Strings in Twistor Space

- String theory can be defined by a two-dimensional field theory whose fields take values in target space:
  - *n*-dimensional flat space
  - 5-dimensional Anti-de Sitter × 5-sphere
  - twistor space: intrinsically four-dimensional ⇒ Topological String Theory
- Spectrum in Twistor space is $N = 4$ supersymmetric multiplet (gluon, four fermions, six real scalars)
- Gluons and fermions each have two helicity states

Amplitudes Beyond MHV

Witten's proposal: hep-ph/0312171

- Each external particle represented by a point in twistor space
- Amplitudes non-vanishing only when points lie on a curve of degree d and genus g, where
  - d = # negative helicities – 1 + # loops
  - $g \leq$ # loops; $g = 0$ for tree amplitudes
- Integrand on curve supplied by a topological string theory
- Obtain amplitudes by integrating over all possible curves $\Rightarrow$ moduli space of curves
- Can be interpreted as D_1-instantons

Simple Cases

Amplitudes with all helicities '+' $\Rightarrow$ degree –1 curves.
No such curves exist, so the amplitudes should vanish.
They do.

Amplitudes with one '–' helicity $\Rightarrow$ degree-0 curves: points.
Generic external momenta, all external points won't coincide (singular configuration, all collinear), $\Rightarrow$ amplitudes must vanish.
They do.

Amplitudes with two '–' helicities (MHV) $\Rightarrow$ degree-1 curves: lines.
As we'll see, this is indeed true.

Other Cases

Amplitudes with three negative helicities (next-to-MHV) live on conic sections (quadratic curves)

Amplitudes with four negative helicities (next-to-next-to-MHV) live on twisted cubics

Fourier transform back to spinors $\Rightarrow$ differential equations in conjugate spinors

Differential Operators

Equation for a line (CP^1): $\epsilon_{IJKL} Z_1^I Z_2^J Z_3^K = 0$

gives us a differential ('line') operator in terms of momentum-space spinors

$$F_{123} = \langle \lambda_1 \, \lambda_2 \rangle \frac{\partial}{\partial \tilde{\lambda}_3} + \langle \lambda_2 \, \lambda_3 \rangle \frac{\partial}{\partial \tilde{\lambda}_1} + \langle \lambda_3 \, \lambda_1 \rangle \frac{\partial}{\partial \tilde{\lambda}_2}.$$

Equation for a plane (CP^2): $\epsilon_{IJKL} Z_1^I Z_2^J Z_3^K Z_4^L = 0$

also gives us a differential ('plane') operator

$$K_{1234} = \langle \lambda_1 \, \lambda_2 \rangle \frac{\partial}{\partial \tilde{\lambda}_{3\dot{a}}} \frac{\partial}{\partial \tilde{\lambda}_4^{\dot{a}}} + \text{ perms}$$

Even String Theorists Can Do Experiments

- Apply F operators to NMHV (3 –) amplitudes: products annihilate them! K annihilates them;
- Apply F operators to N^2MHV (4 –) amplitudes: longer products annihilate them! Products of K annihilate them;

