LECTURE NOTES

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Preface

This "Lecture notes" is a basic material written as a basis for the lectures ***The Hardy inequality-prehistory, history and current status*** and ***The interplay between Convexity, Interpolation and Inequalities*** presented at my visit at Collège de France in November 2015 on invitation by Professor Pierre-Louis Lions.

I cordially thank Professor Pierre-Louis Lions and Collège de France for this kind invitation. I also thank Professor Natasha Samko, Luleå University of Technology, for some related late joint research and for helping me to finalize this material.

I hope this material can serve not only as a basis of these lectures but also as a source of inspiration for further research in this area. In particular, a number of open questions are pointed out.

The material is closely connected to the following books:

[1] A. Kufner and L.E. Persson, *Weighted Inequalities of Hardy Type*, World Scientific Publishing Co. Inc., River Edge, NJ, 2003.

[2] A. Kufner, L. Maligranda and L.E. Persson, *The Hardy Inequality. About its History and Some Related Results*, Vydavatelsky Servis Publishing House, Pilsen, 2007.

[3] L. Larsson, L. Maligranda, J. Pečaric and L.E. Persson, *Multiplicative Inequalities of Carlson Type and Interpolation*, World Scientific Publishing Co., New Jersey-London-Singapore-Beijing-Shanghai-Hong Kong-Chennai, 2006.

[4] C. Niculescu and L.E. Persson, *Convex Functions and their Applications- A Contemporary Approach*. Canad. Math. Series Books in Mathematics, Springer. 2006.

[5] V. Kokilashvili, A. Meskhi and L.E. Persson, *Weighted Norm Inequalities for Integral transforms with Product Weights*, Nova Scientific Publishers, Inc., New York, 2010.

But also some newer results and ideas can be found in this Lecture Notes, in particular from the following manuscript:

[6] L.E. Persson and N. Samko, *Classical and New Inequalities via Convexity and Interpolation*, book manuscript, in preparation.

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LECTURE I

The Hardy inequality: Prehistory, history and current status

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1 The prehistory of the Hardy inequality

We consider the following statements of the Hardy inequality: the discrete inequality asserts that if $\{a_n\}_1^\infty$ is a sequence of non-negative real numbers then

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{n=1}^{n}a_i\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty}a_n^p, \quad p>1, \tag{1.1}$$

the continuous inequality informs us that if f is a non-negative p-integrable function on $(0,\infty)$, then f is integrable over the interval $(0,x)$ for each positive x and

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)dy\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx), \quad p>1. \tag{1.2}$$

The development of the famous Hardy inequality in both discrete and continuous forms during the period 1906 to 1928 has its own history or, as we call it, prehistory. Contributions of mathematicians other than G.H.Hardy, such as E.Landau, G.Pòlya, E.Schur and M.Riesz, are important here.

This prehistory was recently described in detail in

$[*]$ A.Kufner, L.Maligranda and L.E.Persson. The prehistory of the Hardy inequality, Amer. Math. Monthly, 113(8):715–732, 2006

In particular, the following is clear:

(a) Inequalities (1.1) and (1.2) are the standard forms of the Hardy inequalities that can be found in many text books on Analysis and were highlighted first in the famous book *Inequalities* by Hardy, Littlewood and Pólya.

(b) By restricting (1.2) to the class of step functions one proves easily that (1.1) implies (1.2).

(c) The constant $(p/(p-1))^p$ in both (1.1) and (1.2) is *sharp*: it cannot be replaced with a smaller number such that (1.1) and (1.2) remain true for all relevant sequences and functions, respectively.

(d) The main motivation for Hardy to begin this dramatic history in 1915 was to find a simpler proof of the Hilbert inequality from 1906:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} \leq \pi \left(\sum_{m=1}^{\infty}a_m^2\right)^{1/2}\left(\sum_{n=1}^{\infty}b_n^2\right)^{1/2} \tag{1.3}$$

(In Hilbert's version of (1.3) the constant 2π appears instead of the sharp one π.) We remark that nowadays the following more general form of (1.3) is also sometimes is referred in the literature as Hilbert's inequality

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} \leq \frac{\pi}{\sin\frac{\pi}{p}} \left(\sum_{m=1}^{\infty}a_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty}b_n^q\right)^{1/q}, \tag{1.4}$$

where $p>1$ and $p'=p/(p-1)$. However, Hilbert was not even close to consider this case (the l_p-spaces appeared only in 1910).

(e) The first weighted version of (1.2) was proved by Hardy himself in 1928:

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)dy\right)^p x^a dx \le \left(\frac{p}{p-1-a}\right)^p \int_0^\infty f^p(x)x^a dx, \tag{1.5}$$

where f is a measurable and non-negative function on $(0,\infty)$ whenever $a < p-1, p > 1$.

1.1 A new look on the inequalities (1.1) and (1.5)

Observation 1.1. We note that for $p > 1$

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)dy\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx,$$

$$\Leftrightarrow$$

$$\int_0^\infty \left(\frac{1}{x}\int_0^x g(y)dy\right)^p \frac{dx}{x} \le 1\cdot \int_0^\infty g^p(x)\frac{dx}{x}, \tag{1.6}$$

where $f(x) = g(x^{1-1/p})x^{-1/p}$.

This means that Hardy's inequality (1.2) is equivalent to (1.6) for $p > 1$ and, thus, that Hardy's inequality can be proved in the following simple way (see form (1.6)): By Jensen's inequality and Fubini's theorem we have that

$$\int_0^\infty \left(\frac{1}{x}\int_0^x g(y)dy\right)^p \frac{dx}{x} \le \int_0^\infty \left(\frac{1}{x}\int_0^x g^p(y)dy\right)\frac{dx}{x} = \int_0^\infty g^p(y)\int_y^\infty \frac{dx}{x^2}dy = \int_0^\infty g^p(y)\frac{dy}{y}.$$

By instead making the substitution $f(t) = g(t^{\frac{p-1-a}{p}})t^{-\frac{1+a}{p}}$ in (1.5) we see that also this inequality is equivalent to (1.6). These facts imply especially the following:

(a) Hardy's inequalities (1.1) and (1.5) hold also for $p < 0$ (because the function $\varphi(u) = u^p$ is convex also for $p < 0$) and hold in the reverse direction for $0 < p < 1$ (with sharp constants $\left(\frac{p}{1-p}\right)^p$ and $\left(\frac{p}{a+1-p}\right)^p$, $a > p-1$, respectively).

(b) The inequalities (1.1) and (1.5) are equivalent.

(c) The inequality (1.6) holds also for $p = 1$ which gives us a possibility to interpolate and get more information about the mapping properties of the Hardy operator.

More information about the development of this idea can be found in

[∗] L.E.Persson and N. Samko, What should have happened if Hardy discovered this?, J. Inequal. Appl. SpringerOpen 2012, 2012:29.

Remark 1.2. In Section 3.1 of this lecture some of these results are presented.

2 On the further development of Hardy type inequalities

Some parts of this development are described in the books:

[A] A.Kufner and L.E.Persson, *Weighted Inequalities of Hardy Type*, World Scientific Publishing Co. Inc., River Edge, NJ, 2003.
[B] A.Kufner, L.Maligranda and L.E.Persson, *The Hardy Inequality. About its History and Some Related Results*, Vydavatelsky Servis Publishing House, Pilsen, 2007.
[C] V.Kokilashvili, A.Meskhi and L.E.Persson, *Weighted Norm Inequalities for Integral Transforms with Product Weights*, Nova Scientific Publishers, Inc., New York, 2010.

Next we present some examples of information from each of the chapters of the book [A].

A1 : Introduction

One important question is the following:
For which weights u and v does it hold that

$$\left(\int_0^b \left(\int_0^x f(t)dt\right)^q u(x)dx\right)^{1/q} \le C\left(\int_0^b f^p(x)v(x)dx\right)^{1/p}, \quad 0 < b \le \infty$$

for some finite constant C ?
During the last 80 years it has been a lot of activities to answer this and more general questions concerning Hardy type inequalities and a lot of interesting results have been developed.
Just as one example we mention the following well known result:

Theorem 2.1. *Let $1 < p \le q < \infty$ and u and v be weight functions on $\mathbb{R}_+$. Then each of the following conditions are necessary and sufficient for the inequality*

$$\left(\int_0^b \left(\int_0^x f(t)\,dt\right)^q u(x)dx\right)^{\frac{1}{q}} \le C\left(\int_0^b f^p(x)v(x)\,dx\right)^{\frac{1}{p}} \tag{2.1}$$

to hold for all positive and measurable functions on $\mathbb{R}_+$:
a) the Muckenhoupt-Bradley-type condition,

$$A_{MB} := \sup_{x>0}\left(\int_x^b u(t)\,dt\right)^{\frac{1}{q}}\left(\int_0^x v(t)^{1-p'}\,dt\right)^{\frac{1}{p'}} < \infty, \tag{2.2}$$

with the estimation $C \in [A_{MB}, \lambda A_{MB}]$ for the best constant C in (2.1)*, where*

$$\lambda = min(p^{1/q}(p')^{1/p'}, q^{1/q}(q')^{1/p'}).$$

b) The condition

$$A_{PS} := \sup_{x>0} V(x)^{-\frac{1}{p}}\left(\int_0^x u(t)V(t)^q\,dt\right)^{\frac{1}{q}} < \infty, \quad V(x) := \int_0^x v(t)^{1-p'}\,dt, \tag{2.3}$$

with $C \in [A_{PS}, p' A_{PS}]$ *for the best constant in* (2.1).

