No ghosts and critical dimension

Paolo Di Vecchia

Niels Bohr Instituttet, Copenhagen and Nordita, Stockholm

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Plan of the talk

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2. Factorization properties of the N-point amplitude
3. Virasoro decoupling conditions
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The N-point amplitude in the operator formalism

- If the intercept of the Regge trajectory is $\alpha_0 = 1$, then the lowest state is a tachyon with mass $m^2 = -\frac{1}{\alpha'}$ and the N-point amplitude for N tachyons is given by:

$$B_N = \int_{-\infty}^{\infty} \frac{\prod_1^N dz_i \theta(z_i - z_{i+1})}{dV_{abc}} \prod_{i<j} (z_i - z_j)^{2\alpha' p_i \cdot p_j} \;\; ; \;\; p_i^2 = -m^2 = \frac{1}{\alpha'}$$

that can be rewritten in the operator formalism as follows:

$$(2\pi)^d \delta(\sum_{i=1}^N p_i) B_N = \int_{-\infty}^{\infty} \frac{\prod_1^N dz_i \theta(z_i - z_{i+1})}{dV_{abc}} \langle 0, 0| \prod_{i=1}^N V(z_i, p_i) |0, 0\rangle$$

[Fubini, Gordon and Veneziano, 1969]

- Here we keep an arbitrary space-time dimension d for future use, but in 1969 d was taken to be $d = 4$ as it was natural for hadrons.
- $V(z_i, p_i)$ is the vertex operator associated to the tachyon state:

$$V(z_i, p_i) =: \mathrm{e}^{ip_i \cdot Q(z_i)} :\equiv \mathrm{e}^{\sqrt{2\alpha'} \sum_{n=1}^{\infty} \frac{p_i \cdot a_n^\dagger}{\sqrt{n}} z^n} \mathrm{e}^{ip_i \cdot \hat{q}} z^{2\alpha' p_i \cdot \hat{p}} \mathrm{e}^{-\sqrt{2\alpha'} \sum_{n=1}^{\infty} \frac{p_i \cdot a_n}{\sqrt{n}} z^{-n}}$$

- $Q_\mu(z)$ is the Fubini-Veneziano-Gervais operator:

$$Q_\mu(z) = \hat{q}_\mu - 2i\alpha' \hat{p}_\mu \log z + i\sqrt{2\alpha'} \sum_{n=1}^{\infty} \left[\frac{a_{n,\mu}}{\sqrt{n}} z^{-n} - \frac{a^\dagger_{n,\mu}}{\sqrt{n}} z^n \right]$$

- The center of mass variables $\hat{p}, \hat{q}$ and the harmonic oscillators satisfy the following commutation relations:

$$[\hat{q}_\mu, \hat{p}_\nu] = i\eta_{\mu\nu} \;\; ; \;\; [a_{n,\mu}, a^\dagger_{m,\nu}] = \delta_{nm}\eta_{\mu\nu} \;\; ; \;\; \eta_{\mu\nu} = (-1, 1, \ldots, 1)$$

- The vacuum $|0,0\rangle$ satisfies:

$$\hat{p}_\mu |0,0\rangle = a_{n,\mu}|0,0\rangle = 0 \;\; ; \;\; n = 1, 2 \ldots$$

Some detail on the previous expressions

- The N-point amplitude can be obtained from the previous vacuum expectation value by bringing all annihilation operators and the term with $\hat{p}$ to the right of the creation operators and of $\hat{q}$.
- This can be done by using the following reordering formula:

$$: \mathrm{e}^{ik\cdot Q(z)} :: \mathrm{e}^{ip\cdot Q(w)} := (z-w)^{2\alpha' k\cdot p} : \mathrm{e}^{ik\cdot Q(z)} \mathrm{e}^{ip\cdot Q(w)} :$$

- It can be obtained using the Baker-Hausdorff relation:

$$\mathrm{e}^{A}\mathrm{e}^{B} = \mathrm{e}^{B}\mathrm{e}^{A}\mathrm{e}^{[A,B]}$$

  that is valid if the commutator $[A, B]$ is a c-number.
- Once this is done all annihilation and creation operators give 1 hitting the oscillator vacuum, also the terms with $\hat{q}$ give 1 hitting the vacuum of momentum and one gets the N-point function times the following matrix element:

$$\langle 0|\mathrm{e}^{i\hat{q}\sum_{i=1}^{N} p_i}|0\rangle = (2\pi)^d \delta(\sum_{i=1}^{N} p_i)$$

