Particules Élémentaires, Gravitation et Cosmologie

Année 2009-'10

Théorie des Cordes: une Introduction

Cours V: 12 février 2010

Dual Resonance Models (cordes sans cordes)

- Généralisation à N-particules
- Opérateurs et factorisation
- Croissance exponentielle de la dégénérescence

Fear of ghosts

- The properties of the Beta-function were very nice and welcome, almost too good to be true.
- There was, however, a big worry based on previous experience: possibly, in order to satisfy all the constraints, the model had to contain "ghosts", i.e. negative-norm states produced with negative probability.
- If so the model would have been inconsistent.
- In order to find out, it was necessary to identify first all the states.

A first look at the spectrum

Consider the N^{th} pole in s, at $\alpha(s) = N$, and its residue.

It is immediately realized that such a residue is a polynomial in t (hence in $\cos\theta$) of degree N. As such it can be expanded in

the first N Legendre polynomials each one corresponding to a definite J for the resonance.

One finds that all J up to N do indeed contribute. Thus the spectrum is degenerate with a degeneracy growing at least linearly (quadratically if we count 2J+1 states for spin J) in N.

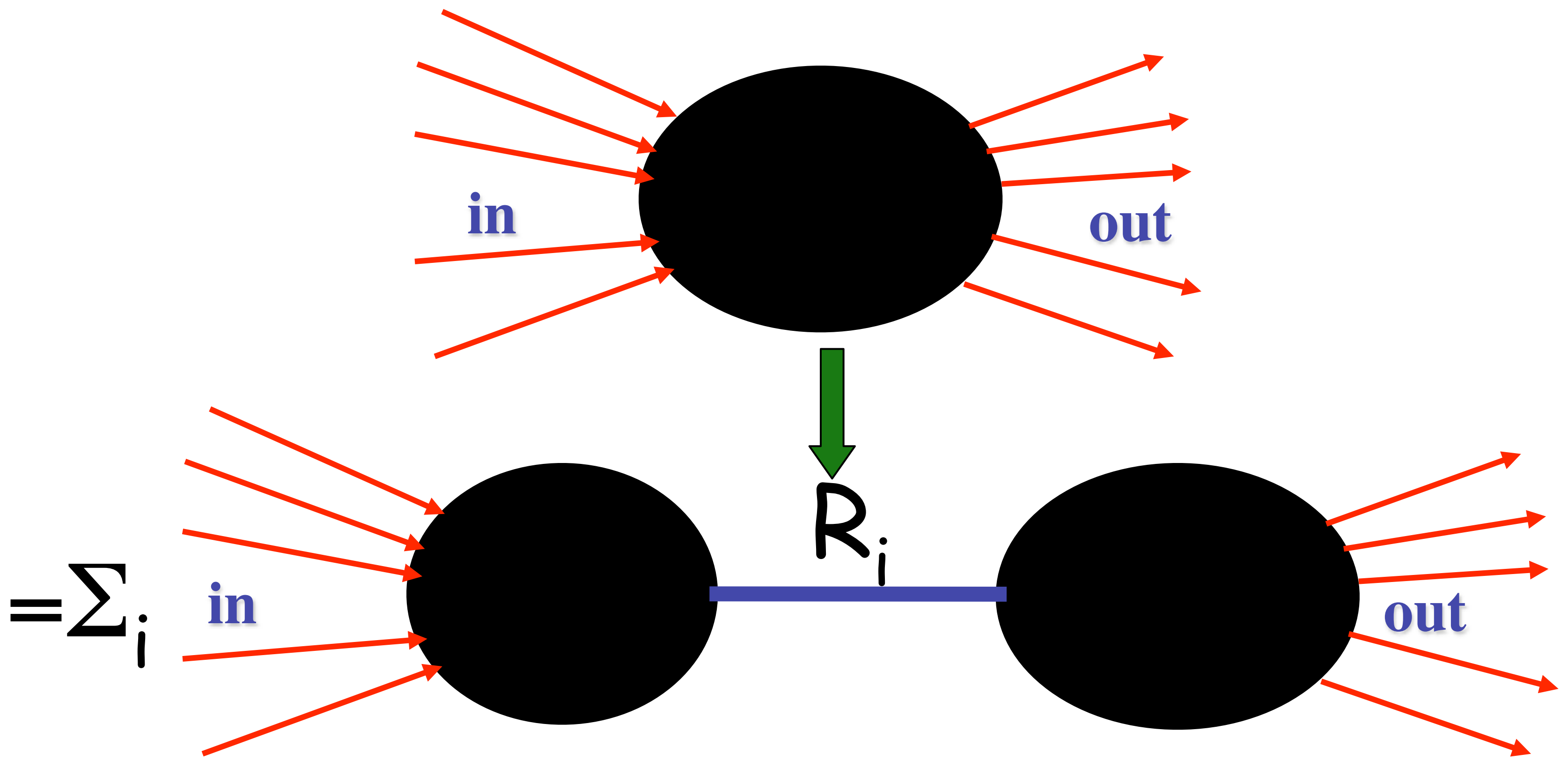
At least, because a single 2->2 scattering process is unable to resolve the degeneracy within a given J.