Remark 2.2. The dramatic history until (2.2) was derived can be found in the book [B]. A simple proof of the characterization (2.2) was given by B.Muckenhoupt in 1972 for $p = q$ and by J.S.Bradley in 1978 for $p \leq q$. In 2002 L.E. Persson and V.D. Stepanov presented an elementary proof of the alternative condition (2.3).

Remark 2.3. It has recently been discovered that also these two conditions to characterize (2.1) are not unique and can even be replaced by infinite many conditions even by four scales of conditions. In Section 3.8 of this lecture also this result will be presented.

A2 : Some weighted norm inequalities

Here we study in particular characterizations of the following more general Hardy-type inequality

$$\|Tf\|_{q,u} \leq C\|f\|_{p,v}, \tag{2.4}$$

where u and v are weight functions and

$$Tf(x) := \int_a^x k(x,y)f(y)dx,$$

$k(x,y)$ denote a positive kernel.

Some facts:

(a) Without restrictions on the kernel $k(x,y)$ the problem is open.

(b) The solution of this problem is known for a number of special cases and parameters.

Remark 2.4. The really newest knowledge can be found in the following review article:
$[*]$ A.Kufner, L.E.Persson and N.Samko, Hardy type inequalities with kernels: the current status and some new results, research report (submitted in 2015).
Some of these newer result will be presented in Sections 3.5 and 3.7 of this lecture.

A3 : The Hardy-Steklov operator

Here we consider the Hardy-Steklov operator $\int\limits_{a(x)}^{b(x)}$ and the "moving averaging operator"

$$S_a^b f(x) = \frac{1}{b(x)-a(x)} \int_{a(x)}^{b(x)} f(t)dt,$$

where $a(x)$ and $b(x)$ are increasing functions on $(0,\infty)$ such that $a(0) = b(0) = 0$, and $a(x) < b(x)$.

Special case: The Steklov operator

$$(S_\gamma f)(x) = \frac{1}{2\gamma}\int_{x-\gamma}^{x+\gamma} f(t)dt.$$

The corresponding mapping properties of the operator S_a^b as those for the Hardy operator H are investigated as characterizations of weighted inequalities.

A4 : Higher order Hardy inequalities

Hardy's inequality in modern form can be formulated

$$\left(\int_0^b (g(x))^q\, u(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_0^b (g'(x))^p v(x)\, dx\right)^{\frac{1}{p}}, \tag{2.5}$$

where $g(a) = 0 \;\; (g(x) = \int_0^x f(t)dt)$.

Remark 2.5. This problem can be handled also when the condition "$g(a) = 0$" is replaced by the condition $g(b) = 0$ (the dual situation). It can also be handled for the case when $g(a) = g(b) = 0$ and $g \in C_0^\infty[a, b]$ but not of course without restrictions in the end points.

The inequality (2.5) with appropriate boundary conditions is referred to as a Hardy inequality of first order.

Hardy's inequality of second order:

$$\left(\int_0^b (g(x))^q\, u(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_0^b (g''(x))^p v(x)\, dx\right)^{\frac{1}{p}}.$$

The first crucial question is : Under which conditions on $g(a), g'(a), g(b)$ and $g'(b)$ is it reasonable to study this Hardy-type inequality? And after that study the corresponding Hardy-type inequality in each of these cases.

Hardy's inequality of n:th order:

$$\left(\int_0^b (g(x))^q\, u(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_0^b (g^{(n)}(x))^p v(x)\, dx\right)^{\frac{1}{p}}.$$

Remark 2.6. A number of open questions remains to be solved in this case.

A5 : Fractional order Hardy inequalities

A very guiding result was the following classical Jakovlev- Grisvard inequality:

Example 2.7. Let $1 < p < \infty$, $0 < \lambda < 1$, $\lambda \neq 1/p$. Then for every $g \in C_0^\infty(0, \infty)$,

$$\left(\int_0^\infty \left| \frac{g(x)}{x^\lambda} \right|^p dx \right)^{1/p} \leq C \left(\int_0^\infty \int_0^\infty \frac{|g(x) - g(y)|^p}{|x - y|^{1+\lambda p}} dx dy \right)^{1/p}. \tag{2.6}$$

Remark 2.8. The inequality (2.6) was independently derived in 1961 by G.N.Jakovlev and in 1963 by P.Grisvard but it seems to be known much earlier by J.L.Lions and others.

It is easy to derive the following complement of (2.6):

Example 2.9. Let $1 < p < \infty$, $0 < \lambda < 1$. Then for every $g \in AC(0, \infty)$,

$$\left(\int_0^\infty \int_0^\infty \frac{|g(x) - g(y)|^p}{|x - y|^{1+\lambda p}} dx dy \right)^{1/p} \leq C \left(\int_0^\infty |g'(x)|^p x^{(1-\lambda)p} dx \right)^{1/p} \tag{2.7}$$

where $C = 2^{1/p} \lambda^{-1} (p(1-\lambda))^{-1/p}$ is the best possible constant.

Remark 2.10. If both the inequalities (2.6) and (2.7) hold, we have a refinement of the classical Hardy inequality (2.5) in differential form:

$$\|g\|_{p,u} \leq C \|g'\|_{p,v}$$

with $(a, b) = (0, \infty)$, $u(x) = x^{-\lambda p}$, $v(x) = x^{(1-\lambda)p}$.

Next we present the following more precise version of the Grisvard-Jakovlev inequality:

Theorem 2.11. *Let $1 \leq p < \infty$, $0 < \lambda < 1$, $\lambda \neq 1/p$. Assume that $\int_0^x g(t)dt$ exists for every $x > 0$ and that either*

$$\frac{1}{p} < \lambda < 1 \quad \textit{and} \quad \lim_{x \to 0} \frac{1}{x} \int_0^x g(t) dt = 0$$

or

$$0 < \lambda < \frac{1}{p} \quad \textit{and} \quad \lim_{x \to \infty} \frac{1}{x} \int_0^x g(t) dt = 0.$$

Then

$$\left(\int_0^\infty \left| \frac{g(x)}{x^\lambda} \right|^p dx \right)^{1/p} \leq C_{\lambda,p} \left(\int_0^\infty \int_0^\infty \frac{|g(x) - g(y)|^p}{|x - y|^{1+\lambda p}} dx dy \right)^{1/p}$$

with
$$C_{\lambda,p} = 2^{-1/p}\left(1 + \frac{p}{|\lambda p - 1|}\right).$$

Remark 2.12. This result from 2000 is due to N.Krugljak, L.Maligranda and L.E.Persson and the original motivation for this result was to try to supplement the theory so that the disturbing condition $\lambda \neq 1/p$ in Example 2.7 can be understood from an interpolation point of view. This theory was further developed by the same authors and led to a satisfactory explanation in terms of interpolation between subspaces. This phenomenon appears also for usual power weighted Hardy inequalities. In Section 3.2 we give a simple explanation of this idea.

The key to prove Theorem 2.11 is the following Lemma of independent interest:

Lemma 2.13. *Let $1 \leq p < \infty$, $\alpha \in \mathbb{R} \setminus \{0\}$. Assume that $\int_0^x g(t)dt$ exists for every $x > 0$ and that either*
$$\alpha > 0 \quad \textit{and} \quad \lim_{x \to 0} \frac{1}{x}\int_0^x g(t)dt = 0$$
or
$$\alpha < 0 \quad \textit{and} \quad \lim_{x \to \infty} \frac{1}{x}\int_0^x g(t)dt = 0.$$
Then
$$\left(\int_0^\infty \left|\frac{g(x)}{x^\alpha}\right|^p \frac{dx}{x}\right)^{1/p} \leq C(\alpha)\left(\int_0^\infty \left|\frac{g(x) - \frac{1}{x}\int_0^x g(t)dt}{x^\alpha}\right|^p \frac{dx}{x}\right)^{1/p}$$
with $C(\alpha) = 1 + 1/|\alpha|$.

By combining this Lemma and using the Minkowski and Hardy inequalities in the reversed direction we obtain the following remarkable mapping property of the $I - H$ operator:

Example 2.14. Let $g \in L^p(x^{-\alpha p - 1})$ with $p \geq 1$ and $\alpha > -1$, $\alpha \neq 0$. Then
$$\left(\int_0^\infty \left|\frac{g(x) - \frac{1}{x}\int_0^x g(y)dy}{x^\alpha}\right|^p \frac{dx}{x}\right)^{1/p} \approx \left(\int_0^\infty \left|\frac{g(x)}{x^\alpha}\right|^p \frac{dx}{x}\right)^{1/p}$$
with the equivalence constants $1 + 1/|\alpha|$ and $(\alpha + 1)/(\alpha + 2)$.