- Since the integrand in the N-point amplitude is projective invariant we can fix for convenience $z_1 = \infty, z_2 = 1$ and $z_N = 0$:

$$A_N = \int_0^1 \prod_{i=3}^{N-1} dz_i \prod_{i=2}^{N-1} \theta(z_i - z_{i+1}) \langle 0, -p_1| \prod_{i=2}^{N-1} V(z_i; p_i)|0, p_N\rangle$$

where $(|0, p\rangle \equiv \mathrm{e}^{ip\cdot\hat{q}}|0, 0\rangle)$

$$\lim_{z_N \to 0} V(z_N; p_N)|0, 0\rangle \equiv |0; p_N\rangle \;\; ; \;\; \langle 0; 0| \lim_{z_1 \to \infty} z_1^2 V(z_1; p_1) = \langle 0, -p_1|$$

- In the operator formalism, for reasons that will become clear in a moment, an infinite set of operators L_n (n is an integer $-\infty < n < \infty$) was introduced.
- It was recognized that the L_n operators satisfy algebra of the conformal transformations in two dimensions, called nowadays Virasoro algebra:

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{d}{12} n(n^2 - 1)\delta_{n+m;0}$$

[Fubini and Veneziano, 1969 and Weis, 1969]

- The Virasoro operators L_n are given by:

$$L_n = \oint_0 dz\, z^{n+1} \left[-\frac{1}{4\alpha'} : \left(\frac{dQ(z)}{dz} \right)^2 : \right] \quad ; \quad \oint_0 \frac{dz}{z} \equiv 1$$

- The vertex operator satisfies the following commutation relation with the generators of the Virasoro algebra:

$$[L_n, V(z,p)] = \frac{d}{dz} \left(z^{n+1} V(z,p) \right)$$

  [Fubini and Veneziano, 1969]
- It is therefore a conformal field with dimension $\Delta = 1$.
- A conformal or primary field $\Phi(z)$ transforms under the conformal transformation generated by the operator L_n as follows:

$$[L_n, \Phi(z)] = z^{n+1} \frac{d\Phi(z)}{dz} + \Delta(n+1) z^n \Phi(z)$$

- In the following we want to rewrite the N-point amplitude in a form that is more convenient to study its factorization properties.

- Under a finite dilatation the vertex operator transforms as follows:

$$z^{L_0-1} V(1,p) z^{-L_0} = V(z,p)$$

- Changing the integration variables as follows:

$$x_i = \frac{z_{i+1}}{z_i} \quad ; \quad i = 2,3\ldots N-2 \quad ; \quad \det \frac{\partial z_i}{\partial x_j} = z_3 z_4 \ldots z_{N-2}$$

$\det \frac{\partial z_i}{\partial x_j}$ is the jacobian of the transformation from z_i to x_i, we get the following expression:

$$A_N \equiv \langle 0, -p_1 | V(1,p_2) D V(1,p_3) \ldots D V(1,p_{N-1}) | 0, p_N \rangle$$

- The propagator D is equal to:

$$D = \int_0^1 dx\, x^{L_0-2} = \frac{1}{L_0 - 1}$$

Factorization properties of the N-point amplitude

- The factorization properties of the amplitude can be studied by inserting in the channel $(1, M)$ described by the Mandelstam variable

$$s = -(p_1 + p_2 + \dots p_M)^2 = -(p_{M+1} + p_{M+2} \dots + p_N)^2 \equiv -P^2$$

the complete and orthonormal set of states

$$A_N = \int \frac{d^d P}{(2\pi)^d} \int \frac{d^d P'}{(2\pi)^d} \sum_{\lambda,\mu} \langle p_{(1,M)} | \lambda, P \rangle \langle \lambda, P | D | \mu, P' \rangle \langle \mu, P' | p_{(M+1,N)} \rangle$$

- where

$$\langle p_{(1,M)} | = \langle 0, -p_1 | V(1, p_2) D V(1, p_3) \dots V(1, p_M)$$

and

$$| p_{(M+1,N)} \rangle = V(1, p_{M+1}) D \dots V(1, p_{N-1}) | p_N, 0 \rangle$$

- The operator L_0 is given in terms of the oscillator number operator:

$$L_0 = \alpha' \hat{p}^2 + R \quad ; \quad R = \sum_{n=1}^{\infty} n a_n^\dagger \cdot a_n$$

- Choosing a complete and orthonormal set of states $|\lambda\rangle$ that are eigenstates of R,
- it is possible to rewrite ($\langle P|P'\rangle = (2\pi)^d \delta^{(d)}(P - P')$)

$$\int \frac{d^d P'}{(2\pi)^d} \langle \lambda, P|D|\mu, P'\rangle = \langle \lambda | \frac{1}{\alpha' P^2 + R - 1} |\mu\rangle = \langle \lambda | \frac{1}{R - \alpha(s)} |\lambda\rangle \; \delta_{\lambda\mu}$$

where $\alpha(s) \equiv 1 + \alpha' s$ and $s \equiv -P^2$.

