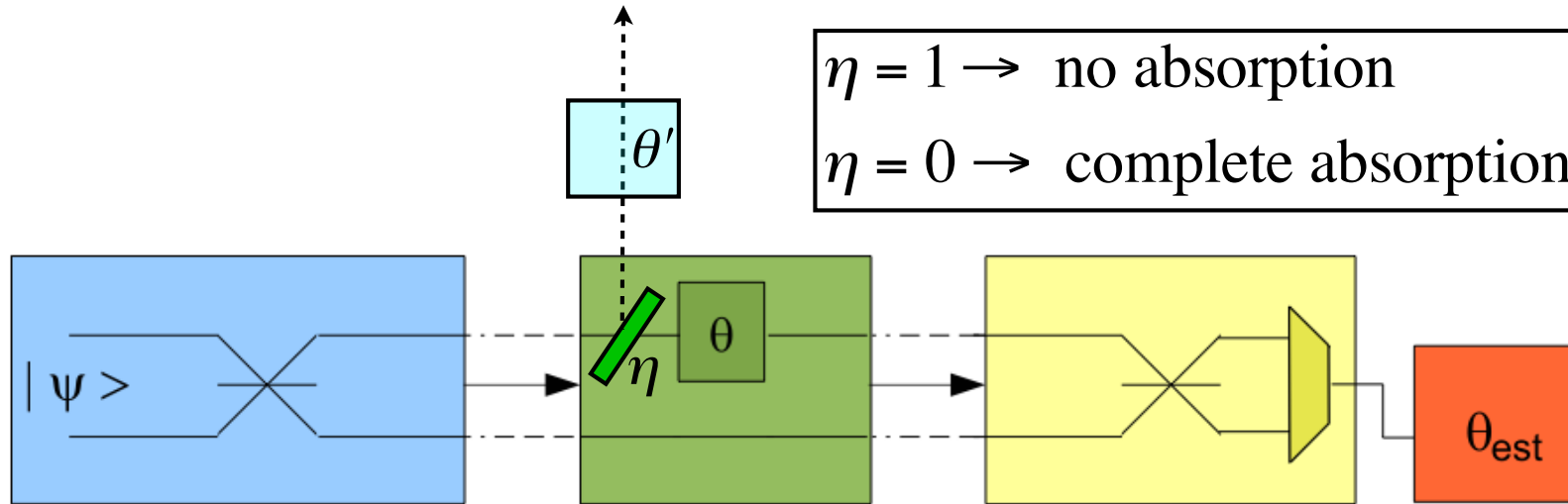
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Formalism is applied to $H = p^2/2m + m\omega_m^2 q^2/2 - qF(t)$, where $F(t)$ is a stationary process: $\langle F(t)F(t') \rangle$ depends only on $\tau = t' - t$. Then the quantum Cramér-Rao bound becomes a spectral uncertainty principle:

$$C(\omega) \left[S_{\Delta q}(\omega) + \frac{\hbar^2}{4S_{\Delta F}(\omega)} \right] \geq \frac{\hbar^2}{4}. \text{ Here } S_x(\omega) = \int_{-\infty}^{+\infty} d\tau \langle x(t)x(t+\tau) \rangle e^{i\omega\tau}$$

and $C(\omega)$ is the power spectral density of the estimation error.

Quantum limits for lossy optical interferometry



One uses here a similar strategy: a phase displacement on the environment so as to remove additional information on the phase θ .