$$F_{512}F_{234}F_{345}F_{451}A_5(1^-,2^-,3^-,4^+,5^+) =$$

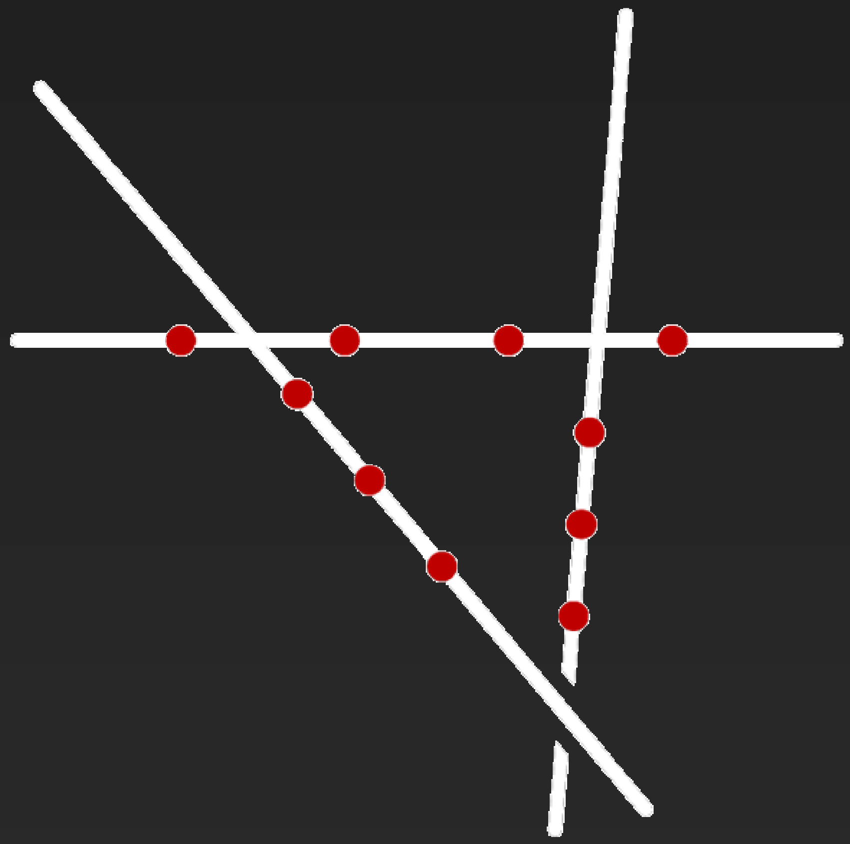
$$F_{512}F_{234}F_{345}F_{451}\frac{[4\,5]^4}{[1\,2]\,[2\,3]\,[3\,4]\,[4\,5]\,[5\,1]} = 0$$

A more involved example

$$F_{612}F_{234}F_{345}F_{561}A_6(1^-,2^-,3^-,4^+,5^+,6^+) = 0$$

Don't try this at home!

Interpretation: twistor-string amplitudes are supported on intersecting line segments



Simpler than expected: what does this mean in field theory?

Cachazo–Svrcek–Witten Construction

hep-th/0403047

Amplitudes can be built up out of vertices ↔ line segments

Intersections ↔ propagators

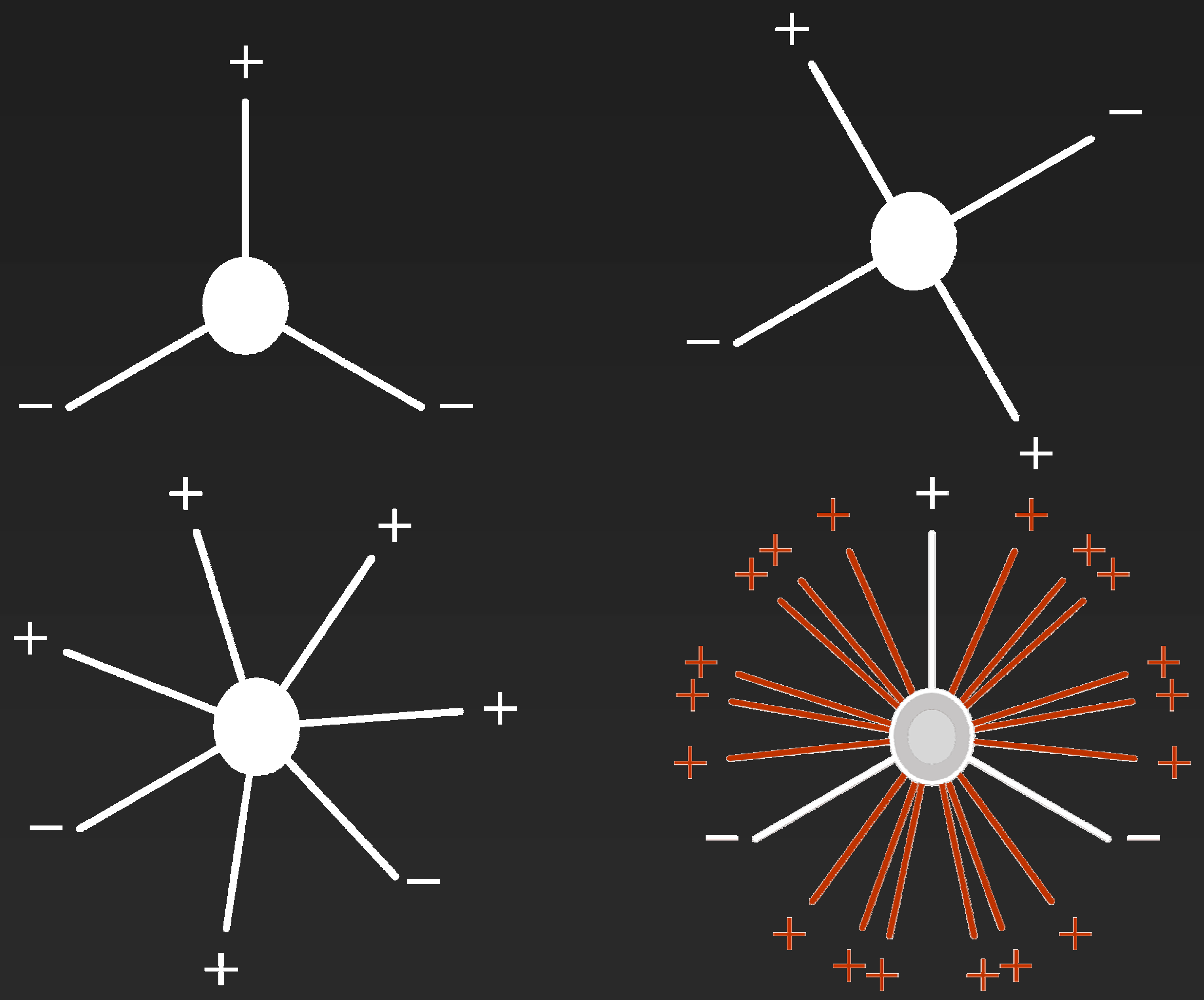
Vertices are off-shell continuations of MHV amplitudes: every vertex has two '–' helicities, and one or more '+' helicities