- Using this equation we get

$$A_N = \sum_{\lambda} \int \frac{d^d P}{(2\pi)^d} \langle p_{(1,M)}|\lambda, P\rangle \langle \lambda | \frac{1}{R - \alpha(s)} |\lambda\rangle \langle \lambda, P | p_{(M+1,N)}\rangle$$

- A_N has a pole in the channel $(1, M)$ when

$$\alpha(-P^2) \equiv 1 - \alpha' P^2 = N \quad ; \quad N = 0, 1, 2 \ldots$$

- The states $|\lambda\rangle$ contributing to its residue are those satisfying the relation:

$$R|\lambda\rangle \equiv \sum_{n=1}^{\infty} n a_n^{\dagger} \cdot a_n |\lambda\rangle = N|\lambda\rangle$$

- The number of independent states $|\lambda\rangle$ contributing to the residue gives the degeneracy of states at the level N.
- A state $|\lambda, P\rangle$ is called an on shell state at the level N if

$$1 - \alpha' P^2 = N \qquad and \qquad R|\lambda, P\rangle = N|\lambda, P\rangle$$

Virasoro decoupling conditions

- Because of manifest relativistic invariance, the space spanned by the complete set of states contains states with negative norm.
- They correspond to those states having an odd number of oscillators with timelike directions.
- But in a quantum theory, because of the probabilistic interpretation of the norm of a state, the states of a system must span a positive definite Hilbert space.
- At this point it seems that there is a contradiction between special relativity and quantum mechanics.
- But there is no contradiction if one finds a mechanism to decouple the non-positive norm states.
- In other words, there must exist a number of relations satisfied by the states $|p_{(1,M)}\rangle$ that decouple a number of states leaving a positive definite Hilbert space.
- It turns out that not all states $|\lambda\rangle$ contribute to the residue of the pole because the state $|p_{(1,M)}\rangle$ satisfies the equation:

$$W_n|p_{(1,M)}\rangle = 0 \;\; ; \;\; n = 1 \ldots \infty \;\; ; \;\; W_n = L_n - L_0 - (n-1)$$

- These are the Virasoro conditions [Virasoro, 1969].
- There is one condition for each negative norm oscillator and, therefore, there is the possibility that the physical subspace is positive definite.
- Let us prove the previous relations.
- The commutation relation of the L_n-operators with the vertex operator implies:

$$W_n V(1, p) = V(1, p)(W_n + n)$$

- that together with the following equation:

$$L_n x^{L_0} = x^{L_0+n} L_n$$

implies:

$$(W_n + n)D = [L_0 + n - 1]^{-1} W_n$$

- From the previous equations one gets:

$$W_n V(1,p) D = V(1,p)[L_0 + n - 1]^{-1} W_n$$

- Therefore one gets:

$$\begin{aligned}
& W_n | p_{(1,M)} \rangle \\
& = W_n V(1,p_1) D V(1,p_2) D \ldots V(1,p_{M-2}) D V(1,p_{M-1}) |0, p_M\rangle \\
& = V(1,p_1)[L_0 + n - 1]^{-1} \ldots V(1,p_{M-2})[L_0 + n - 1]^{-1} \\
& \times W_n V(1,p_{M-1}) |0, p_M\rangle \\
& = V(1,p_1)[L_0 + n - 1]^{-1} \ldots V(1,p_{M-2})[L_0 + n - 1]^{-1} V(1,p_{M-1}) \\
& \times (L_n - L_0 + 1) |0, p_M\rangle = 0
\end{aligned}$$

because

$$L_n |0, p_M\rangle = (L_0 - 1)|0, p_M\rangle = 0$$

Characterization of the physical states

- The subspace of the states, contributing to the residue of the pole $\alpha(-P^2) = N$, is spanned by the orthonormal set of states:

$$|\lambda, P\rangle = \prod_n \prod_{\mu_n} \frac{(a^\dagger_{n,\mu_n})^{m_{n,\mu_n}}}{\sqrt{m_{n,\mu_n}!}} |0, P\rangle \quad ; \quad 1 - \alpha' P^2 = N$$