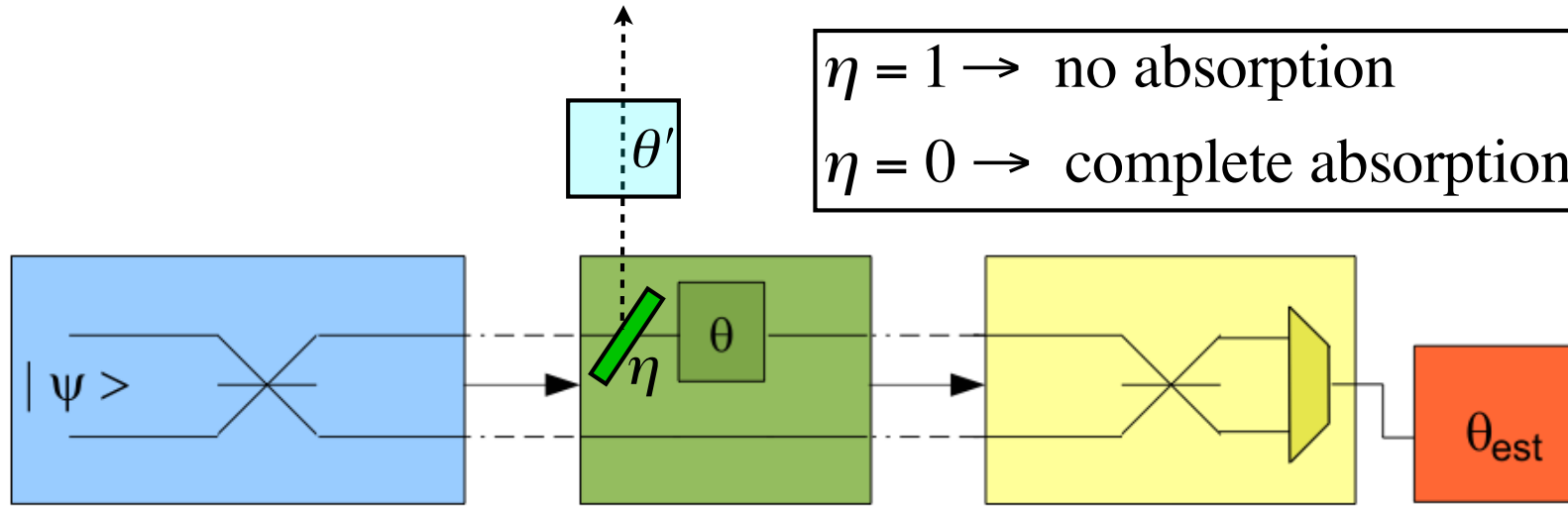
Minimization of the quantum Fisher information of system + environment yields an upper bound for the Fisher information of the system:

$$C_Q(\hat{\rho}_0) = \frac{4\eta\langle\hat{n}\rangle_0\Delta^2\hat{n}_0}{(1-\eta)\Delta^2\hat{n}_0 + \eta\langle\hat{n}\rangle_0}$$

Note that if $(1-\eta)\Delta^2\hat{n}_0 \ll \eta\langle\hat{n}\rangle_0$ then $C_Q \rightarrow \Delta^2\hat{n}_0$, the quantum Fisher information for pure states. On the other hand, in the high-dissipation limit $\eta \ll 1$, one has $(1-\eta)\Delta^2\hat{n}_0 \gg \eta\langle\hat{n}\rangle_0$, yielding a standard-limit scaling:

$$\delta\theta \geq \sqrt{(1-\eta)/4\eta\langle\hat{n}\rangle_0}$$

Quantum limits for lossy optical interferometry



States with well-defined total photon number:

$$|\psi_0\rangle = \sum_{n=0}^N \beta_n |n, N-n\rangle$$

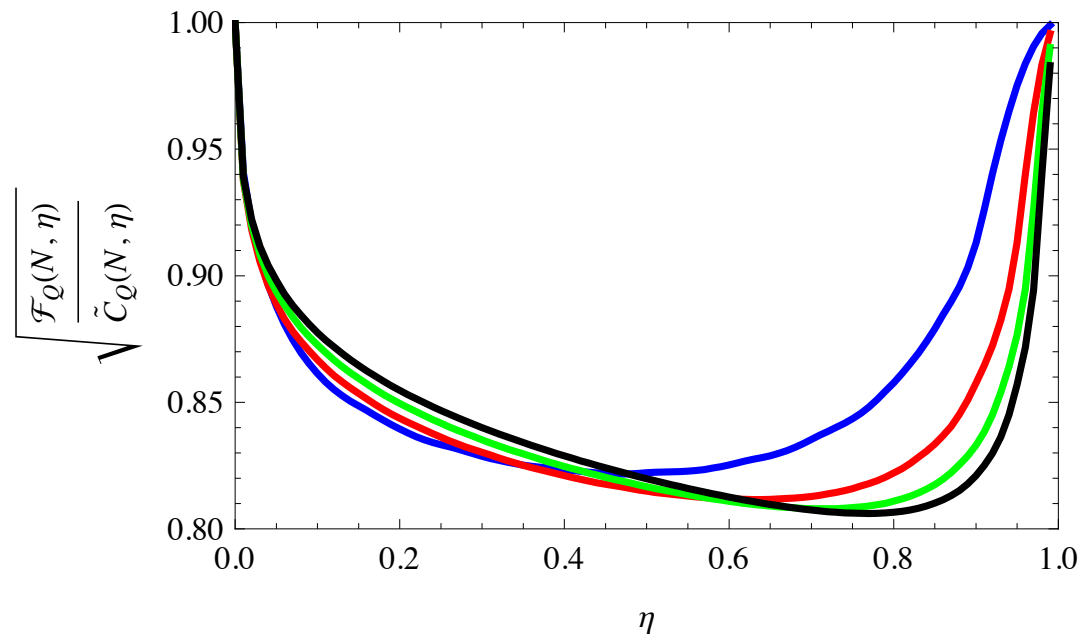
$$2\delta\theta \geq \left[1 + \sqrt{1 + \frac{1-\eta}{\eta} N} \right] / N$$

$$N \ll \frac{\eta}{1-\eta} \Rightarrow \sqrt{v}\delta\theta \geq 1/N \rightarrow \text{Heisenberg scaling}$$

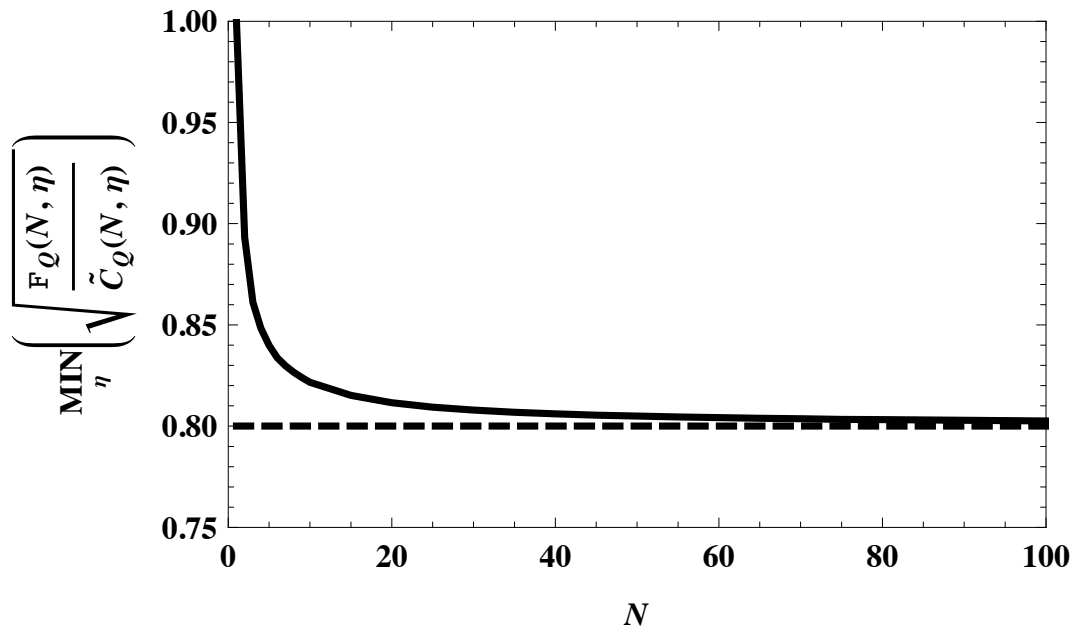
$$N \gg \frac{\eta}{1-\eta} \Rightarrow \delta\theta \geq \frac{\sqrt{1-\eta}}{2\sqrt{v\eta N}} \rightarrow \text{Standard scaling}$$