Includes a three-point vertex

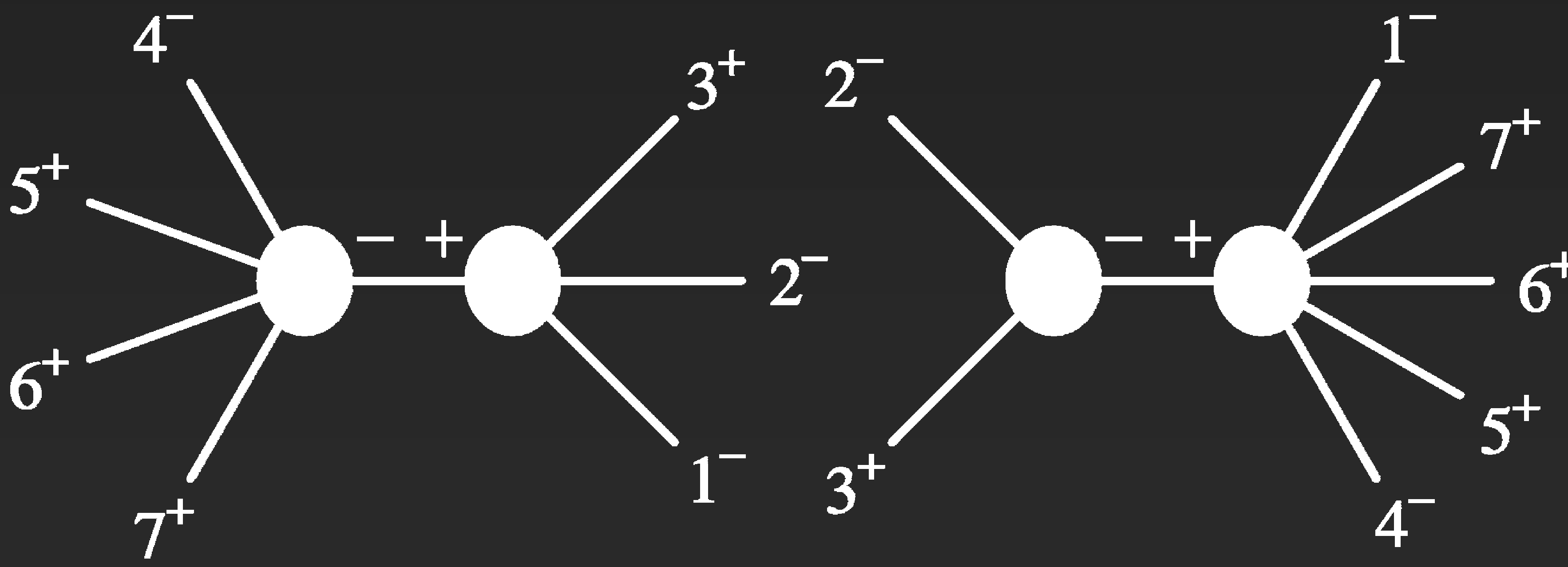
Propagators are scalar ones: i/K^2; helicity projector is in the vertices

Draw all tree diagrams with these vertices and propagator

Different sets of diagrams for different helicity configurations



Seven-Point Example



How Do We Know It's Right?