- satisfying the condition:

$$(L_0 - 1)|\lambda, P\rangle = 0 \Longleftrightarrow R|\lambda, P\rangle \equiv \sum_{n=1}^{\infty} n a^\dagger_n \cdot a_n |\lambda, P\rangle = N|\lambda, P\rangle$$

- We call them on shell states at the level N.
- Consider the (off shell by n units) states $|\psi, P\rangle$ at the level $N - n$ satisfying the equation:

$$(L_0 + n - 1)|\psi, P\rangle = 0 \quad ; \quad 1 - \alpha' P^2 = N$$

$$R|\psi, P\rangle = \sum_{n=1}^{\infty} n a^\dagger_n \cdot a_n |\psi, P\rangle = (N - n)|\psi, P\rangle$$

- and from them, acting with L_{-n}, construct the states on shell at the level N:

$$L_{-n}|\psi, P\rangle \quad ; \quad (L_0 - 1)L_{-n}|\psi, P\rangle = 0 \;\; ; \;\; [L_0, L_{-n}] = nL_{-n}$$

- we immediately see that they are decoupled from the physical states $|p_{(1,W)}\rangle$

$$\langle\psi, P|W_n|p_{(1,W)}\rangle = \langle\psi, P|(L_n - L_0 - n + 1)|p_{(1,W)}\rangle = 0$$

- The on shell physical states are defined as those orthogonal to the previous states:

$$\langle\psi, P|L_n|Phys., P\rangle = 0 \Longrightarrow L_n|Phys., P\rangle = (L_0 - 1)|Phys., P\rangle = 0$$

  [Del Giudice and Di Vecchia, 1970]

- These equations do not completely define the physical subspace because there could be states that are physical (satisfying the previous equations), but that are decoupled from the states $|p_{(1,M)}\rangle$.

- A set of them at the level N can be generated as follows.
- Let us consider a physical state $|\psi_1, P\rangle$ at the level $N-1$ that is off shell by one unit:

$$L_0|\psi_1, P\rangle = L_n|\psi_1, P\rangle = 0 \;\; ; \;\; n = 1, 2 \ldots \;\; ; \;\; 1 - \alpha' P^2 = N$$

- Starting from any of the previous states we can construct an on shell physical state at the level N as follows:

$$L_{-1}|\psi_1, P\rangle \Longrightarrow L_n(L_{-1}|\psi_1, P\rangle) = (L_0 - 1)(L_{-1}|\psi_1, P\rangle) = 0 \qquad (1)$$

- that is decoupled from the states $|p_{(1,M)}\rangle$:

$$\langle\psi_1, P|L_1|p_{(1,M)}\rangle = 0 \qquad (2)$$

- All those states have zero norm:

$$\langle\psi_1, P|L_1 L_{-1}|\psi_1, P\rangle = \langle\psi_1, P|\left(2L_0 + L_{-1}L_1\right)|\psi_1, P\rangle = 0$$

- It can be shown that Eqs. (1) and (2) can be satisfied only by zero norm states.

- In conclusion, the on shell physical subspace consists of the states satisfying the equations:

$$L_n|Phys., P\rangle = (L_0 - 1)|Phys., P\rangle = 0$$

- and that are not decoupled from all states $|p_{(1,M)}\rangle$:

$$\langle Phys., P|p_{(1,M)}\rangle \neq 0$$

Analysis of the first few levels

- The ground state is a tachyon $|0, P\rangle$ that satisfies the physical conditions if $1 - \alpha' P^2 = 0$.
- The first excited level ($N = 1$) corresponds to a massless gauge field.
- The most general state at this level has the form

$$\epsilon^\mu a^\dagger_{1\mu}|0, P\rangle \quad ; \quad P^2 = 0$$

- In this case the only condition that we must impose is:

$$L_1 \epsilon^\mu a^\dagger_{1\mu}|0, P\rangle = 0 \Longrightarrow P \cdot \epsilon = 0$$

- In the frame of reference where the momentum of the photon is given by $P^\mu \equiv (P, 0....0, P)$, the most general state satisfying the physical conditions is:

$$\epsilon^i a^\dagger_{1i}|0, P\rangle + \epsilon(a^\dagger_{1;0} - a^\dagger_{1;d-1})|0, P\rangle \;\; ; \;\; i = 1 \ldots d-2$$