For N sufficiently large, $1/\sqrt{N}$ behavior is always reached!

How good is this bound?



Comparison between the numerical maximum value of \mathcal{F}_Q and the upper bound $\tilde{\mathcal{C}}_Q$ as a function of η , for $N = 10$ (blue), $N = 20$ (red), $N = 30$ (green), and $N = 40$ (black).



Behavior of the minimum for all values of η , as a function of N

Phase diffusion in optical interferometer

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Quantum Metrological Limits via a Variational Approach

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The process of phase diffusion in an optical interferometer can be described by the following master equation:

$$\dot{\rho} = \Gamma \mathcal{L}[a^\dagger a] \rho, \quad \mathcal{L}[O] \rho = 2O\rho O^\dagger - O^\dagger O \rho - \rho O^\dagger O$$

The corresponding solution is

$$\rho(t) = \sum_{m,n} e^{-\beta^2(n-m)^2} \rho_{n,m}(0) |n\rangle \langle m|, \quad \beta = \Gamma t$$

which shows that the coherences decay exponentially with the square of time, while the populations remain constant.

Phase diffusion in optical interferometer (2)

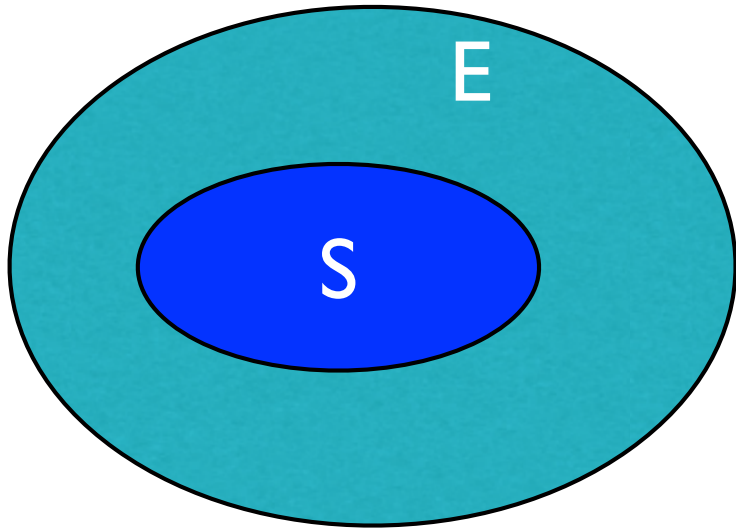
A possible purification consists in coupling the photon number operator to the center-of-mass motion of one of the mirrors of the interferometer: the photon-number dependent motion of the mirror, induced by an interaction that corresponds to radiation pressure, induces a dephasing:

$$|\Phi_{S,E}(\phi)\rangle = e^{-i\phi\hat{n}_S} e^{i(2\beta)\hat{n}_S\hat{x}_E} |\psi_S\rangle |0_E\rangle$$

Radiation pressure Ground state of mirror
(harmonic oscillator)

Upon tracing out the mirror, one gets the master equation in the previous slide. This purification does not help, however, since the corresponding quantum Fisher information is $C_Q = 4\Delta^2 n$, which coincides with that for the diffusion-free interferometer. This is a trivial upper bound. One is interested in getting a tighter bound, that should depend on the diffusion constant.