To fully disentangle the spectrum we need to construct more general scattering amplitudes and use a basic property of each single intermediate state known as factorization. Each state contributes to the residue by the product of its couplings to the "initial" and "final" states.

This is what unitarity of the S-matrix reduces to in the single-particle-exchange approximation.

Thus counting states amounts to answering the following question:

Q: How many terms are needed (in the sum over i) in order to have, for all **in** and **out** states,



In principle we could stick to 2-->2 processes varying the species of the initial and final particles. However, we do not have (yet) at our disposal those other possible initial and final particles (finding them is precisely the problem we want to solve!).

The simplest extension turned out to be in the direction of increasing the number of particles participating in the scattering process, without changing their nature.
This turns out to be good enough for our purposes.

DRM=Multiparticle generalizations of the B-function

Which properties should our multiparticle amplitudes satisfy? How should we generalize the duality properties of two-body scattering that allowed us to find the solution?

We will insist on having poles (and only poles) in the appropriate Mandelstam variables as well as the appropriate crossing symmetries, but will not impose Regge behaviour (which, btw, can be generalized to multiparticle processes). Actually, (multi)Regge behaviour will come out as a bonus.

The other crucial input will be imposing “Planar Duality”.

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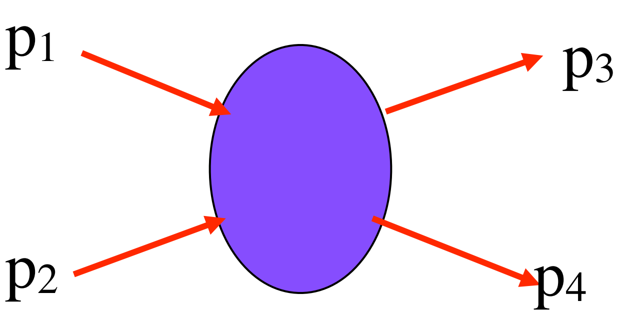
NB. There is also a notion of “Non-Planar-Duality”, embodied in the Shapiro-Virasoro model, later interpreted as describing the interaction of closed strings.

Planar Duality (to be related to open strings)

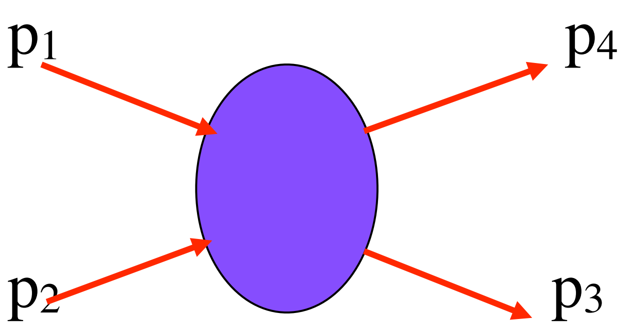
The Beta-function model for the 4-point function (2->2 scattering) exhibits "planar duality" i.e. duality w.r.t. the channels put in evidence by each particular (cyclic=anticyclic) order of the external lines. There are 3 of them (3 pairs of Mandelstam variables):

$$s = -(p_1 + p_2)^2 \quad , \quad t = -(p_1 - p_3)^2 \quad , \quad u = -(p_1 - p_4)^2$$

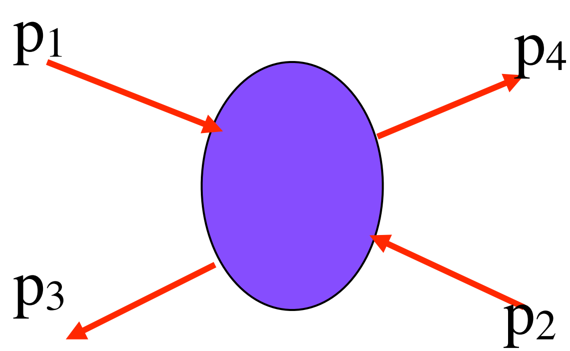
s-t duality



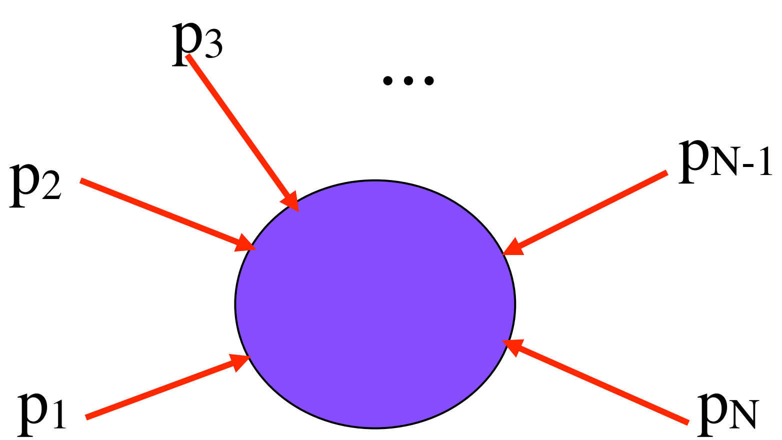
s-u duality



t-u duality



Consider now a process involving N > 4 external spinless particles. The corresponding (connected) amplitude is called an N-point function A_N. There are (N-1)!/2 distinct terms (distinct cyclic orderings) that have to be added at the end (with some specific numerical weights). Consider the term corresponding to the "trivial" cyclic ordering:



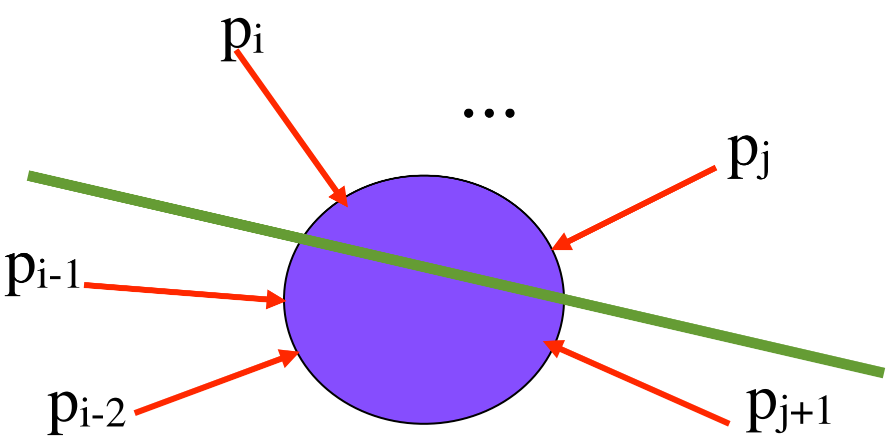
It will be given by an analytic function with poles in the Mandelstam invariants corresponding to its "planar channels".

Useful convention: all momenta are incoming so that 4-momentum conservation reads: $\Sigma_i \; p_i = 0$.

=> some of the $p_{i0} = E_i$ must be negative.

They correspond to outgoing particles w/ 4-momentum $-p_i$

planar channels are defined by a partition of the external legs in two sets of adjacent legs each containing at least two particles

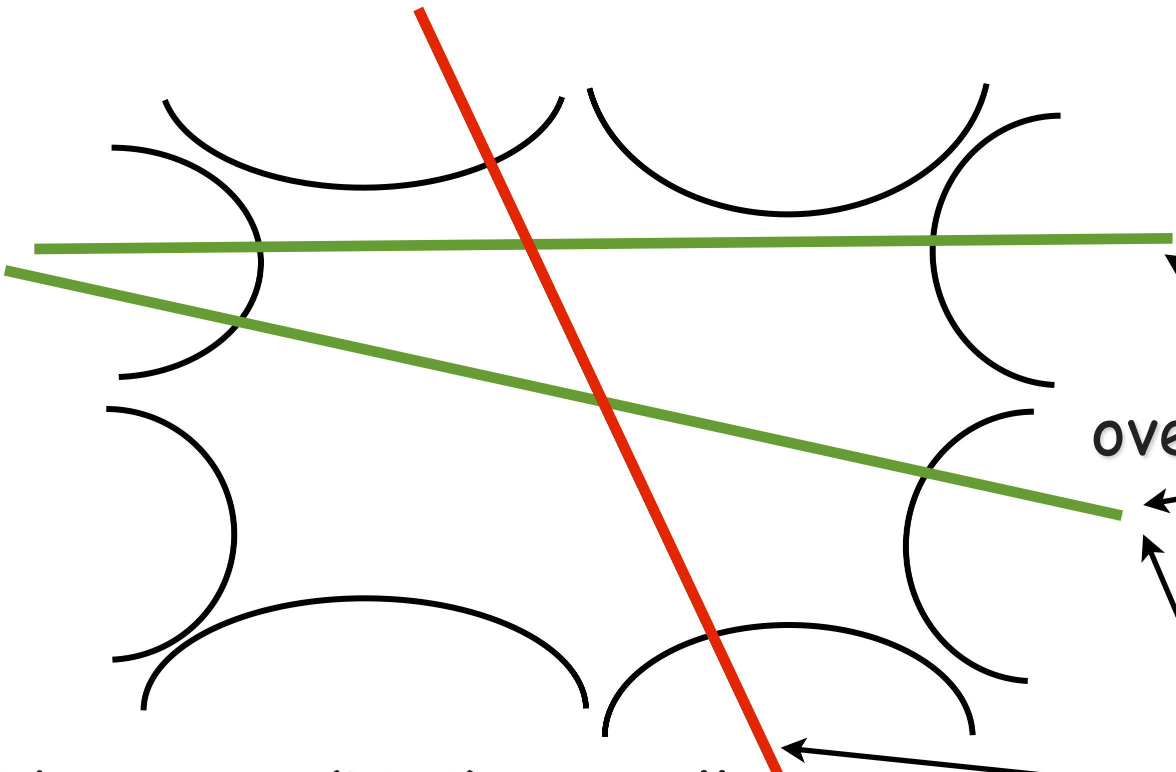


Poles appear in the corresponding Mandelstam variables:

$s_{ij} = -(p_i + p_{i+1} + ... + p_j)^2 = -(p_{j+1} + p_{j+2} + ... + p_{i-1})^2$

Their total number is $N(N-3)/2$ (= 2, 5, 9, ...)