No derivation from Lagrangian

Physicists' proof:

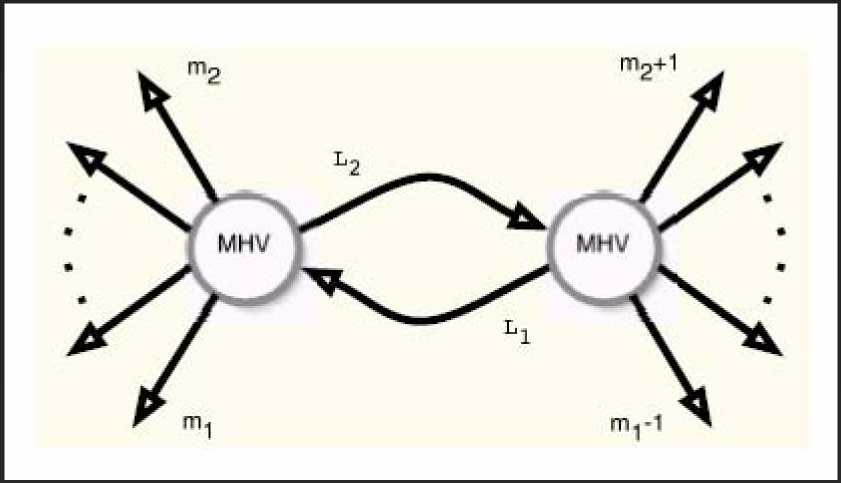
- Correct factorization properties (collinear, multiparticle)
- Compare numerically with conventional recurrence relations through n=11 to 20 digits

Practical Applications

- Compact expressions for amplitudes suitable for numerical use
- Compact analytic expressions suitable for use in computing loop amplitudes
- Extend older loop results for MHV to non-MHV

From Trees To Loops

- Sew together two MHV vertices



Brandhuber, Spence, & Travaglini (2004)

- Simplest off-shell continuation lacks $i\varepsilon$ prescription
- Use alternate form of continuation

$$K = k^\flat + \frac{K^2}{2\eta\cdot K}\eta, \qquad (k^\flat)^2 = 0 \quad \Rightarrow \quad L = \ell + z\eta, \qquad \ell^2 = 0$$

to map the calculation on to the cut + dispersion integral

Brandhuber, Spence, & Travaglini (2004)

- Reproduces MHV loop amplitudes

originally calculated by Dixon, Dunbar, Bern, & DAK (1994)

Unitarity Method for Higher-Order Calculations

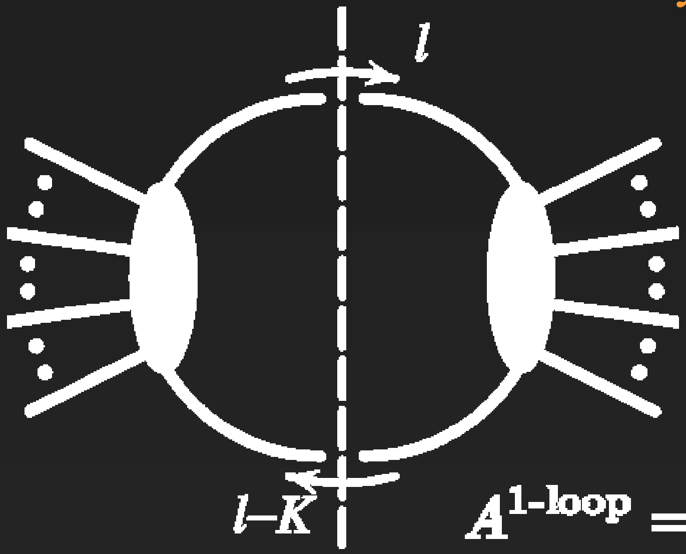
Bern, Dixon, Dunbar, & DAK (1994)