Remark 2.15. In fact for the special case $\alpha = -1/2$ and $p = 2$ we can even prove the following more precise isometry of the $I - H$ operator in L^2 :
$$\int_0^\infty \left(g(x) - \frac{1}{x}\int_0^x g(y)dy\right)^2 dx = \int_0^\infty g^2(x)dx.$$

A6 : Integral operators on the cone of monotone functions

Recall that, for $0 < p < \infty$, the weighted Lorentz spaces $\Lambda^p(w)$, $1 < p < \infty$, are defined as follows:

$$\Lambda^p(w) := \Big\{ f \ \|f^*\|_{p,w} = \Big(\int_0^\infty (f^*(t))^p w(t)dt \Big)^{1/p} < \infty \Big\},$$

where f is a measurable function on a measure space X (for example $\mathbb{R}^n$), f^* being the equimeasurable decreasing rearrangement of $|f|$ defined by

$$f^*(t) := \inf\{y > 0 : \lambda_f(y) \leq t\}.$$

Here λ_f is the *distribution function* defined by

$$\lambda_f(y) := \text{meas}\{x \in X : |f(x)| > y\}.$$

Note that $\|f^*\|_{p,w}$ is a norm on $\Lambda^p(w)$ if and only if w is decreasing. But the expression $\|f^{**}\|_{p,w}$ with

$$f^{**}(x) = \frac{1}{x} \int_0^x f^*(t)dt$$

gives a norm which is equivalent to $\|f^*\|_{p,w}$.

In what follows, we take the measure in the definition of $\lambda_f(y)$ to be the *Lebesgue measure*.

Recall that the rearrangement of the Hardy-Littlewood maximal function Mf is equivalent to the Hardy (averaging) operator of the rearrangement of $|f|$. To be more precise, if

$$(Mf)(x) \ = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(z)|dz, \quad x \in \mathbb{R}^n,$$

where Q is a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes and $|Q|$ is its Lebesgue measure, then

$$(Mf)^*(t) \approx \frac{1}{t} \int_0^t f^*(s)ds, \quad t > 0.$$

Hence, to prove that

$$M : \Lambda^p(v) \to \Lambda^q(u), \quad 1 < p, q < \infty,$$

is a *bounded* mapping, or, in other words, to characterize the weight functions u and v for which M is bounded between Lorentz spaces, it is equivalent to prove that the Hardy (averaging) operator H, defined now by

$$(Hf)(t) := \frac{1}{t} \int_0^t f(s)ds, \quad t \geq 0,$$

and considered on the cone of non-negative decreasing functions (notation: $0 \leq f \downarrow$), is bounded from $L^p(v)$ to $L^q(u)$, $1 < p, q < \infty$. This means that one desires to characterize the weight functions u and v for which the (Hardy) inequality

$$\Big(\int_0^\infty \Big(\frac{1}{t} \int_0^t f(s)ds \Big)^q u(t)dt \Big)^{1/q} \leq C \Big(\int_0^\infty f^p(t)v(t)dt \Big)^{1/p}$$

holds whenever $0 \leq f \downarrow$.

A useful duality principle is given by

$$\sup_{f\geq 0} \frac{\int_0^\infty f(x)g(x)dx}{(\int_0^\infty f^p(x)v(x)dx)^{1/p}} = \|g\|_{p',v^{1-p'}}, \tag{2.8}$$

where $p>1,\ p'=p/(p-1)$ and v is a measurable locally integrable weight function. Another key result is the following duality principle of E. Sawyer (from 1990):

Theorem 2.16. *Suppose $1<p<\infty$. Let g,v be non-negative measurable functions on $(0,\infty)$ with v locally integrable. Then*

$$\sup_{0\leq f\downarrow} \frac{\int_0^\infty f(x)g(x)dx}{(\int_0^\infty f^p(x)v(x)dx)^{1/p}}$$

$$\approx \Big(\int_0^\infty \Big(\int_0^x g(t)dt\Big)^{p'} \Big(\int_0^x v(t)dt\Big)^{-p'} v(x)dx\Big)^{1/p'} + \frac{\int_0^\infty g(x)dx}{(\int_0^\infty v(x)dx)^{1/p}}. \tag{2.9}$$

Remark 2.17. Let us consider an operator T defined by

$$(Tf)(x) = \int_0^\infty k(x,y)f(y)dy,$$

where k is a non-negative kernel. In order to characterize the weight functions u and v for which the inequality

$$\Big(\int_0^\infty (Tf)^q(x)u(x)dx\Big)^{1/q} \leq C\Big(\int_0^\infty f^p(x)v(x)dx\Big)^{1/p} \tag{2.10}$$

with $1<p,q<\infty$ holds for all f, $0\leq f\downarrow$, we can use the duality principles (2.8) and (2.9). They show that (2.10) is equivalent to the inequality

$$\Big(\int_0^\infty \Big(\int_0^x (\widetilde{T}g)(t)dt\Big)^{p'} \Big(\int_0^x v(t)dt\Big)^{-p'} v(x)dx\Big)^{1/p'}$$

$$\leq C\Big(\int_0^\infty g^{q'}(x)u^{1-q'}(x)dx\Big)^{1/q'},$$

where $\widetilde{T}$ is the conjugate of T and g is an *arbitrary* non-negative measurable function.

Example 2.18. If $T=H$ is the Hardy averaging operator

$$(Hf)(x) = \frac{1}{x}\int_0^x f(t)dt,$$

then its conjugate is given by

$$(\widetilde{H}g)(y) = \int_y^\infty \frac{g(t)}{t}dt.$$

A simple calculation shows that

$$\int_0^x (\widetilde{H}g)(y)dy = \int_0^x g(t)dt + x\int_x^\infty \frac{g(t)}{t}dt,$$

so that $\int_0^x (\widetilde{H}g)(y)dy$ is essentially the sum of the Hardy operator and the adjoint of the Hardy averaging operator. Therefore, to characterize (2.10) is equivalent to characterize two Hardy-type inequalities from our previous Sections.

In connection to the results presented in this Section there are many open questions. We just mention the following ones:

Open problem 1 Find necessary and sufficient conditions on the weights $u = u(x)$, $0 \le x \le b$, and $v = v(x, y)$, $0 \le x, y \le b$, so that

$$\Big(\int_0^b |g(x)|^p u(x)dx\Big)^{1/p} \le K\Big(\int_0^b\int_0^b \frac{|g(x)-g(y)|^p}{|x-y|^{1+\lambda p}}v(x,y)dxdy\Big)^{1/p}$$

holds for some finite $K > 0$.

Open problem 2 Find necessary and sufficient conditions on the weights $v = v(x)$, $0 \le x \le b$, and $u = u(x, y)$, $0 \le x, y \le b$, so that

$$\Big(\int_0^b\int_0^b \frac{|g(x)-g(y)|^q}{|x-y|^{1+\lambda p}}u(x,y)dxdy\Big)^{1/q} \le K\Big(\int_0^b |g'(x)|^p v(x)dx\Big)^{1/p}$$

holds for some finite $K > 0$.

Open problem 3 Find necessary and sufficient conditions on the (averaging) operator T such that

$$\|g\|_2 = \|(I-T)g\|_2$$

holds for all $g \in L^2 = L^2(0,\infty)$.

Remark 2.19. In Section 3.6 we present some multidimensional inequalities involving kernel type operators and decreasing functions (and with sharp constant in each case).

3 Examples of complementary and newer results

3.1 Further consequences of the new look presented in Section 1.1

For the finite interval case we need the following extention of our basic observation in Section 1.1.

Lemma 3.1. *Let g be a non-negative and measurable function on $(0,\ell), 0<\ell\leq\infty$.*
a) If $p<0$ or $p\geq 1$, then

$$\int_0^\ell \left(\frac{1}{x}\int_0^x g(y)dy\right)^p \frac{dx}{x} \leq 1\cdot \int_0^\ell g^p(x)\left(1-\frac{x}{\ell}\right)\frac{dx}{x}. \tag{3.1}$$

(In the case $p<0$ we assume that $g(x)>0, 0<x\leq\ell$).

b) If $0<p\leq 1$, then (3.1) holds in the reversed direction.

c) The constant $C=1$ is sharp in both a) and b).