Planar duality is very natural from a duality diagram viewpoint



compatible, non
overlapping channels

There are (N-3) mutually
compatible channels

incompatible,
overlapping channels

Chan's form for the N-point function (1968)

Chan's form for the N-point function is the most direct generalization of the integral form of the Beta function (now called B_4):

$$B_4\left(-\alpha(s), -\alpha(t)\right) = \int_0^1 dx\; x^{-1-\alpha(s)}(1-x)^{-1-\alpha(t)}$$

$$B_N = \int_0^1 \prod_{ij} du_{ij} u_{ij}^{-1-\alpha(s_{ij})} \delta(\dots)$$

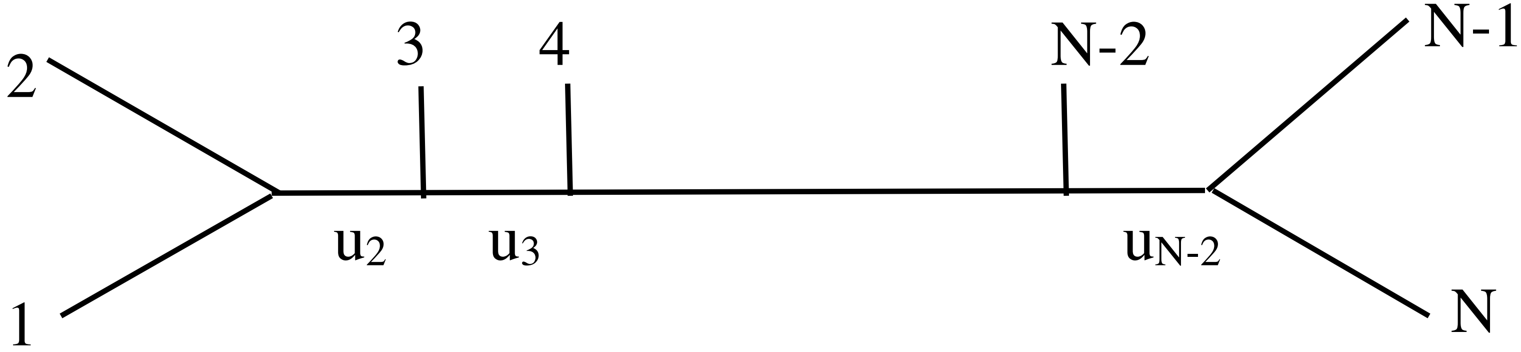
where the δ -functions eliminate all but (N-3) integration variables (the maximal number of compatible poles) via the constraints:

$$1 - u_P = \prod_{\bar{P}} u_{\bar{P}}$$

where the product extends to all channels overlapping with P.

$$B_N = \int_0^1 \prod_{ij} du_{ij} u_{ij}^{-1-\alpha(s_{ij})} \delta(\dots) \qquad 1 - u_P = \prod_{\bar{P}} u_{\bar{P}}$$

Quite amazingly the constraint can be "easily" solved in terms of a natural subset:



where $u_i = u_{1i}$. After using the δ-functions, one finds

(the ' means "over the remaining channels"):

$$B_N = \prod_{i=2}^{N-2} \left[\int_0^1 du_i u_i^{-1-\alpha(s_i)} (1-u_i)^{-1-\alpha(s_{i,i+1})} \right] \prod_{ij}{}' (1 - u_i \dots u_{j-1})^{-\gamma_{ij}}$$

$$\gamma_{ij} = \alpha(s_{ij}) + \alpha(s_{i+1,j-1}) - \alpha(s_{i+1,j}) - \alpha(s_{i,j-1}) = -2\alpha' p_i p_j$$