- But the state

$$(a^\dagger_{1;0} - a^\dagger_{1;d-1})|0, P\rangle \sim P \cdot a^\dagger_1|0, P\rangle \sim L_{-1}|0, P\rangle$$

has zero norm ($P^2 = 0$) and is decoupled from the state $|p_{(1,M)}\rangle$:

$$\langle 0, P|P \cdot a_1|p_{(1,M)}\rangle = 0$$

- This condition implies that the amplitude M_μ involving M tachyon and one massless state is gauge invariant

$$P^\mu M_\mu = 0$$

- Gauge invariance prevents the presence of non-positive norm states in electrodynamics.
- In conclusion, at this level the only physical components are the $d - 2$ transverse components corresponding to the physical degrees of freedom of a massless spin 1 state in d space-time dimensions.

- The most general state at the level $N = 2$ is given by:

$$[\alpha^{\mu\nu} a^\dagger_{1,\mu} a^\dagger_{1,\nu} + \beta^\mu a^\dagger_{2,\mu}]|0, P\rangle$$

- In the center of mass frame where $P^\mu = (M, \vec{0})$ we get the following most general physical state $(1 - \alpha' P^2 = 2)$:

$$|Phys >= \alpha^{ij}[a^\dagger_{1,i} a^\dagger_{1,j} - \frac{1}{(d-1)} \delta_{ij} \sum_{k=1}^{d-1} a^\dagger_{1,k} a^\dagger_{1,k}]|0, P\rangle +$$

$$+\beta^i [a^\dagger_{2,i} - a^\dagger_{1,0} a^\dagger_{1,i}]|0, P\rangle +$$

$$+\alpha \left[\sum_{i=1}^{d-1} a^\dagger_{1,i} a^\dagger_{1,i} + \frac{d-1}{5} (a^{\dagger 2}_{1,0} - 2 a^\dagger_{2,0}) \right] |0, P\rangle$$

where the indices i, j run over the $d - 1$ space components.

- The first term corresponds to a spin 2 in d dimensional space-time and has a positive norm being made with space indices.

- The second term has zero norm, is orthogonal to the other physical states and it is decoupled from the states $|p_{(1,M)}\rangle$ since it can be written as

$$L_{-1} a^{+}_{1,i} |0, P\rangle$$

- The last state is spinless and has a norm given by:

$$2(d-1)(26-d)$$

- If $d < 26$ it corresponds to a physical spin zero particle with positive norm.
- If $d > 26$ it is a ghost.
- If $d = 26$ it has a zero norm, is also orthogonal to the other physical states and it is decoupled from the states $|p_{(1,M)}\rangle$since it can be written as:

$$(2L_2^{\dagger} + 3L_1^{\dagger 2})|0, P > \quad ; \quad 1 - \alpha' P^2 = 2$$

- Can we generalize the previous analysis to an arbitrary level?

Some detail of the calculations at the level $N = 2$

- Before we go, let us give here some detail of the calculations at the level $N = 2$.
- At this level where $\sqrt{\alpha'}M = 1$, we need only the following expressions for L_1 and L_2:

$$L_1 = \sqrt{2}\left(-a_{1,0} + a_2 \cdot a_1^\dagger\right) \quad ; \quad L_2 = -2a_{2,0} + \frac{1}{2}a_1 \cdot a_1$$

- At this level we have the following physical states:

$$|A\rangle \equiv \left(a_{2,i}^\dagger - a_{1,0}^\dagger a_{1,i}^\dagger\right)|0\rangle$$

$$|A_{ij}\rangle \equiv \left(a_{1,i}^\dagger a_{1,j}^\dagger - \frac{\delta_{ij}}{d-1}\sum_{k=1}^{d-1} a_{1,k}^\dagger a_{1,k}^\dagger\right)|0\rangle$$

$$|B\rangle \equiv \left[\sum_{k=1}^{d-1} a_{1,k}^\dagger a_{1,k}^\dagger + \frac{d-1}{5}\left((a_{1,0}^\dagger)^2 - 2a_{2,0}^\dagger\right)\right]$$

- Using the algebra of the harmonic oscillators it is easy to show that :

$$L_1|A\rangle = L_2|A\rangle = 0$$
$$L_1|A_{ij}\rangle = L_2|A_{ij}\rangle = 0$$
$$L_1|B\rangle = L_2|B\rangle = 0$$

- It is not necessary to impose the vanishing of the L_n with $n > 2$ because they are automatically satisfied as a consequence of the Virasoro algebra.