By using this Lemma and straightforward calculations the following can be proved:

Theorem 3.2. *Let $0<\ell\leq\infty$, let $p\in\mathbb{R}_+\setminus\{0\}$ and let f be a non-negative function. Then*
a) the inequality

$$\int_0^\ell \left(\frac{1}{x}\int_0^x f(y)dy\right)^p x^a dx \leq \left(\frac{p}{p-1-a}\right)^p \int_0^\ell f^p(x)x^a\left[1-\left(\frac{x}{\ell}\right)^{\frac{p-a-1}{p}}\right]dx \tag{3.2}$$

holds for all measurable functions f, each $\ell, 0<\ell\leq\infty$ and all a in the following cases:

$$(a_1) \qquad p\geq 1, a<p-1,$$

$$(a_2) \qquad p<0, a>p-1.$$

b) For the case $0<p<1, a<p-1$, inequality (3.2) holds in the reversed direction under the conditions considered in a).
c) The inequality

$$\int_\ell^\infty \left(\frac{1}{x}\int_x^\infty f(y)dy\right)^p x^{a_0}dx \leq \left(\frac{p}{a_0+1-p}\right)^p \int_\ell^\infty f^p(x)x^{a_0}\left[1-\left(\frac{\ell}{x}\right)^{\frac{a_0+1-p}{p}}\right]dx \tag{3.3}$$

holds for all measurable functions f, each $\ell, 0\leq\ell<\infty$ and all a in the following cases:

$$(c_1) \qquad p\geq 1, a_0>p-1,$$

$$(c_2) \qquad p < 0, a_0 < p - 1.$$

d) *For the case* $0 < p \leq 1$*, inequality* (3.3) *holds in the reversed direction under the conditions considered in c).*

e) *All inequalities above are sharp.*

f) *Let* $p \geq 1$ *or* $p < 0$*. Then, the statements in a) and c) are equivalent for all permitted* a *and* a_0 *because they are in all cases equivalent to* (3.1) *via substitutions.*

g) *Let* $0 < p < 1$*. Then, the statements in b) and d) are equivalent for all permitted* a *and* a_0*.*

3.2 The failure of the Hardy inequality and interpolation of intersections

There are many examples of inequalities which hold except for one or more values of the parameters involved, Sometimes this phenomenon can be explained via interpolation. We give the following example:

Example 3.3. : The (classical) Hardy inequality in differential form

$$\left(\int_0^\infty \left|\frac{1}{x}g(x)\right|^p x^\beta dx\right)^{1/p} \leq C\left(\int_0^\infty |g'(x)|^p x^\beta dx\right)^{1/p} \tag{3.4}$$

holds with $p \geq 1$ and $g \in C_0^\infty(0,\infty)$ for every $\beta \neq p-1$ but does not hold for $\beta = p-1$.

In order to be able to understand this phenomenon we first note that inequality (3.4) can be rewritten as

$$\left(\int_0^\infty \left|\frac{1}{x}\int_0^x f(y)dy\right|^p x^\beta dx\right)^{1/p} \leq C\left(\int_0^\infty |f(x)|^p x^\beta dx\right)^{1/p} \tag{3.5}$$

for $\beta < p-1$ and as

$$\left(\int_0^\infty \left|\frac{1}{x}\int_x^\infty f(y)dy\right|^p x^\beta dx\right)^{1/p} \leq C\left(\int_0^\infty |f(x)|^p x^\beta dx\right)^{1/p} \tag{3.6}$$

for $\beta > p-1$. When (3.4) is written in this form we see that it is impossible to interpolate between $\beta < p-1$ (see (3.5)) and $\beta > p-1$ (see (3.6)) to obtain the inequality for $\beta = p-1$. The reason is that we have in fact two different operators involved:

$$(Hf)(x) = \frac{1}{x}\int_0^x f(y)dy \ \text{ for } \beta < p-1,$$

$$(\mathcal{H}f)(x) = -\frac{1}{x}\int_x^\infty f(y)dy \ \text{ for } \beta > p-1.$$

in order that these two operators coincide it is necessary that

$$\frac{1}{x}\int_0^x f(y)dy = -\frac{1}{x}\int_x^\infty f(y)dy$$

i.e. that

$$\int_0^\infty f(y)dy = 0.$$

Denote by N the set of locally integrable functions satisfying this condition. In order to interpolate the Hardy operator on weighted Lebesgue spaces we have to consider not the spaces themselves but their intersection with N.

Remark 3.4.: Notice that for weighted Lebesgue spaces the following interpolation formula holds:

$$(L^p(v_1), L^p(v_2))_{\theta,p} = L^p(v_1^{1-\theta} v_2^{\theta}),$$

i.e., the interpolation with two different weight functions v_1, v_2 produces a new weight function $v_\theta = v_1^{1-\theta} v_2^{\theta}$. In the case mentioned in Example 3.3 we need to interpolate between $L^p(v_1) \cap N$ and $L^p(v_2) \cap N$ with $v_1(x) = x^\gamma, \gamma < \beta - 1$, and $v_2(x) = x^\delta, \delta > \beta - 1$.

These observations lead to the investigation of interpolation spaces of the type

$$\left(N \cap L^p(x^\beta), N \cap L^p(x^\gamma)\right)_{\lambda,p}.$$

In our mentioned paper from 2000 (see also the book [A]) we in particular proved the following:

Theorem 3.5. *Let $1 \le p < \infty$ and $\gamma < p - 1 < \beta$. Then*

$$\left(N \cap L^p(x^\beta), N \cap L^p(x^\gamma)\right)_{\lambda,p} = N \cap L^p(x^{(1-\lambda)\beta + \lambda\gamma})$$

if $\lambda \neq (\beta + 1 - p)/(\beta - \gamma)$ and

$$\left(N \cap L^p(x^\beta), N \cap L^p(x^\gamma)\right)_{\lambda,p} = C^p(x^{p-1}) \cap L^p(x^{p-1})$$

if $\lambda = (\beta + 1 - p)/(\beta - \gamma)$.
Here $C^p(v)$ denotes the Cesàro function space of non-absolute type:

$$C^p(v) = \Bigg\{ g(x), x \in (0,\infty) :$$

$$\|g\|_{C^p(v)} := \left(\int_0^\infty \left| \frac{1}{x} \int_0^x g(y)dy \right|^p v(x)dx \right)^{1/p} < \infty \Bigg\}.$$

Finally, let us note that Theorem 3.5 is only a special case of a more general result (with power weights replaced by general weights), which allows to describe situations where some Hardy inequality fails not just for one value of the parameter but even for an interval of parameters. For illustration let us give a concrete example.

Example 3.6. : Let $v_0(x) = \max\{x^{\alpha_0}, x^{\alpha_1}\}$ with $0 \le \alpha_0 \le \alpha_1$ and $v_1(x) = \min\{x^{-\beta_0}, x^{-\beta_1}\}$ with $0 < \beta_0 \le \beta_1$. Assume that $\alpha_0/\alpha_1 \le \beta_0/\beta_1$. Then, for

$$\lambda \in (0,1) \setminus \left[\frac{\alpha_0}{\alpha_0 + \beta_0}, \frac{\alpha_1}{\alpha_1 + \beta_1} \right]$$

and $f \in N$ we have both the Hardy inequalities

$$\left(\int_0^\infty \left|\frac{1}{x}\int_0^x f(y)dy\right|^p v_0^{1-\lambda}(x)v_1^\lambda(x)dx\right)^{1/p} \leq C\left(\int_0^\infty |f(x)|^p\, v_0^{1-\lambda}(x)v_1^\lambda(x)dx\right)^{1/p}$$

and

$$\left(\int_0^\infty \left|\frac{1}{x}\int_x^\infty f(y)dy\right|^p v_0^{1-\lambda}(x)v_1^\lambda(x)dx\right)^{1/p} \leq C\left(\int_0^\infty |f(x)|^p\, v_0^{1-\lambda}(x)v_1^\lambda(x)dx\right)^{1/p}.$$

It follows that

$$\left(N\cap L^1(v_0), N\cap L^1(v_1)\right)_{\lambda,1} = N\cap L^1(v_0^{1-\lambda}v_1^\lambda).$$

Moreover, for $\lambda \in \left[\frac{\alpha_0}{\alpha_0+\beta_0}, \frac{\alpha_1}{\alpha_1+\beta_1}\right]$ none of these Hardy inequalities is true and we only have that

$$\left(N\cap L^1(v_0), N\cap L^1(v_1)\right)_{\lambda,1} = N\cap C^1(v_0^{1-\lambda}v_1^\lambda)\cap L^1(v_0^{1-\lambda}v_1^\lambda).$$

Remark 3.7.: In the same paper also the exception $\lambda \neq 1/p$ in the Jakovlev-Grisvard inequality (2.6) was explained in a similar way from an interpolation point of view.

3.3 A further development of Bennett's inequalities with two sharp constants

There exists very few Hardy type inequalities with sharp constant in the limit case and when the interval $(0,\infty)$ is replaced by a finite interval $(0,\ell), \ell < \infty$. We continue by giving two such examples (Bennett's inequalities from 1973), which have direct applications e.g. to Interpolation Theory.

Proposition A: Let $\alpha > 0, 1 \leq p \leq \infty$ and f be a non-negative and measurable function on $[0,1]$. Then

$$\left(\int_0^1 [\log(e/x)]^{\alpha p-1}\left(\int_0^x f(y)dy\right)^p \frac{dx}{x}\right)^{1/p} \leq$$
$$\alpha^{-1}\left(\int_0^1 x^p[\log(e/x)]^{(1+\alpha)p-1} f^p(x)\frac{dx}{x}\right)^{1/p}, \tag{3.7}$$

and

$$\left(\int_0^1 [\log(e/x)]^{-\alpha p-1}\left(\int_x^1 f(y)dy\right)^p \frac{dx}{x}\right)^{1/p} \leq$$
$$\alpha^{-1}\left(\int_0^1 x^p[\log(e/x)]^{(1-\alpha)p-1} f^p(x)\frac{dx}{x}\right)^{1/p} \tag{3.8}$$

with the usual modification if $p=\infty$.