Minimization procedure



Remember that there is always an unitary operator acting only on E that connects two different purifications of ρ_S .

$$\text{Given } |\Phi_{S,E}(x)\rangle = \hat{U}_{S,E}(x) |\psi\rangle_S |0\rangle_E,$$

$$i \frac{d|\Phi_{S,E}(x)\rangle}{dx} = \hat{H}_{S,E}(x) |\Phi_{S,E}(x)\rangle,$$

then any other purification can be written as:

$$|\Psi_{S,E}(x)\rangle = u_E(x) |\Phi_{S,E}(x)\rangle$$

$$\text{Define } \hat{h}_E(x) = i \frac{d\hat{u}_E^\dagger(x)}{dx} \hat{u}_E(x)$$

Minimize now C_Q over all Hermitian operators $h_E(x)$ that act on E.

Choose: $\hat{u}_E(\phi; \lambda) = e^{i\phi\lambda\hat{p}_E/(2\beta)}$ and apply it to

$$|\Phi_{S,E}(\phi)\rangle = e^{-i\phi\hat{n}_S} e^{i(2\beta)\hat{n}_S\hat{x}_E} |\psi_S\rangle |0_E\rangle$$

with $\lambda \rightarrow$ Variational parameter

$$\text{Get then } C_Q = (1 - \lambda)^2 4\Delta n^2 + \lambda^2 / (2\beta^2)$$

Phase uncertainty due to phase diffusion

Minimization over λ yields:

$$\delta\phi_{pd} \geq \sqrt{\frac{1}{\nu} \left(\frac{1}{4\Delta^2 n} + 2\beta^2 \right)}$$

Intrinsic quantum feature Phase diffusion

for $\lambda_{\text{opt}} = 8\Delta^2 \hat{n} \beta^2 / (1 + 8\Delta^2 \hat{n} \beta^2)$.

An important property of the bound shown above is the presence of a constant term. This means that the presence of phase diffusion is, in general, more detrimental to phase-shift estimation than the presence of photon losses, for which the uncertainty goes to zero as the average number of photons goes to infinity.

This result is now compared to the numerical calculations, done for Gaussian states, by Genoni, Olivares, and Paris - PRL 106, 153603 (2011).

Phase uncertainty due to phase diffusion (2)

$$\delta\phi_{pd} \geq \sqrt{\frac{1}{\nu} \left(\frac{1}{4\Delta^2 n} + 2\beta^2 \right)} \quad \beta = \Gamma t$$

Very close to numerical value obtained by Genoni, Olivares, and Paris for Gaussian state - PRL 106, 153603 (2011)

For Gaussian states:

$$\Delta^2 n \leq 2N(N+1)$$

(N is the average photon number)

Then:

$$C_Q^{\text{opt}} \leq C_Q^{\text{max}} \equiv \left[2\beta^2 + \frac{1}{8N(N+1)} \right]^{-1}$$