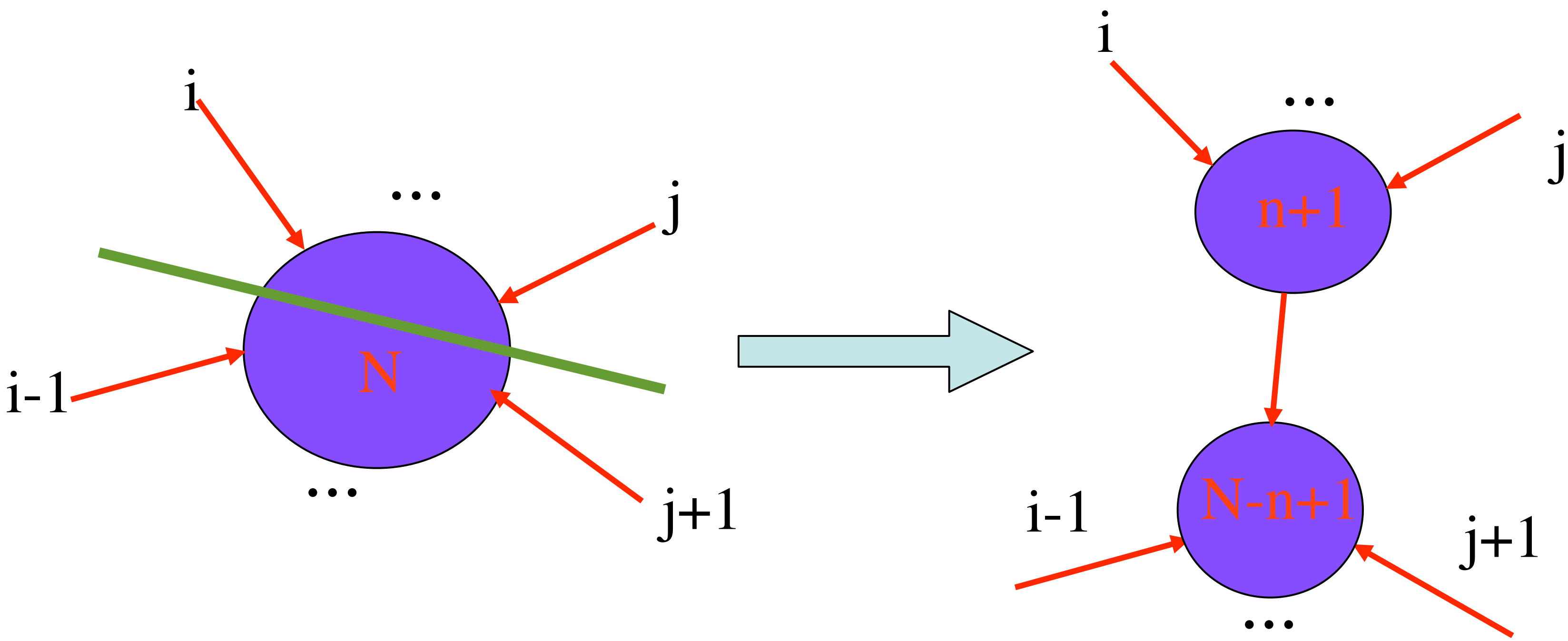
Although this is not explicit, by its construction B_N is invariant under cyclic permutations of the external lines (the analogue of s-t duality for B_4)

$$B_N = \int_0^1 \prod_{ij} du_{ij} u_{ij}^{-1-\alpha(s_{ij})} \delta(\ldots) \qquad 1 - u_P = \prod_{\bar{P}} u_{\bar{P}}$$

If we go on the lowest lying pole in one of the privileged channels this amounts to setting the corresponding $u_i = 0$. But then all the u's in overlapping channels go to 1 making the residue at the pole independent of the corresponding Mv's.

The rest simply gives the product of two lower B_N's

The lowest pole is consistent with being due to exchanging the external particle itself!



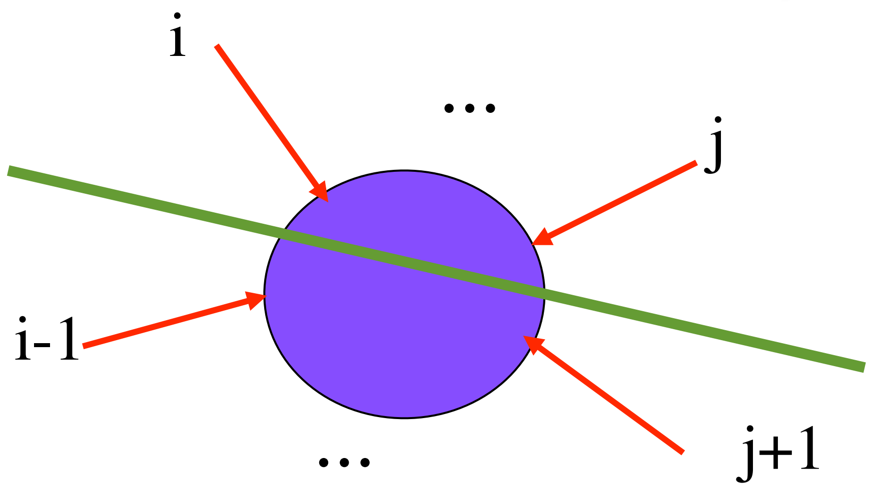
This is the simplest example of factorization telling us that the lowest state is non-degenerate.