- Proven utility as a tool for explicit calculations
  - Fixed number of external legs
  - All-n equations
- Tool for formal proofs
- Yields explicit formulae for factorization functions
- *I*-duality: phase space integrals ⇔ loop integrals

cf. Melnikov & Anastasiou

- Color ordering

Unitarity-Based Calculations

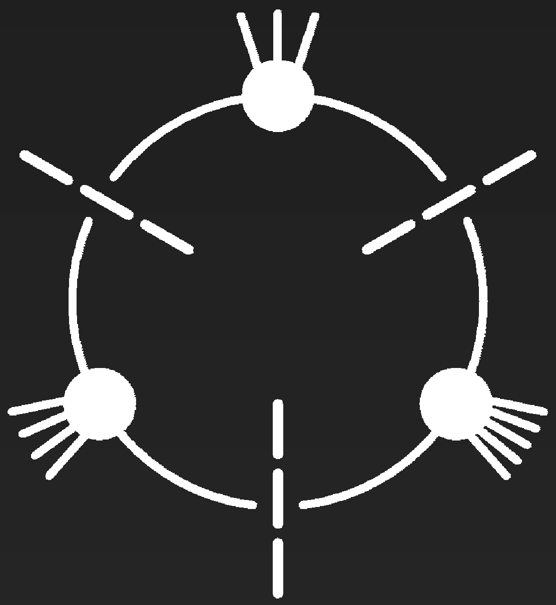


Bern, Dixon, Dunbar, & DAK (1994)

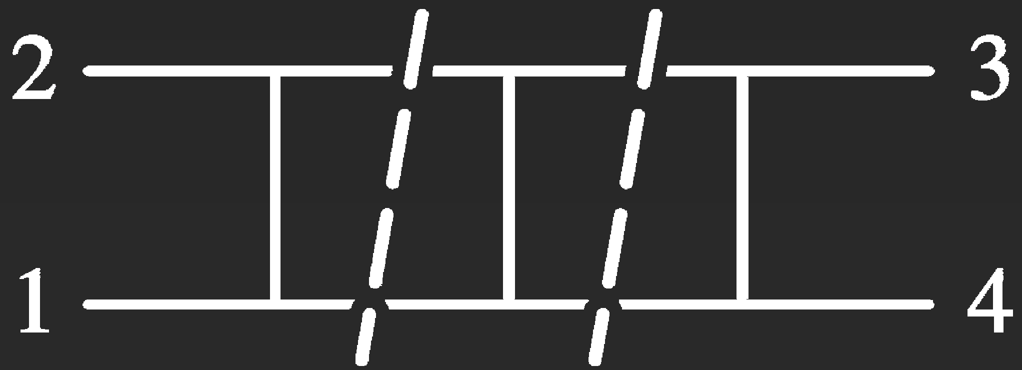
$$A^{\text{1-loop}} = \sum_{\text{cuts } K^2} \int \frac{d^{4-2\epsilon}\ell}{(2\pi)^{4-2\epsilon}} \frac{i}{\ell^2} A^{\text{tree}}_{\text{left}} \frac{i}{(\ell - K)^2} A^{\text{tree}}_{\text{right}}$$

- At one loop in D=4 for SUSY ⇒ full answer
  (also for N =4 two-particle cuts at two loops)
- In general, work in D=4-2ϵ ⇒ full answer
  van Neerven (1986): dispersion relations converge
- Merge channels: find function w/given cuts in all channels
- 'Generalized cuts': require more than two propagators to be present

Generalized Cuts



- Isolate different contributions at higher loops as well



Twistor-space structure

- Can be analyzed with same differential operators
- 'Anomaly' in the analysis; once taken into account, again support on simple sets of lines

Cachazo, Svrček, & Witten; Bena, Bern, DAK, & Roiban (2004)