The next refinements of the inequalities (3.7) and (3.8) in Proposition A was proved in 2014 by S. Barza, L.E. Persson and N. Samko:

Theorem 3.8. *Let $\alpha, p > 0$ and f be a non-negative and measurable function on $[0,1]$.*
(a) *If $p > 1$, then*

$$\alpha^{p-1}\left(\int_0^1 f(x)dx\right)^p + \alpha^p\int_0^1 [\log(e/x)]^{\alpha p-1}\left(\int_0^x f(y)dy\right)^p \frac{dx}{x} \leq \leq \int_0^1 x^p[\log(e/x)]^{(1+\alpha)p-1} f^p(x)\frac{dx}{x} \tag{3.9}$$

and

$$\alpha^{p-1}\left(\int_0^1 f(x)dx\right)^p + \alpha^p\int_0^1 [\log(e/x)]^{-\alpha p-1}\left(\int_x^1 f(y)dy\right)^p \frac{dx}{x} \leq \leq \int_0^1 x^p[\log(e/x)]^{(1-\alpha)p-1} f^p(x)\frac{dx}{x}. \tag{3.10}$$

Both constants α^{p-1} and α^p in (3.9) *and* (3.10) *are sharp. Equality is never attained unless f is identically zero.*
(b) *If $0 < p < 1$, then both* (3.9) *and* (3.10) *hold in the reverse direction and the constants in both inequalities are sharp. Equality is never attained unless f is identically zero.*
(c) *If $p = 1$ we have equality in* (3.9) *and* (3.10) *for any measurable function f and any $\alpha > 0$.*

3.4 The sharp constant for the power weighted case when $1 < p < q$

By applying the general results (see Theorem 2.1 and the corresponding dual result) for the power weighted case we get the following:

Example 3.9. The inequality

$$\left(\int_0^\infty\left(\int_0^x f(t)dt\right)^q x^\alpha dx\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty f^p(x)x^\beta dx\right)^{\frac{1}{p}}$$

holds for $1 < p \leq q < \infty$, if and only if

$$\beta < p-1 \quad \text{and} \quad \frac{\alpha+1}{q} = \frac{\beta+1}{p} - 1.$$

Example 3.10. The inequality

$$\left(\int_0^\infty \left(\int_x^\infty f(t)dt\right)^q x^\alpha dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty f^p(x)x^\beta dx\right)^{\frac{1}{p}}$$

holds for $1 < p \le q < \infty$, if and only if

$$\beta > p-1 \quad \text{and} \quad \frac{\alpha+1}{q} = \frac{\beta+1}{p} - 1.$$

For the next result we need the following Lemma.

Lemma 3.11. *Let $1 < p < q < \infty$. The following statements (a) and (b) hold and are equivalent:*
(a) The inequality

$$\left(\int_0^\infty \left(\int_0^x f(t)dt\right)^q x^\alpha dx\right)^{1/q} \le C\left(\int_0^x f^p(x)x^\beta dx\right)^{1/p} \tag{3.11}$$

holds for all measurable functions $f(t)$ on $(0,\infty)$ if and only if

$$\beta < p-1 \text{ and } \frac{\alpha+1}{q} = \frac{\beta+1}{p} - 1. \tag{3.12}$$

(b) The inequality

$$\left(\int_0^\infty \left(\int_x^\infty f(t)dt\right)^q x^{\alpha_0} dx\right)^{1/q} \le C\left(\int_0^\infty f^p(x)x^{\beta_0} dx\right)^{1/p} \tag{3.13}$$

holds for all measurable functions $f(t)$ on $(0,\infty)$ if and only if

$$\beta_0 > p-1, \frac{\alpha_0+1}{q} = \frac{\beta_0+1}{p} - 1. \tag{3.14}$$

Moreover, it yields that
(c) the formal relation between the parameters β and β_0 is $\beta_0 = -\beta - 2 + 2p$ and in this case the best constants C in (3.11) *and* (3.13) *are the same.*

The next result was recently proved in 2015 by L.E.Persson and S.Samko and thus finally an old question, where G.A.Bliss in 1930 found the best constant for the case $\beta = 0$ in (3.11), was finally solved.

Theorem 3.12. *Let $1 < p < q < \infty$ and the parameters α and β satisfy* (3.12). *Then the sharp constant in* (3.11) *is $C = C^*_{pq}$, where*

$$C^*_{pq} = \left(\frac{p-1}{p-1-\beta}\right)^{\frac{1}{p'}+\frac{1}{q}} \left(\frac{p'}{q}\right)^{\frac{1}{p}} \left(\frac{\frac{q-p}{p}\Gamma\left(\frac{pq}{q-p}\right)}{\Gamma\left(\frac{p}{q-p}\right)\Gamma\left(\frac{p(q-1)}{q-p}\right)}\right)^{\frac{1}{p}-\frac{1}{q}}. \tag{3.15}$$

Equality in (3.11) *occurs exactly when* $f(x) = \frac{cx^{-\frac{\beta}{p-1}}}{\left(dx^{\frac{p-1-\beta}{p-1}\cdot\left(\frac{q}{p}-1\right)}+1\right)^{\frac{q}{q-p}}}$. *Moreover,*

$$C^*_{pq} \to \frac{p}{p-1-\beta} \text{ as } q \to p.$$

By using this result and Lemma 3.11 we obtain the following sharp constant in (3.13):

Theorem 3.13. *The sharp constant in* (3.13) *with parameters satisfying* (3.14) *for the case* $1 < p < q < \infty$ *is* $C^\sharp_{p,q}$, *where* $C^\sharp_{p,q}$ *coincides with the constant* $C^*_{p,q}$ *with* β *replaced by* $-\beta_0 - 2 + 2p$.
Equality in (3.13) *occurs if and only if* $f(x)$ *is of the form*

$$f(x) = \frac{cx^{\beta_0/p-1}}{(dx^{(\frac{\beta_0+1-p}{p-1})(\frac{q}{p}-1)}+1)^{\frac{q}{q-p}}} \text{ a.e..}$$

Moreover, we have the continuity between sharp constants when $q \to p$, *i.e.*

$$C^\sharp_{p,q} \to \frac{p}{\beta_0+1-p} \text{ as } q \to p.$$

Remark 3.14. In the same paper also the sharp constants in the corresponding multi-dimensional Hardy type inequalities were derived.

3.5 Another new result for the kernel operator case

The following new result was recently proved for the general kernel operator case (in the previously mentioned review article by A.Kufner, L.E.Persson and N.Samko from 2015):

Theorem 3.15. *Let* $1 < p \le q < \infty$, $a < b \le \infty$, u *and* v *are weights. Let* $k(x,y)$ *be a non-negative kernel.*
(a) *Then* (2.4) *holds if*

$$A_s := \sup_{a<y<b} \left(\int_y^b k^q(x,y)u(x)V^{(\frac{q(p-s-1)}{p})}(x)dx \right)^{1/q} V^{s/p}(y) < \infty, \tag{3.16}$$

for any $s < p-1$.
(b) *The condition* (3.16) *can not be improved in general for* $s > 0$ *because for product kernels it is even necessary and sufficient for* (2.4) *to hold.*
(c) *For the best constant* C *in* (2.4) *we have the following estimate*

$$C \le \inf_{s<p-1} \left(\frac{p}{p-s-1} \right)^{1/p'} A_s.$$

Here and the sequel we use the following notations

$$U(x) := \int_x^b u(y)dy, \qquad V(x) := \int_a^x v^{1-p'}(y)dy, \tag{3.17}$$

Remark 3.16. This result opens a possibility that the condition (3.16) can be a candidate to solve the open question we have pointed out in Section **A2**.

Remark 3.17. In Section 3.6 we present some multidimensional inequalities involving kernel type operators and decreasing functions (and with sharp constant in each case).

3.6 Some multidimensional weighted inequalities involving generalized kernel operators and non-increasing functions

Let $(\mathcal{M}_j, \mu_j)$, $j = 1, 2$, denote two σ-finite measure spaces. Further, for every $x \in \mathcal{M}_j$ let $d\sigma_j^x(y)$ denote a positive measure on $\mathbb{R}_+^n$ and define T_j by

$$(T_j f)(x) := \int\limits_{\mathbb{R}_+^n} f(y) d\sigma_j^x(y), \quad j = 1, 2.$$

In the next Theorem by S.Barza, L.E.Persson and S.Soria from 2000 we give sharp estimates of the type

$$\Big(\int_{\mathcal{M}_1} (T_1 f)^q(x) d\mu_1(x) \Big)^{1/q} \le C \Big(\int_{\mathcal{M}_2} (T_2 f)^p(x) d\mu_2(x) \Big)^{1/p}.$$

Definition 3.18. We say that a set $D \in \mathbb{R}_+^n$ is decreasing (and write $D \in \Delta_d$) if the function χ_D is decreasing (separately in each variable). Similarly, we say that a set $I \in \mathbb{R}_+^n$ is increasing (and write $I \in \Delta_i$) if the function χ_I is increasing (in each variable).