Koba-Nielsen form for the N-point function

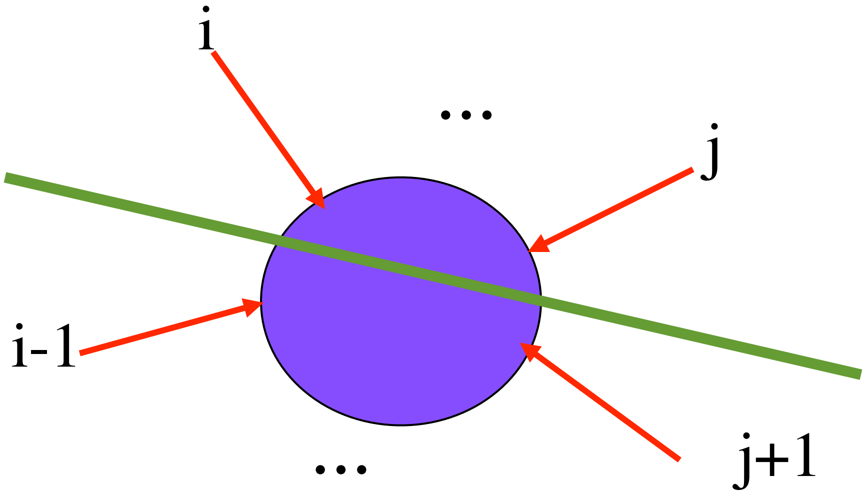
(taking already the special case of $\alpha(0) = 1$)

For further developments a more useful form for the N-point function was given by Koba & Nielsen (1968). It has the advantage of treating all the external particles on the same footing. Their construction is as follows:

Associate with each external particle a (KN) real variable z_i ($i = 1,2, \ldots N$) and to each planar channel a particular anharmonic ratio of the z's:



$$(z_i, z_{i-1}, z_j, z_{j+1}) \equiv \frac{(z_i - z_j)(z_{i-1} - z_{j+1})}{(z_{i-1} - z_j)(z_i - z_{j+1})}$$



$$(z_i, z_{i-1}, z_j, z_{j+1}) \equiv \frac{(z_i - z_j)(z_{i-1} - z_{j+1})}{(z_{i-1} - z_j)(z_i - z_{j+1})}$$

B_N is then given by (a,b,c are chosen arbitrarily):

$$B_N = \int_{-\infty}^{+\infty} dV(z) \prod_{i,j} (z_i, z_{i-1}, z_j, z_{j+1})^{-1-\alpha(s_{ij})}$$

$$dV(z) = \frac{\prod dz_i \theta(z_i - z_{i+1})}{\prod (z_i - z_{i+2}) dV_{abc}}$$

$$dV_{abc} = \frac{dz_a dz_b dz_c}{(z_b - z_a)(z_c - z_b)(z_a - z_c)}$$

$$B_N = \int_{-\infty}^{+\infty} dV(z) \prod_{i,j} (z_i, z_{i-1}, z_j, z_{j+1})^{-1-\alpha(s_{ij})}$$

$$dV(z) = \frac{\prod dz_i \theta(z_i - z_{i+1})}{\prod (z_i - z_{i+2}) dV_{abc}}$$

$$dV_{abc} = \frac{dz_a dz_b dz_c}{(z_b - z_a)(z_c - z_b)(z_a - z_c)}$$

Integrand and integration measure are invariant under projective O(2,1) transformations:

$$z_i \to \frac{\alpha z_i + \beta}{\gamma z_i + \delta} \;;\; \alpha\delta - \beta\gamma = 1$$

Without dividing by dV_{abc} one would get infinity.

3 z's can be fixed arbitrarily leaving N-3 int. variables.

$$
\begin{aligned}
B_N &= \int_{-\infty}^{+\infty} dV(z) \prod_{i,j} (z_i, z_{i-1}, z_j, z_{j+1})^{-1-\alpha(s_{ij})} \\
dV(z) &= \frac{\prod dz_i \theta(z_i - z_{i+1})}{\prod (z_i - z_{i+2}) dV_{abc}} \\
dV_{abc} &= \frac{dz_a dz_b dz_c}{(z_b - z_a)(z_c - z_b)(z_a - z_c)}
\end{aligned}
$$

Using relations such as:

$$
\gamma_{ij} = \alpha(s_{ij}) + \alpha(s_{i+1,j-1}) - \alpha(s_{i+1,j}) - \alpha(s_{i,j-1}) = -2\alpha' p_i p_j
$$

we collect all the factors that contain a given $(z_i - z_j)$ and obtain (for $\alpha(0) = 1$!) the standard KN form:

$$
B_N = \int_{-\infty}^{+\infty} \frac{\prod dz_i \theta(z_i - z_{i+1})}{dV_{abc}} \prod_{j>i} (z_i - z_j)^{2\alpha' p_i \cdot p_j}
$$

$$B_N = \int_{-\infty}^{+\infty} \frac{\prod dz_i \theta(z_i - z_{i+1})}{dV_{abc}} \prod_{j>i} (z_i - z_j)^{2\alpha' p_i \cdot p_j}$$

Note that the integrand is now independent of the cyclic ordering of the external lines. This only appears in the integration measure through the ordering of the z's (again only for $\alpha(0) = 1$).