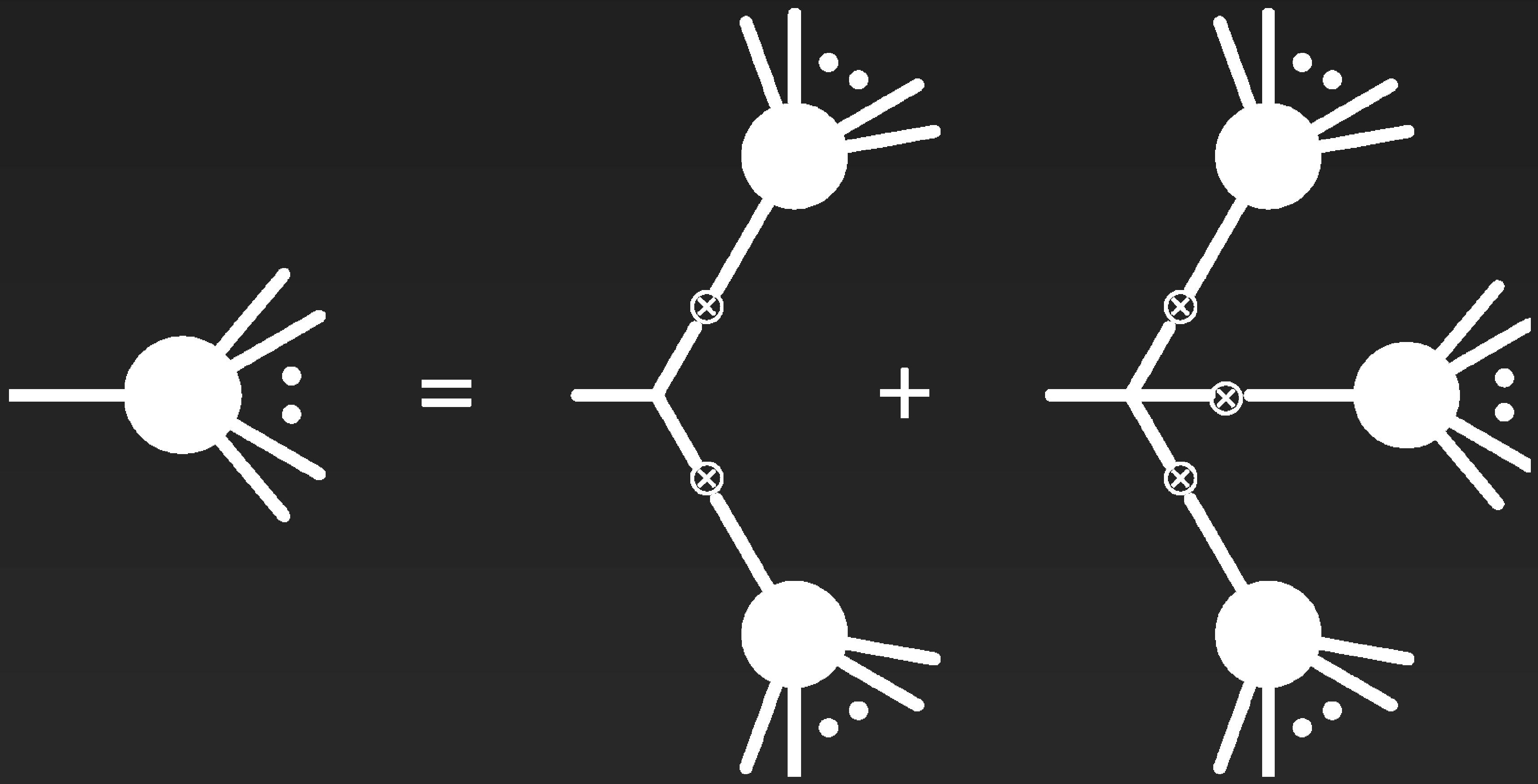
- Non-supersymmetric amplitudes also supported on simple sets

Cachazo, Svrček, & Witten (2004)

Emerging Techniques

- Build on unitarity-based technique & integral basis
- Computation of coefficients via 'holomorphic anomaly'
- Computation of coefficients using complex momenta
- Tree-level and rational one-loop amplitudes via on-shell recurrence relations in complex momenta

Recurrence Relations



Berends & Giele (1988); DAK (1989)