- We have already seen that, starting from any physical state $|\psi_1, P\rangle$ off shell by one unit ($1 - \alpha' P^2 = N$):

$$L_0|\psi_1, P\rangle = L_n|\psi_1, P\rangle = 0 \quad ; \quad n = 1, 2 \ldots$$

$$R|\psi_1, P\rangle \equiv \sum_{n=1}^{\infty} n a_n^\dagger \cdot a_n |\psi_1, P\rangle = N - 1$$

we can always construct the following on shell zero norm physical state:

$$|\psi, P\rangle = L_{-1}|\psi_1, P\rangle \Longrightarrow (L_0 - 1)|\psi, P\rangle = L_n|\psi, P\rangle = 0 \; ; \; n = 1, 2 \ldots$$

- It is zero norm and decoupled from $|p_{(1,M)}\rangle$:

$$\langle\psi_1, P|L_1 L_{-1}|\psi_1, P\rangle = \langle\psi_1, P|(2L_0 + L_{-1}L_1)|\psi_1, P\rangle = 0$$

$$\langle\psi_1, P|L_1|p_{(1,M)}\rangle = 0$$

- We can see the appearance of the critical dimension $d = 26$ as the dimension for which we can construct an additional set of zero norm states.
- In fact, starting from the physical state $|\psi_2, P\rangle$ (but off shell by two units) satisfying the equations:

$$(L_0 + 1)|\psi_2, P\rangle = L_n|\psi_2, P\rangle = 0 \ ; \ \ n = 1, 2 \dots$$

$$R|\psi_2, P\rangle = \sum_{n=1}^{\infty} n a_n^\dagger \cdot a_n |\psi_2, P\rangle = N - 2 \ ; \ \ 1 - \alpha' P^2 = N$$

- we can construct the state:

$$|\psi, P\rangle \equiv (2L_{-2} + 3L_{-1}^2)|\psi_2, P\rangle$$

- that is a zero norm on shell physical state:

$$(L_0 - 1)|\psi, P\rangle = L_n|\psi, P\rangle = 0 \ ; \ \ n = 1, 2 \dots$$

$$\langle \psi_2, P|(2L_2 + 3L_1^2)|p_{(1,M)}\rangle$$

$$= \langle \psi_2, P|(2(L_0 + 1) + 3L_0(L_0 + 1)|p_{(1,M)}\rangle = 0$$

- But then what are the real physical states with positive norm?

Vertex operators for excited states

- The previous analysis was done starting from the N-tachyon amplitude.
- It could have been done starting from an amplitude involving any physical state.
- We can associate to any physical state $|\alpha, P\rangle$ its corresponding vertex operator $V_\alpha(z, P)$ that is a conformal field with dimension $\Delta = 1$:

$$[L_n, V_\alpha(z, P)] = \frac{d}{dz}\left(z^{n+1} V_\alpha(z, P)\right)$$

- It reproduces the physical state in the limits:

$$\lim_{z \to 0} V_\alpha(z, P)|0, 0\rangle \equiv |\alpha; P\rangle \;\; ; \;\; \langle 0; 0| \lim_{z \to \infty} z^2 V_\alpha(z, P) = \langle \alpha, -P|$$
$$L_n|\alpha, P\rangle = (L_0 - 1)|\alpha, P\rangle = 0 \;\; ; \;\; n = 1, 2 \ldots$$

- It satisfies the hermiticity relation:

$$V_{\alpha}^{\dagger}(z, P) = V_{\alpha}(\frac{1}{z}, -P)(-1)^{m} \;\; ; \;\; 1 - \alpha' P^2 = m$$

[Campagna, Fubini, Napolitano and Sciuto, 1970]
[Clavelli and Ramond, 1970]

- In terms of these vertices one can write the most general amplitude involving physical states:

$$(2\pi)^d \delta(\sum_{i=1}^{N} p_i) B_N(\alpha_1, p_1; \ldots \alpha_N, p_N)$$

$$= \int_{-\infty}^{\infty} \frac{\prod_1^N dz_i \theta(z_i - z_{i+1})}{dV_{abc}} \langle 0, 0| \prod_{i=1}^{N} V_{\alpha_i}(z_i, p_i) |0, 0\rangle$$

- It has precisely the same form as the *N*-tachyon amplitude except that the vertex operators depend on the physical states involved.
- There is a complete democracy among the physical states, as advocated by the followers of S-matrix theory.