A convenient choice for the 3 fixed z's is:

$$z_a = z_1 = \infty \; ; \; z_b = z_2 = 1 \; ; \; z_c = z_N = 0$$

$$B_N = \prod_{3}^{N-1} \left[\int_0^1 dz_i \theta(z_i - z_{i+1}) \right] \prod_{i=2}^{N-1} \prod_{j=i+1}^{N} (z_i - z_j)^{2\alpha' p_i \cdot p_j}$$

$$B_N = \prod_3^{N-1} \left[\int_0^1 dz_i \theta(z_i - z_{i+1}) \right] \prod_{i=2}^{N-1} \prod_{j=i+1}^{N} (z_i - z_j)^{2\alpha' p_i \cdot p_j}$$

This was the starting point of the original study of the spectrum (FV & BM, 1969).

By far the simplest way to describe it is by introducing (FGV, N, 1969) an operator formalism (which also leads straight into string theory!).

$$[q_\mu \ , \ p_\nu] = i\eta_{\mu\nu} \ , \quad [a_{n,\mu} \ , \ a^\dagger_{m,\nu}] = \delta_{n,m}\eta_{\mu\nu} \ , \quad \eta_{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1)$$

$$(n = 1, 2, \ldots \ ; \ \mu = 0, 1, 2, \ldots D-1)$$

They look like innocent NR-QM operators, in particular those of an infinite set of decoupled harmonic oscillators with one crucial difference: Also time-components (and an indefinite metric) appear! We are relativistic!

We will show in a moment that a sufficient set of states for factorization consists of the eigenstates of momentum and of the occupation numbers of those harmonic oscillators i.e.

$$|N_{n,\mu}, k\rangle \sim \prod_{n,\mu} \left(a^{\dagger}_{n,\mu}\right)^{N_{n,\mu}} e^{iqk}|0\rangle \quad ; \quad a_{n,\mu}|0\rangle = p_{\mu}|0\rangle = 0$$

$$-\alpha' k^2 = \alpha' M^2 = -1 + \sum_{n,\mu} n \; a^{\dagger}_{n,\mu} \; a^{\mu}_{n}$$

Because of the "wrong" sign of the timelike c.r., states created by an odd number of timelike operators are ghosts.
Was the DRM doomed? One (tiny?) hope remained: all those states were sufficient but perhaps only a (ghost-free?) subset was necessary.
This will be the main topic of today's seminar by PdV.

Proof of factorization

We shall now rewrite the KN form of B_N using our operators. Two essential ingredients are:

1) a "field operator" $Q_\mu(z)$ and

2) a "vertex operator" $V(z, k)$

$$
\begin{aligned}
Q_\mu(z) &= Q_\mu^{(0)}(z) + Q_\mu^{(+)}(z) + Q_\mu^{(-)}(z) \;\; ; \;\; Q_\mu^{(0)}(z) = q_\mu - 2i\alpha' p_\mu log z \\
Q_\mu^{(+)}(z) &= i\sqrt{2\alpha'} \sum_{n=1}^{\infty} \frac{a_{n,\mu}}{\sqrt{n}} z^{-n} \;\; ; \;\; Q_\mu^{(-)}(z) = -i\sqrt{2\alpha'} \sum_{n=1}^{\infty} \frac{a_{n,\mu}^\dagger}{\sqrt{n}} z^{n}
\end{aligned}
$$

$$
V(z,k) =: e^{ik\cdot Q(z)} : \;\; \equiv \;\; e^{ik\cdot Q^{(-)}(z)} \; e^{ik\cdot q} \; e^{2\alpha' k\cdot p log z} \; e^{ik\cdot Q^{(+)}(z)}
$$

They satisfy the following operator identities:

$$Q_\mu(z) = Q_\mu^{(0)}(z) + Q_\mu^{(+)}(z) + Q_\mu^{(-)}(z) \quad ; \quad Q_\mu^{(0)}(z) = q_\mu - 2i\alpha' p_\mu log z$$

$$Q_\mu^{(+)}(z) = i\sqrt{2\alpha'} \sum_{n=1}^{\infty} \frac{a_{n,\mu}}{\sqrt{n}} z^{-n} \quad ; \quad Q_\mu^{(-)}(z) = -i\sqrt{2\alpha'} \sum_{n=1}^{\infty} \frac{a_{n,\mu}^\dagger}{\sqrt{n}} z^{n}$$

$$V(z,k) =: e^{ik\cdot Q(z)} : \; \equiv \; e^{ik\cdot Q^{(-)}(z)} \; e^{ik\cdot q} \; e^{2\alpha' k\cdot p log z} \; e^{ik\cdot Q^{(+)}(z)}$$

$$[Q_\mu^{(+)}(z), Q_\nu^{(-)}(w)] = -2\alpha' log\left(1 - \frac{w}{z}\right)\eta_{\mu\nu}$$

$$V(z,k)V(w,k') =: V(z,k)V(w,k') : (z-w)^{2\alpha' k\cdot k'}$$

leading easily to:

$$\langle 0| \prod_{i=1}^{N} V(z_i, p_i)|0\rangle = (2\pi)^D \delta^{(D)}(\sum p_i) \prod_{i>j} (z_i - z_j)^{2\alpha' p_i\cdot p_j}$$