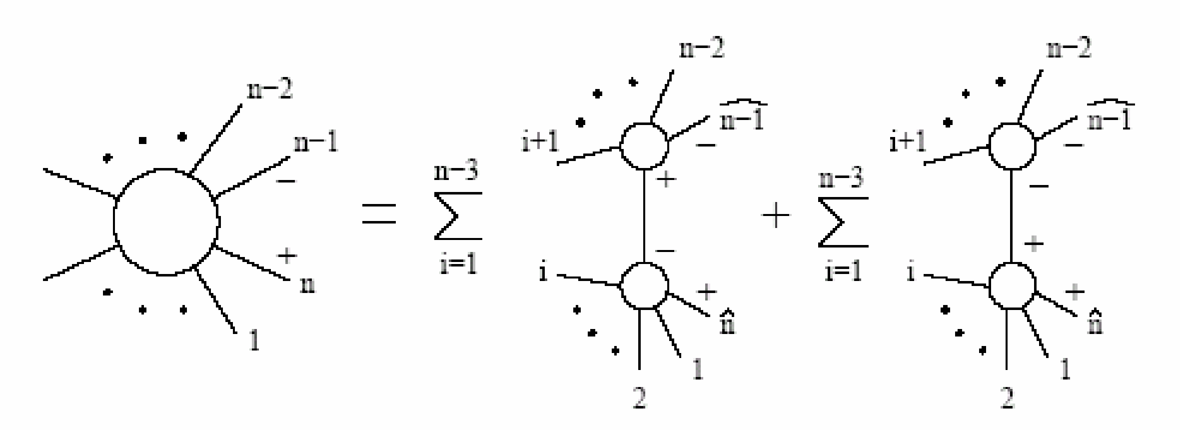
$\Rightarrow$ Polynomial complexity per helicity

On-Shell Recurrence Relations

Britto, Cachazo, Feng (2004)

- Amplitudes written as sum over ‘factorizations’ into *on-shell* amplitudes — but evaluated for *complex* momenta

$$A_n \sim \sum_j A_{j+1}(\ldots, -\hat{K}_{1\cdots j}) \frac{1}{K^2_{1\cdots j}} A_{n-j+1}(\hat{K}_{1\cdots j}, \ldots)$$



$$k_l^\mu \text{unchanged}, \qquad 1 < l < n,$$

$$k_1^\mu \to \hat{k}_1^\mu = k_1^\mu - \frac{K_{1\dots j}^2}{2\left\langle n^- \right| K_{1\dots j} \left| 1^- \right\rangle} \left\langle n^- \right| \gamma^\mu \left| 1^- \right\rangle,$$

$$k_n^\mu \to \hat{k}_n^\mu = k_n^\mu + \frac{K_{1\dots j}^2}{2\left\langle n^- \right| K_{1\dots j} \left| 1^- \right\rangle} \left\langle n^- \right| \gamma^\mu \left| 1^- \right\rangle,$$

$$\hat{K}_{1\dots j} = K_{1\dots j}^\mu - \frac{K_{1\dots j}^2}{2\left\langle n^- \right| K_{1\dots j} \left| 1^- \right\rangle} \left\langle n^- \right| \gamma^\mu \left| 1^- \right\rangle,$$

- Massless momenta: $\hat{k}_1^2, \hat{k}_n^2, \hat{K}_{1\dots j}^2 = 0$

Proof Ingredients

Britto, Cachazo, Feng, Witten (2004)

- Complex shift of momenta (j, l)

$$p_j^\mu \to p_j^\mu(z) = p_j^\mu - \frac{z}{2}\langle j^- |\, \gamma^\mu \,| l^- \rangle,$$

$$p_l^\mu \to p_l^\mu(z) = p_l^\mu + \frac{z}{2}\langle j^- |\, \gamma^\mu \,| l^- \rangle.$$

- Purely rational function
- Behavior as z $\to \infty$: need $A(z) \to 0$
- Knowledge of factorization: at tree level, tracks known multiparticle-pole and collinear factorization

On-Shell Recurrence at Loop Level

Bern, Dixon, DAK (2005)

- Subtleties in factorization: factorization in complex momenta is not exactly the same as for real momenta
- Study finite amplitudes: purely rational A(+…+), A(–+…+)
- Obtain recurrence relations which agree with known results (Chalmers, Bern, Dixon, DAK; Mahlon)
- and yield simpler forms
- Simpler forms involve spurious singularities

Challenges Ahead

- Twistor string side: understand one-loop structure (conformal supergravitons); connected-curve picture?
- Tree-level gauge theory: what relates all the different representations of amplitudes?
- Loop-level gauge theory: compute new QCD amplitudes
- Interface with non-QCD parts of amplitudes
  - Higgs + Gluons (Dixon, Glover, & Khoze)
  - W + QCD (Bern, Forde, Mastrolia, & DAK)
- Keep the virtuous circle twistor string gauge theory going