- The vertex operator associated to the massless vector state is, however, somewhat special and will play an important role in the proof of the no-ghost theorem.
- It is given by:

$$V_\epsilon(z,k) \equiv \epsilon \cdot \frac{dQ(z)}{dz} e^{ik\cdot Q(z)} \quad ; \quad k\cdot\epsilon = k^2 = 0$$

DDF operators

- We want to construct an infinite set of physical states starting from the vertex operator for the massless spin 1 state.
- The starting point is the DDF operator defined in terms of the vertex operator corresponding to the massless gauge field:

$$A_{i,n} = \frac{i}{\sqrt{2\alpha'}} \oint_0 \frac{dz}{2\pi i} \epsilon_i^\mu P_\mu(z) e^{ik\cdot Q(z)}$$

where

$$P(z) \equiv \frac{dQ(z)}{dz} = -i\sqrt{2\alpha'}\left[\sqrt{2\alpha'}\frac{\hat{p}_0}{z} + \sum_{n=1}^{\infty}\sqrt{n}\left(a_n z^{n-1} + a_n^\dagger z^{-n-1}\right)\right]$$

- The index i runs over the $d-2$ transverse directions that are orthogonal to the momentum k.
- DDF stands for [Del Giudice, Di Vecchia and Fubini, 1971] who constructed this operator.

- The zero mode part of $Q(z) = \cdots - 2\alpha' i\hat{p} \log z \ldots$ has a logarithmic singularity at $z = 0$.
- The contour integral is well defined only if we constrain the momentum of the state, on which $A_{i,n}$ acts, to satisfy the relation:

$$2\alpha' p \cdot k = n$$

  where n is a non-vanishing integer.
- The DDF operators commutes with the gauge operators L_m:

$$[L_m, A_{n;i}] = 0$$

  because the vertex operator transforms as a total derivative under the action of L_n.
- They satisfy the algebra of the harmonic oscillator as we are now going to show.

- We get

$$[A_{n,i}, A_{m,j}] = -\frac{1}{2\alpha'} \oint_0 \frac{d\zeta}{2\pi i} \oint_\zeta \frac{dz}{2\pi i} \epsilon_i \cdot P(z) e^{ik\cdot Q(\zeta)} \epsilon_j \cdot P(\zeta) e^{ik'\cdot Q(\zeta)}$$

where

$$2\alpha' p \cdot k = n \;\; ; \;\; 2\alpha' p \cdot k' = m$$

- k and k' are supposed to be in the same direction, namely

$$k_\mu = n\hat{k}_\mu \quad ; \quad k'_\mu = m\hat{k}_\mu$$

with

$$2\alpha' p \cdot \hat{k} = 1$$

- Finally the polarizations are normalized as:

$$\epsilon_i \cdot \epsilon_j = \delta_{ij}$$

- Since $\hat{k} \cdot \epsilon_i = \hat{k} \cdot \epsilon_j = \hat{k}^2 = 0$ a singularity for $z = \zeta$ can appear only from the contraction of the two terms $P(\zeta)$ and $P(z)$ that is given by:

$$\langle 0, 0|\epsilon_i \cdot P(z)\epsilon_j \cdot P(\zeta)|0, 0\rangle = -\frac{2\alpha' \delta_{ij}}{(z-\zeta)^2}$$

- From it we get:

$$[A_{n,i}, A_{m,j}] = \delta_{ij} in \oint_0 d\zeta \hat{k} \cdot P(\zeta) e^{i(n+m))\hat{k}\cdot Q(\zeta)} =$$

$$= in\delta_{ij}\delta_{n+m;0} \oint_0 \frac{d\zeta}{2\pi i} \hat{k} \cdot P(\zeta) \;\; ; \;\; P(\zeta) = -2i\alpha' \frac{\hat{p}}{z} + \ldots$$

- We have used the fact that the integrand is a total derivative and therefore one gets a vanishing contribution unless $n + m = 0$.
- We get:

$$[A_{n,i}, A_{m,j}] = n\delta_{ij}\delta_{n+m;0} \;\; ; \;\; i, j = 1 \ldots d-2$$