Consequently, recalling

$$B_N = \int_{-\infty}^{+\infty} \frac{\prod dz_i \theta(z_i - z_{i+1})}{dV_{abc}} \prod_{j>i} (z_i - z_j)^{2\alpha' p_i \cdot p_j}$$

we have the elegant result:

$$(2\pi)^D \delta^{(D)}(\sum p_i) B_N = \int_{-\infty}^{+\infty} \frac{\prod dz_i \theta(z_i - z_{i+1})}{dV_{abc}} \langle 0 | \prod_{i=1}^{N} V(z_i, p_i) | 0 \rangle$$

This looks already nicely factorized. To complete the proof we use the fact that the operator

$$L_0 = \alpha' p^2 + \sum_{n,\mu} n \; a^\dagger_{n,\mu} \; a^\mu_n$$

acts on Q as z d/dz Q giving:

$$V(z,k) = z^{L_0 - \alpha' k^2} V(1,k) z^{-L_0} = z^{L_0 - 1} V(1,k) z^{-L_0}$$

Using this repeatedly and performing the explicit integrals on z_{i+1}/z_i we finally arrive at the desired fully factorized form:

$$
\begin{aligned}
(2\pi)^D \delta^{(D)}(\sum p_i) B_N &= \langle p_1 | V(1, p_2)\ D\ V(1, p_3)\ D\ V(1, p_4)\ D \ldots D\ V(1, p_{N-1}) | p_N \rangle \\
D &= \frac{1}{L_0 - 1}
\end{aligned}
$$

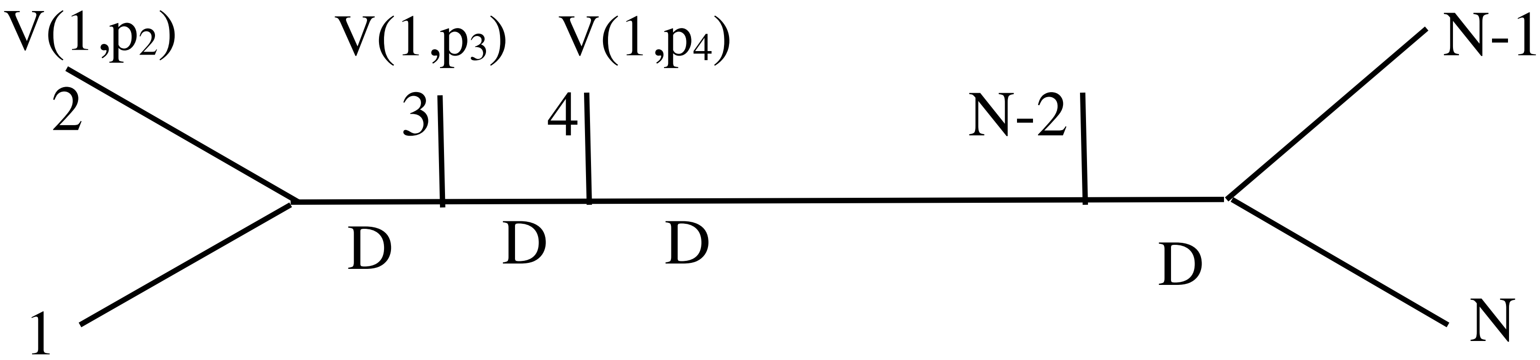
In order to factorize this amplitude it's enough to introduce a complete set of harmonic oscillator states before and after a given "propagator" D. This will provide a pole at:

$$
L_0 = 1 \Rightarrow -\alpha' p^2 = \alpha' M^2 = -1 + \sum_{n,\mu} n\ a^{\dagger}_{n,\mu}\ a^{\mu}_{n} = -1 + \sum_{n,\mu} n N_{n,\mu}
$$

$$(2\pi)^D \delta^{(D)}\left(\sum p_i\right) B_N \quad = \quad \langle p_1 | V(1,p_2)\ D\ V(1,p_3)\ D\ V(1,p_4)\ D \ldots D\ V(1,p_{N-1}) | p_N \rangle$$

$$D \quad = \quad \frac{1}{L_0 - 1}$$

$$1 = \sum \int dk |N_{n,\mu}, k\rangle \langle N_{n,\mu}, k|$$



$$L_0 = 1 \Rightarrow -\alpha' p^2 = \alpha' M^2 = -1 + \sum_{n,\mu} n\ a^\dagger_{n,\mu}\ a^\mu_n = -1 + \sum_{n,\mu} n N_{n,\mu}$$