We need the following constant:

$$C_n = C_n(p, q, T_1, T_2) = \sup_{D \in \Delta_d} \frac{(\int_{\mathcal{M}_1} (T_1 \chi_D)^q(x) d\mu_1(x))^{1/q}}{(\int_{\mathcal{M}_2} (T_2 \chi_D)^p(x) d\mu_2(x))^{1/p}}. \tag{3.18}$$

Theorem 3.19. *Let C_n be defined by* (3.18) *and let f be non-negative and decreasing.*
(i) *Let $0 < p \le 1 < q < \infty$. Then the inequality*

$$\Big(\int_{\mathcal{M}_1} (T_1 f)^q(x) d\mu_1(x) \Big)^{1/q} \le C \Big(\int_{\mathcal{M}_2} (T_2 f)^p(x) d\mu_2(x) \Big)^{1/p}$$

holds with $C > 0$ independent of f if and only if $C_n < \infty$.
(ii) *Let $0 < \max(1, p) \leq q < \infty$ and $T_1 = I_d$ (identity operator). Then the inequality*

$$\Big(\int_0^\infty f^q(x) d\mu_1(x) \Big)^{1/q} \leq C \Big(\int_{\mathcal{M}_2} (T_2 f)^p(x) d\mu_2(x) \Big)^{1/p}$$

holds with $C > 0$ independent of f if and only if $C_n < \infty$.
(iii) *Let $0 < p \leq \min(1, q) < \infty$ and $T_2 = I_d$. Then the inequality*

$$\Big(\int_{\mathcal{M}_1} (T_1 f)^q(x) d\mu_1(x) \Big)^{1/q} \leq C \Big(\int_0^\infty f^p(x) d\mu_2(x) \Big)^{1/p}$$

holds with $C > 0$ independent of f if and only if $C_n < \infty$.
(iv) *Let $0 < p \leq q < \infty$ and $T_1 = T_2 = I_d$. Then the inequality*

$$\Big(\int_0^\infty f^q(x) d\mu_1(x) \Big)^{1/q} \leq C \Big(\int_0^\infty f^p(x) d\mu_2(x) \Big)^{1/p}$$

holds with $C > 0$ independent of f if and only if $C_n < \infty$.
(v) *In all cases, $C = C_n$ is the sharp constant.*

3.7 Some sharp inequalities for multidimensional integral operators with homogeneous kernel

We consider the inequality

$$\left(\int_{\mathbb{R}^n} |\mathbf{K} f(x)|^p \, dx \right)^{\frac{1}{p}} \leq C_{k,p} \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{\frac{1}{p}}, \quad 1 \leq p \leq \infty, \tag{3.19}$$

for multidimensional integral operators

$$\mathbf{K} f(x) := \int_{\mathbb{R}^n} k(x, y) f(y) \, dy \tag{3.20}$$

with a kernel $k(x, y)$.

We assume the following:
1^0. the kernel $k(x, y)$ is *homogeneous of degree $-n$*, i.e.

$$k(tx, ty) = t^{-n} k(x, y), \qquad t > 0, \quad x, y \in \mathbb{R}^n, \tag{3.21}$$

2^0. it is *invariant with respect to rotations*, i.e.

$$k[\omega(x), \omega(y)] = k(x, y), \qquad x, y \in \mathbb{R}^n \tag{3.22}$$

for all rotation $\omega(x)$ in $\mathbb{R}^n$. Let

$$\varkappa_p = \int_{\mathbb{R}^n} |k(\sigma, y)| \frac{dy}{|y|^{\frac{n}{p}}}, \quad \sigma \in \mathbb{S}^{n-1}, \tag{3.23}$$

where $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$. In the sequel

$$|\mathbb{S}^{n-1}| = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$$

denotes its surface measure.

The following result was proved in 2015 by D.Lukkassen, L.E.Persson, S.Samko and P.Wall:

Theorem 3.20. *Let $1 \le p \le \infty$ and $\varkappa_p$ be defined by* (3.23)*. Moreover, let the kernel $k(x,y)$ satisfy the assumptions (3.21)-(3.22). If*

$$\varkappa_p < \infty,$$

then the inequality (3.19) *(with $\mathbf{K}f(x)$ defined by* (3.20)*) holds with $C(k,p) = \varkappa_p$. If $k(x,y) \ge 0$, then the condition $\varkappa_p < \infty$ is also necessary for* (3.19) *to hold and $\varkappa_p$ is the sharp constant.*

The following multidimensional Hilbert type inequality is obtained from Theorem 3.20 by just calculating the integral in (3.23).

Example 3.21. (Hilbert type inequality) Let $\lambda > 0, \alpha > 0$ and $1 \le p < \infty$. Then

$$\int\limits_{\mathbb{R}^n} \left| |x|^{\beta+\lambda\alpha-n} \int\limits_{\mathbb{R}^n} \frac{f(y)\,dy}{|y|^{\beta}\left(|x|^{\lambda}+|y|^{\lambda}\right)^{\alpha}} \right|^p dx \le \varkappa_{p,\beta} \int\limits_{\mathbb{R}^n} |f(x)|^p dx$$

holds if and only if $\beta < \frac{n}{p'}$ and $\alpha\lambda > \frac{n}{p'} - \beta$ and

$$\varkappa_{p,\beta} = \int\limits_{\mathbb{R}^n} \frac{|y|^{-\beta--\frac{n}{p}}dy}{(1+|y|^{\lambda})^{\alpha}} = \frac{|\mathbb{S}^{n-1}|}{\lambda} \int\limits_0^{\infty} \frac{\varrho^{\frac{1}{\lambda}\left(\frac{n}{p'}-\beta\right)-1}}{(1+\varrho)^{\alpha}} = \frac{|\mathbb{S}^{n-1}|}{\lambda} B\left(\frac{1}{\lambda}\left(\frac{n}{p'}-\beta\right), \alpha - \frac{1}{\lambda}\left(\frac{n}{p'}-\beta\right)\right)$$

is the sharp constant.

Our next interest is the best constant in the following multidimensional Hardy inequalities with power weights:

$$\left\| |x|^{\alpha-n} \int\limits_{|y|<|x|} \frac{f(y)\,dy}{|y|^{\alpha}} \right\|_{L^p(\mathbb{R}^n)} \le C_1(p,\alpha)\|f\|_{L^p(\mathbb{R}^n)}, \quad 1 \le p < \infty, \;\; \alpha < \frac{n}{p'}, \tag{3.24}$$

$$\left\| |x|^{\beta-n} \int\limits_{|y|>|x|} \frac{f(y)\,dy}{|y|^{\beta}} \right\|_{L^p(\mathbb{R}^n)} \le C_2(p,\beta)\|f\|_{L^p(\mathbb{R}^n)}, \quad 1 \le p < \infty, \;\; \beta > \frac{n}{p'}. \tag{3.25}$$

The best constant for (3.24) was calculated before only in the non-weighted case $\alpha = 0$, where it was shown that

$$C_1(p, 0) = |B(0, 1)|p',$$

where $|B(0, 1)| = \frac{|\mathbb{S}^{n-1}|}{n} = \frac{2\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})}$ is the volume of the unit ball. The weighted case with general weights was studied before, but by using Theorem 3.20 the sharp constant can never be obtained. However, by applying our result we obtain the following:

Proposition 3.22. *The sharp constants for* (3.24) *and* (3.25) *are given by*

$$C_1(p, \alpha) = \frac{|\mathbb{S}^{n-1}|}{\frac{n}{p'} - \alpha}, \quad \text{resp.} \quad C_2(p, \beta) = \frac{|\mathbb{S}^{n-1}|}{\beta - \frac{n}{p'}}.$$

Remark 3.23. The result in Theorem 3.20 can be used to obtain most of the results which in the literature are called Hardy-Hilbert-type inequalities both in one- and multi-dimensional cases.

3.8 Some new scales of conditions to characterize the modern forms of Hardy's inequality

We have recently proved that the conditions $A_{MB} < \infty$ and $A_{PS} < \infty$ in Theorem 2.1 can be replaced by infinite many equivalent conditions even by scales of conditions as presented below. We refer to a review article by A.Kufner, L.E.Persson and N.Samko from 2013, and references therein.

Theorem 3.24. *Let* $1 < p \le q < \infty$, $0 < s < \infty$, *and define, for the weight functions* u, v, *the functions* U *and* V *by* (3.17). *Then* (2.1) *can be characterized by any of the conditions* $A_i(s) < \infty$, *where* $A_i(s), i = 1, 2, 3, 4$ *are defined by:*

$$A_1(s) := \sup_{0<x<b} \left(\int_x^b u(t) V^{q(\frac{1}{p'} - s)}(t) dt \right)^{1/q} V^s(x);$$

$$A_2(s) := \sup_{0<x<b} \left(\int_0^x v^{1-p'}(t) U^{p'(\frac{1}{q} - s)}(t) dt \right)^{1/p'} U^s(x);$$

$$A_3(s) := \sup_{0<x<b} \left(\int_0^x u(t) V^{q(\frac{1}{p'} + s)}(t) dt \right)^{1/q} V^{-s}(x);$$

$$A_4(s) := \sup_{0<x<b} \left(\int_x^b v^{1-p'}(t) U^{p'(\frac{1}{q} + s)}(t) dt \right)^{1/p'} U^{-s}(x).$$

Remark 3.25. Note that

$$A_{MB} = A_1\left(\frac{1}{p'}\right), \quad A_{PS} = A_3\left(\frac{1}{p}\right).$$

Also all other known alternative conditions are just points on these cases.