Comparison with numerical results

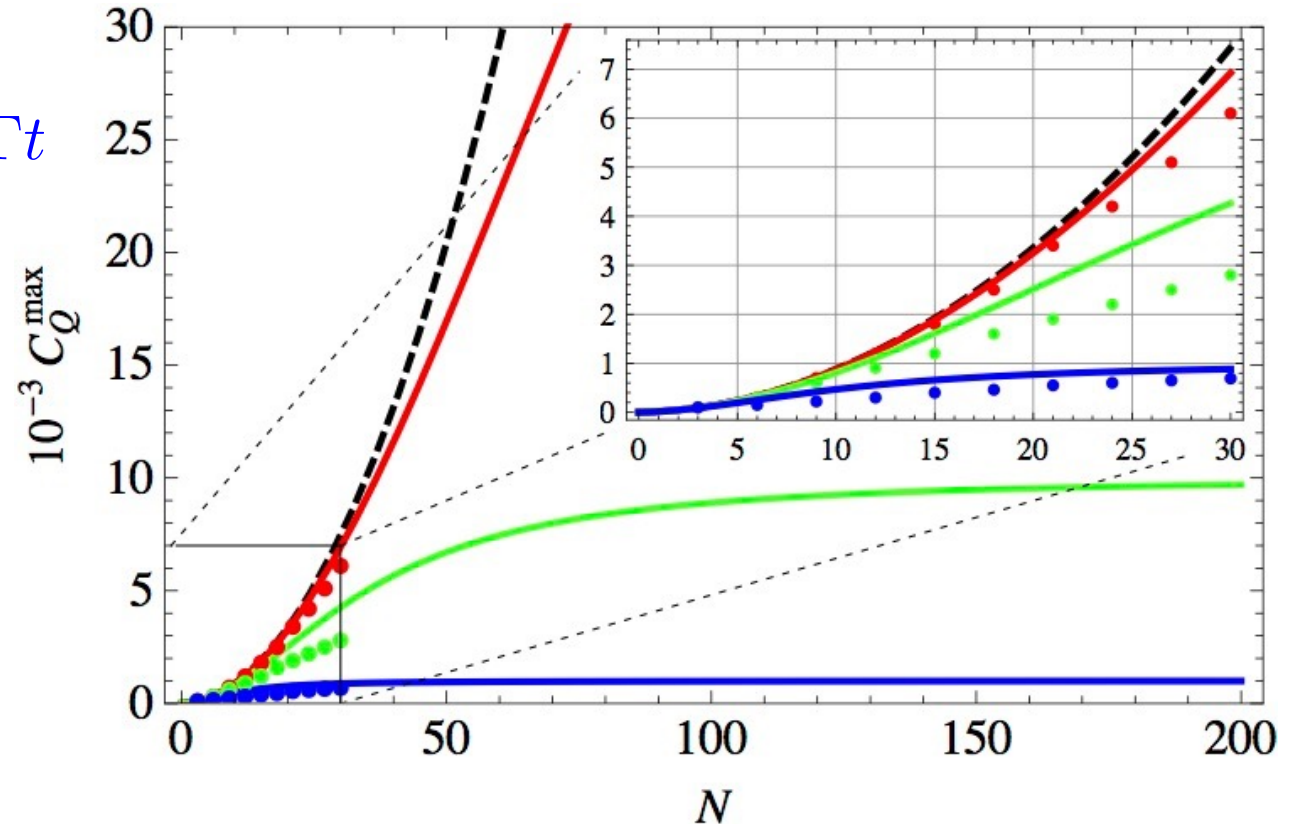


FIG. 1 (color online). Comparison between upper bound C_Q^{max} and the maximum quantum Fisher information $\mathcal{F}_Q^{\text{max}}$ in Ref. [14] as a function of the average number of photons N . The dots stand for the values obtained in Ref. [14], the dashed line corresponds to the noiseless case ($\beta^2 = 0$), and the full lines correspond to C_Q^{max} . The inset displays the two quantities up to $N = 30$, which was the range considered in Ref. [14]. From bottom to top, $\beta^2 = 5 \times 10^{-4}; 5 \times 10^{-5}; 5 \times 10^{-6}$.

Sommaire de la quatrième leçon

Lundi, 29 Février, 2016

La prochaine leçon introduira l'application de la métrologie quantique au problème de la relation d'incertitude entre l'énergie et le temps. On introduira des éléments de géométrie des états quantiques, lesquels vont permettre de mieux comprendre et de généraliser cette relation d'incertitude pour les systèmes ouverts. Ainsi, on obtiendra un traitement valable pour évolutions unitaires et aussi pour systèmes en présence de bruit. Cette théorie va être exemplifiée en considérant la relaxation d'un atome vers l'état fondamental et la décohérence induite par diffusion de la phase.