- In terms of this infinite set of transverse oscillators we can construct an orthonormal set of states:

$$|i_1, N_1; i_2, N_2; \dots i_m, N_m\rangle = \prod_h \frac{1}{\sqrt{\lambda_h!}} \prod_{k=1}^{m} \frac{A_{i_k,-N_k}}{\sqrt{N_k}} |0, p\rangle$$

  where λ_h is the multiplicity of the operator $A_{i_h,-N_h}$ in the product.
- They all have positive definite norm and satisfy the on shell physical conditions:

$$(L_0 - 1)|i_1, N_1; i_2, N_2; \dots i_m, N_m\rangle = L_n|i_1, N_1; i_2, N_2; \dots i_m, N_m\rangle = 0$$

  for $n = 1, 2 \dots$, because the DDF oscillators commute with any Virasoro operator and the tachyon state $|0, p\rangle$ satisfies the previous conditions .
- The momentum of the state and its mass are given by

$$P = p - \sum_{i=1}^{m} \hat{k} N_i \;\; ; \;\; 1 - \alpha' P^2 = \sum_k N_k = N$$

The no-ghost theorem

- Going back to level $N = 2$ we have the following DDF states contributing at this level:

$$A_{-1,i}A_{-1,j}|0,p\rangle \quad ; \quad A_{-2,i}|0,p\rangle \ ; \ i,j = 1\ldots d-2$$

- Therefore the number of states contributing is equal to

$$\frac{(d-2)(d-1)}{2} + d - 2 = \frac{(d-2)(d+1)}{2} \tag{3}$$

- that is equal to the number of components of the state:

$$[a^{\dagger}_{1,I}a^{\dagger}_{1,J} - \frac{1}{(d-1)}\delta_{IJ}\sum_{K=1}^{d-1} a^{\dagger}_{1,K}a^{\dagger}_{1,K}]|0,P\rangle \quad ; \quad I,J = 1\ldots d-1$$

given by:

$$\frac{(d-1)d}{2} - 1 = \frac{(d-2)(d+1)}{2}$$

describing a spin 2 in $d-1$ space dimensions.

- This state is the only physical state at the level $N = 2$ if $d = 26$.
- For $d = 26$ the DDF states are a complete set of states at the level $N = 2$.
- It turns out, after a detailed analysis, that, if $d = 26$, they are indeed a complete set of states at an arbitrary level N. [Goddard and Thorn, 1972 and Brower, 1972]
- Since they span a positive definite Hilbert space, this means that the dual resonance model is ghost-free if $d = 26$.
- It can be shown that this is also true for any $d < 26$.
- However, in this case there are additional operators to be included besides the DDF ones.
- The states produced by these additional operators are called Brower states [Brower, 1972].
- They are needed already at the level $N = 2$ to take care of the additional scalar state not taken into account by the DDF states.

$d = 26$ from the non-planar loop

- Historically, the critical dimension was not found as described before.
- It was first found in the study of one-loop amplitudes.
- The Veneziano model and its extension, the N-point function, satisfies all the axioms of S matrix theory except unitarity.
- In fact, unitarity in a model with only resonances imposes that the total width of a resonance Γ must be the sum of the partial widths over all the possible decay channels:

$$\Gamma = \sum_n \Gamma_n$$

- If the model is ghost-free, all partial widths are positive definite and a sum of positive numbers cannot give zero unless $\Gamma_n = 0$ for any n.
- In the Veneziano model, the total width $\Gamma = 0$, but the partial widths are non zero $\Longrightarrow$ unitarity is violated !

- Immediately after the discovery of the Veneziano model, it was proposed to make it unitary by adding to it the contribution of loop diagrams.
- Unitarity is, in fact, implemented in this way in perturbative field theory.
- The tree diagrams are not unitary and unitarity is implemented order by order in perturbation theory by adding loop diagrams.
- By doing so, one generates the branch points required by unitarity and corresponding, for instance, to the two- three- etc. particle thresholds.
- At one-loop level in the DRM, two kinds of loop diagrams appear: the planar and the non-planar.
- They correctly generate the branch cuts required by unitarity, but the non-planar one showed additional branch cuts violating unitarity.

- In 1970, Lovelace noticed that these branch cuts become poles if the dimension of the space-time $d = 26$.
- And poles create no problem with unitarity.
- They are just additional states appearing at one-loop level.
- Today we know that, while the original poles correspond to the excitation of an open string, the new poles correspond instead to the excitation of a closed string.
- They both lie on linear Regge trajectories given respectively, by:

$$\alpha_{open}(s) = 1 + \alpha' s \quad ; \quad \alpha_{closed}(s) = 2 + \frac{\alpha'}{2} s$$

- At that time, practically nobody took Lovelace's observation seriously.
- But this has been the first evidence of the existence of a critical dimension.