The main result for the case $1 < q < p < \infty$ (Theorem 3.26) is taken from a paper 2007 by L.E. Persson, V. Stepanov and P. Wall. For simplicity we here only consider the case $b = \infty$.

Let $1/r := 1/q - 1/p$. We now introduce the following scales of constants related to previous constants and their dual ones:

For $s > 0$ we define the following functionals:

$$B_{MR}^{(1)}(s) := \left(\int_0^\infty \left[\int_t^\infty uV^{q(1/p'-s)}\right]^{r/p} V^{q(1/p'-s)+rs}(t)u(t)\,dt\right)^{1/r},$$

$$B_{PS}^{(1)}(s) := \left(\int_0^\infty \left[\int_0^t uV^{q(1/p'+s)}\right]^{r/p} u(t)V^{q(1/p'+s)-sr}(t)\,dt\right)^{1/r},$$

$$B_{MR}^{(2)}(s) := \left(\int_0^\infty \left[\int_0^t U^{p'(1/q-s)}dV\right]^{r/p'} U^{rs-1}(t)u(t)\,dt\right)^{1/r},$$

$$B_{PS}^{(2)}(s) := \left(\int_0^\infty \left[\int_t^\infty U^{q(1/p'+s)}dV\right]^{r/p} U^{q(1/p'+s)-rs}(t)\,dV(t)\right)^{1/r}.$$

The main theorem in this case reads:

Theorem 3.26. *a) Let $0 < q < p < \infty$, $1 < p < \infty$ and $q \neq 1$. Then the Hardy inequality* (2.4) *with $b = \infty$ holds for some finite constant $C > 0$ if and only if any of the constants $B_{MR}^{(1)}(s)$ or $B_{PS}^{(1)}(s)$ is finite for some $s > 0$. Moreover, for the best constant C in* (2.4) *we have*

$$C \approx B_{MR}^{(1)}(s) \approx B_{PS}^{(1)}(s).$$

b) Let $1 < q < p < \infty$. Then the Hardy inequality (2.4) *with $b = \infty$ holds for some finite constant $C > 0$ if and only if any of the constants $B_{MR}^{(2)}(s)$ or $B_{PS}^{(2)}(s)$ is finite for some $s > 0$. Moreover, for the best constant C in* (2.4) *we have*

$$C \approx B_{MR}^{(2)}(s) \approx B_{PS}^{(2)}(s).$$

Remark 3.27. Note that Theorem 3.26 is a generalization of the original results of Maz'ya-Rozin in 70:th and Persson-Stepanov from 2002 since $B_{MR}^{(1)}(\frac{1}{p'}) = B_{MR}$ and $B_{PS}^{(1)}(\frac{1}{p}) = B_{PS}$.

3.9 More on multidimensional Hardy-type inequalities

In this Section by a weight we mean a non-negative, measurable and locally integrable function on $\mathbb{R}^n_+$, $n \in \mathbb{Z}$. The main information in this section can be found in recent papers by L.E.Persson and his students A.Wedestig (PhD 2004) and E.Ushakova (PhD 2006). We refer to the book [C] and the review article by L.E.Persson and N.Samko from 2011, where also complementary information can be found.

Some two-dimensional results

We first recall that the following two-dimensional inequality, which was proved by E.T. Sawyer in 1985:

Theorem 3.28. *Let $1 < p \leq q < \infty$ and u and v be weights on $\mathbb{R}^2_+$. Then the inequality*

$$\left(\int_0^\infty\int_0^\infty\left(\int_0^{x_1}\int_0^{x_2} f(t_1,t_2)dt_1dt_2\right)^q u(x_1,x_2)\,dx_1dx_2\right)^{\frac{1}{q}} \leq$$

$$\leq C\left(\int_0^\infty\int_0^\infty f^p(x_1,x_2)v(x_1,x_2)\,dx_1dx_2\right)^{\frac{1}{p}} \tag{3.26}$$

holds for all non-negative and measurable functions on $\mathbb{R}^2_+$, if and only if the following three conditions are satisfied:

$$\sup_{(y_1,y_2)\in\mathbb{R}^2_+}\left(\int_{y_1}^\infty\int_{y_2}^\infty u(x_1,x_2)dx_1dx_2\right)^{\frac{1}{q}}\left(\int_0^{y_1}\int_0^{y_2} v(x_1,x_2)^{1-p'}dx_1dx_2\right)^{\frac{1}{p'}} < \infty, \tag{3.27}$$

$$\sup_{(y_1,y_2)\in\mathbb{R}^2_+}\frac{\left(\int_0^{y_1}\int_0^{y_2}\left(\int_0^{x_1}\int_0^{x_2} v(t_1,t_2)^{1-p'}dt_1dt_2\right)^q u(x_1,x_2)dx_1dx_2\right)^{\frac{1}{q}}}{\left(\int_0^{y_1}\int_0^{y_2} v(x_1,x_2)^{1-p'}dx_1dx_2\right)^{\frac{1}{p}}} < \infty, \tag{3.28}$$

$$\sup_{(y_1,y_2)\in\mathbb{R}^2_+}\frac{\left(\int_{y_1}^\infty\int_{y_2}^\infty\left(\int_{x_1}^\infty\int_{x_2}^\infty u(t_1,t_2)dt_1dt_2\right)^{p'} v(x_1,x_2)^{1-p'}dx_1dx_2\right)^{\frac{1}{q}}}{\left(\int_{y_1}^\infty\int_{y_2}^\infty u(x_1,x_2)dx_1dx_2\right)^{\frac{1}{q'}}} < \infty. \tag{3.29}$$

All three conditions (3.27)-(3.29) are independent and no one may be removed.

Remark 3.29. Note that (3.27) corresponds to the Muckenhoupt-Bradley condition (2.2), (3.33) corresponds to the condition (2.3) and (3.29) corresponds to the dual condition of (2.3).

According to Theorem 3.24 and Remark 3.25 all these conditions are equivalent in the one-dimensional case but it is not so in the two-dimensional case.

One of the recent progresses related to Theorem 3.28 was obtained in A. Wedestig's PhD thesis from 2004. It was shown there that in the case where the weight $v(x_1, x_2)$ on the right-hand side of (3.26) has the form of the product $v_1(x_1)v_2(x_2)$, then only one condition appears (but this condition is not unique and can in fact be given in infinite many forms). Namely, the following statement holds:

Theorem 3.30. *Let $1 < p \leq q < \infty$ and let u be a weight on $\mathbb{R}^2_+$ and v_1 and v_2 be weights on $\mathbb{R}_+$. Then the inequality*

$$\left(\int_0^\infty\int_0^\infty\left(\int_0^{x_1}\int_0^{x_2} f(t_1,t_2)dt_1dt_2\right)^q u(x_1,x_2)\,dx_1dx_2\right)^{\frac{1}{q}} \leq$$

$$\leq C\left(\int_0^\infty\int_0^\infty f^p(x_1,x_2)v_1(x_1)v_2(x_2)\,dx_1dx_2\right)^{\frac{1}{p}} \tag{3.30}$$

holds for all non-negative and measurable functions f on $\mathbb{R}^2_+$, if and only if

$$A_W(s_1,s_2) := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} (V_1(t_1))^{\frac{s_1-1}{p}} (V_2(t_2))^{\frac{s_2-1}{p}} \times$$

$$\left(\int_{t_1}^\infty\int_{t_2}^\infty u(x_1,x_2)(V_1(x_1))^{q\frac{p-s_1}{p}}(V_2(x_2))^{q\frac{p-s_2}{p}}\,dx_1dx_2\right)^{\frac{1}{q}} < \infty$$

holds for some $s_1, s_2 \in (1,p)$ (and, hence, for all $s_1, s_2 \in (1,p)$), where $V_i(t_i) := \int_0^{t_i} v_i(\xi)^{1-p'}d\xi,\ i = 1,2$. Moreover, for the best constant C in (3.30) it yields that $C \approx A_W(s_1,s_2)$.

A limit result of Theorem 3.30 is the following two-dimensional Pólya-Knopp type inequality, which was also proved in the same PhD thesis:

Theorem 3.31. *Let $0 < p \leq q < \infty$ and u and v be weights on $\mathbb{R}^2_+$. Then the inequality*

$$\left(\int_0^\infty\int_0^\infty\left[\exp\left(\frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2}\log f(t_1,t_2)dt_1dt_2\right)\right]^q u(x_1,x_2)dx_1dx_2\right)^{\frac{1}{q}}$$

$$\leq C\left(\int_0^\infty\int_0^\infty f^p(x_1,x_2)v(x_1,x_2)dx_1dx_2\right)^{\frac{1}{p}} \tag{3.31}$$

holds for all non-negative and measurable functions f on $\mathbb{R}_+^2$ if and only if

$$\sup_{y_1>0,y_2>0} y_1^{\frac{s_1-1}{p}} y_2^{\frac{s_2-1}{p}} \left(\int_{y_1}^{\infty} \int_{y_2}^{\infty} x_1^{-\frac{s_1 q}{p}} x_2^{-\frac{s_2 q}{p}} w(x_1,x_2)dx_1dx_2 \right)^{\frac{1}{q}} < \infty,$$

holds for some $s_1 > 1, s_2 > 1$ (and thus for all $s_1 > 1, s_2 > 1$) and where

$$w(x_1,x_2) := u(x_1,x_2) \left[\exp \left(\frac{1}{x_1x_2} \int_0^{x_1} \int_0^{x_2} \log \frac{1}{v(t_1,t_2)} dt_1dt_2 \right) \right]^{\frac{q}{p}}.$$

Remark 3.32. Observe that this limit inequality indeed holds for all weights (and not only for product weights on the right hand side) and also for $0 < p \leq 1$. The reason for this comes from the useful technical details when we perform the limit procedure, e.g. that we first do a substitution so we only need to use the case when the weight in the right hand side in (3.30) is equal to 1. Also here we have a good estimate of the best constant C in (3.31).

Remark 3.33. The corresponding statements as those in Theorems 3.30 and 3.31 hold also for any dimension n. However, in our next Subsection we will present some results mainly from the PhD thesis of E.Ushakova from 2006, where also the case with product weights on the left hand side was considered. The proofs there are completely different from those before and the obtained characterizations are different.

Some more multidimensional results

In the sequel we assume that f is a non-negative and measurable function.

Let $x = (x_1, ..., x_n), t = (t_1, ..., t_n) \in \mathbb{R}_+^n, n \in Z_+$ and $1 < p \leq q < \infty$. We consider the $n-$ dimensional Hardy type operator

$$(H_n f)(x) = \int_0^{x_1} \cdots \int_0^{x_n} f(t)dt$$

and study the inequality

$$\left(\int_{\mathbb{R}_+^n} (H_n f)^q(x)u(x)dx \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}_+^n} f^p(x)v(x)dx \right)^{\frac{1}{p}}. \tag{3.32}$$

Sometimes we assume that one of the involved weight functions v and u is of product type, i.e. that

$$u(x) = u_1(x_1)u_2(x_2)\cdots u_n(x_n), \qquad (LP)$$

or

$$v(x) = v_1(x_1)v_2(x_2)\cdots v_n(x_n). \qquad (RP)$$

Moreover,

$$U(t) = U(t_1, \cdot\cdot\cdot, t_n) := \int_{t_1}^{\infty} \cdots \int_{t_n}^{\infty} u(x)dx$$

and

$$V(t) = V(t_1, \cdot\cdot\cdot, t_n) := \int_0^{t_1} \cdots \int_0^{t_n} (v(x))^{1-p'}dx.$$

The next Statement gives a necessary condition for (3.32) to hold with help of some $n-$dimensional versions of the constants A_{MB} and A_{PS} in Theorem 2.1.

Theorem 3.34. *Let* $1 < p \leq q < \infty$ *and assume that* (3.32) *holds for all non-negative and measurable functions* f *on* $\mathbb{R}_+^n$ *with a finite constant* C*, which is independent on* f*. Then*

$$A_{MB}^{(n)} := \sup_{t_i>0} (U(t_1, \cdots, t_n))^{1/q}(V(t_1, \cdot\cdot\cdot, t_n))^{1/p'} < \infty,$$

and

$$A_{PS}^{(n)} := \sup_{t_i>o} (V(t_1, \cdots, t_n))^{-1/p} \left(\int_{t_1}^{\infty} \cdots \int_{t_n}^{\infty} u(x)V^q(x)dx \right)^{1/q} < \infty.$$

Our next result is that in the case of product weights on the right hand side we get a complete characterization of (3.32).

Theorem 3.35. *Let* $1 < p \leq q < \infty$ *and the weight* v *be of product type* (RP)*. Then* (3.32) *holds for all non-negative and measurable functions* f *on* $\mathbb{R}_+^n$ *with some finite constant* C*, which is independent on* f*, if and only if* $A_{MB}^{(n)} < \infty$ *or* $A_{PS}^{(n)} < \infty$*. Moreover,* $C \approx A_{MB}^{(n)} \approx A_{PS}^{(n)}$ *with constants of equivalence only depending on the parameters* p *and* q *and the dimension* n*.*

Note that here it yields that $V(t_1, \cdot\cdot\cdot, t_n) = V_1(x_1)V_2(x_2) \cdots V_n(x_n)$, where $V_i(t_i) := \int_0^{t_i} (v_i(x_i))^{1-p'}dx_i, i = 1, \cdot\cdot\cdot, n$. For a proof we refer to the mentioned PhD thesis (see also the book [C]).

We can also consider the case when u is of product type (LP) and where we need the dual of the constants $A_{MB}^{(n)}$ and $A_{PS}^{(n)}$:

$$A_{MB}^{*(n)} := \sup_{t_i>0} \big(U_1(t_1) \cdots U_n(t_n)\big)^{1/q} \big(V(t_1, \cdots, t_n)\big)^{1/p'} < \infty,$$

and

$$A_{PS}^{*(n)} := \sup_{t_i>0} \big(U_1(t_1) \cdots$$

$$\cdots U_n(t_n)\big)^{-1/q'} \left(\int_{t_1}^{\infty} \cdots \int_{t_n}^{\infty} v^{1-p'}(x) \big(U_1(x_1) \cdots U_n(x_n)\big)^{p'} dx \right)^{1/p'}.$$

Theorem 3.36. *Let $1 < p \leq q < \infty$ and the weight u be of product type (LP). Then* (3.32) *holds for all non-negative and measurable functions f on $\mathbb{R}^n_+$ with some finite constant C, which is independent of f, if and only if $A^{*(n)}_M < \infty$ or $A^{*(n)}_{PS} < \infty$. Moreover, $C \approx A^{*(n)}_M \approx A^{*(n)}_{PS}$ with constants of equivalence only depending on the parameters p and q and the dimension n.*

Also the case $1 < q < p < \infty$ can be considered and the following multidimensional versions of the usual Mazya-Rosin and Persson-Stepanov constants in one dimension can be defined:

$$B^{(n)}_{MR} := \left(\int_{\mathbb{R}^n_+} (U(t))^{r/q} (V_1(t_1))^{r/q'} \cdots (V_n(t_n))^{r/q'} dV_1(t_1) \cdots dV_n(t_n) \right)^{1/r},$$

$$B^{(n)}_{PS} := \left(\int_{\mathbb{R}^n_+} \left(\int_0^{t_1} \cdots \int_0^{t_n} u(x)(V_1(x_1) \cdots V_n(x_n))^q dx \right)^{r/q} \times \right.$$

$$\left. \times \Big(V_1(t_1) \cdots V_n(t_n) \Big)^{-r/q} dV_1(t_1) \cdots dV_n(t_n) \right)^{1/r}.$$

Here, as usual, $1/r = 1/q - 1/p$. For technical reasons we also need the following additional condition:

$$V_1(\infty) = \cdots = V_n(\infty) = \infty.$$

Theorem 3.37. *Let $1 < q < p < \infty$ and $1/r = 1/q - 1/p$. Assume that the weight v is of product type (RP). Then* (3.32) *holds for all non-negative and measurable functions f on $\mathbb{R}^n_+$ with some finite constant C, which is independent on f, if and only if $B^{(n)}_{MR} < \infty$, or $B^{(n)}_{PS} < \infty$. Moreover, $C \approx B^{(n)}_{MR} \approx B^{(n)}_{PS}$ with constants of equivalence depending only on p and q and the dimension n.*

Remark 3.38. Also for $1 < p < q < \infty$ the case when the left hand side is of product type can be considered and a theorem similar to Theorem 3.37 can be proved by using some dual forms of the constants $B^{(n)}_{MR}$ and $B^{(n)}_{PS}$.

We finalize this Section by shortly discussing some limit multidimensional (Pólya-Knopp type) inequalities. Consider the inequality

$$\left(\int_{\mathbb{R}^n_+} (G_n f)^q(x) u(x) dx \right)^{1/q} \leq C \left(\int_{\mathbb{R}^n_+} f^p(x) v(x) dx \right)^{1/p}, \tag{3.33}$$

where the $n-$ dimensional geometric mean operator G_n is defined by

$$(G_n f)(x) = \exp \left(\frac{1}{x_1 \cdots x_n} \int_0^{x_1} \cdots \int_0^{x_n} ln f(x_1, \ldots, x_n) \, dx_1 \ldots dx_n \right).$$

We denote

$$A_G^{(n)} := \sup_{t_i>0}(t_1 \cdots t_n)^{-1/p} \left(\int_0^{t_1} \cdots \int_0^{t_n} w(x)dx \right)^{1/q}$$

with

$$w(x) := \big((G_n v)(x)\big)^{-q/p} u(x).$$

Theorem 3.39. *Let $0 < p \leq q < \infty$. Then* (3.33) *holds for all non-negative and measurable functions on $\mathbb{R}_+^n$ if and only if $A_G^{(n)} < \infty$. Moreover, $C \approx A_G^{(n)}$ with constants of equivalence depending only on the parameters p and q and the dimension n.*

Remark 3.40. Our proof shows that Theorem 3.39 may be regarded as a natural limit case of Theorem 3.35 characterized by the condition $A_{PS}^{(n)} < \infty$. For $n = 2$ we get another characterization than that in Theorem 3.31. Note that also in this case the limit result holds in a wider range of parameters and for general weights.

Remark 3.41. A similar result can be derived also for the case $0 < q < p < \infty$ now as a limiting case of Theorem 3.37.