Particules Elémentaires, Gravitation et Cosmologie
Année 2008-'09

Gravitation et Cosmologie: le Modèle Standard

Cours 4: 16 janvier 2009

Einstein's Equations
and first consequences

- Construction of an invariant action
- Einstein's equations and conservation laws
- The Schwarzschild solution
- Motion in Schwarzschild and two tests of GR

A GCT-invariant action

Instead of following the historical route let us arrive at Einstein's Equations via the action principle.
Like for the non-gravitational interactions we will impose that the action obey the same symmetries that we wish our theory to have. In the case of the SM the basic symmetry to be imposed was the gauge symmetry.
Similarly, for gravity, the relevant symmetry is the one under GCT.
From our previous mathematical discussion we know how to construct quantities which are invariant under GCT.
We shall proceed in two steps: 1) Suitably modify the SM action; 2) Add a pure gravitational term. This will give EEs.

Modifying the SM action

According to the EP we should modify the SM action in such a way that: i) it becomes invariant under GCT; ii) it reduces to the SM action when we go to locally-inertial-coordinates ($g_{\mu\nu}$ --> $\eta_{\mu\nu}$)

This is relatively easy: it basically consists of:

1. Adding a $(-g)^{1/2}$ to the integration measure d^4x
2. Convert derivatives into "covariant derivatives" also w.r.t. GCT (i.e. using Γ, there is a little technical subtlety for fermions but it can be done)

Adding a pure-gravity term

Like again with the SM we shall limit ourselves to terms with up to two derivatives (= low-energy, large-distance approximation). It is easy to find an invariant term with no derivatives. It is simply:

$$S = \lambda \int d^4x \sqrt{-g(x)}$$

where λ is a constant

There is no invariant term with one derivative and a unique one with two derivatives. It contains the curvature scalar and is called the Einstein-Hilbert term

$$S = -\frac{1}{16\pi G} \int d^4x \sqrt{-g(x)} R(x)$$

where G will be identified with Newton's constant

This is basically it! Summarizing, we can write the full action of all interactions (!!) simply as:

$$\begin{aligned} S &= S_{\rm SM}^{\rm (gen.\ cov.)} + S_{\rm gr} \\ S_{\rm gr} &= -\frac{1}{16\pi G}\int d^4x\sqrt{-g(x)}\,(R(x) - 2\Lambda) \\ \Lambda &\equiv 8\pi G\lambda \end{aligned}$$

It is quite amazing how the EP and invariance under GCT tell us how to add gravity to our chosen theory of the other interactions! This makes it even more fustrating the fact that we are unable to fully quantize such a simple and beautiful generalization of the SM.

Einstein's equations

In order to get Einstein's Equations (EE) we simply have to put to zero the variation of the action. Consider first the variation of S_{gr}. With the help of a few formulae for the variation of det g and R we arrive at:

$$\delta S_{\rm gr} = \frac{1}{16\pi G}\int d^4x\sqrt{-g(x)}\left(R^{\mu\nu}(x) - \frac{1}{2}g^{\mu\nu}R(x) - g^{\mu\nu}\Lambda\right)\delta g_{\mu\nu}(x)$$

There is actually a tricky point here: the variation of R gives the above result modulo some boundary terms. In the usual variational principle boundary terms are neglected since $\delta g_{\mu\nu}=0$ on the boundary. But here we also get derivatives of $\delta g_{\mu\nu}$ on the boundary. Need to add surface term to S_{gr} (Hawking-Gibbons). Unimportant here.

The variation of the SM action is well defined but complicated. One replaces the SM action by an effective model generally called the matter action, S_m. Then one simply defines the energy momentum tensor of matter from the variation of S_m:

$$\delta S_m = \frac{1}{2} \int d^4x \sqrt{-g(x)} T^{\mu\nu}(x) \delta g_{\mu\nu}(x)$$

Imposing now: $\delta \left(S_m + S_{\rm gr} \right) = 0$ we get:

$$R^{\mu\nu}(x) - \frac{1}{2} g^{\mu\nu} R(x) - g^{\mu\nu} \Lambda = -8\pi G T^{\mu\nu}$$

These are just Einstein's equations in the presence of a cosmological constant (apparently needed experimentally!)

Covariant conservation of $T_{\mu\nu}$

There is an important consequence of the independent invariance of S_m and S_{gr}. On the equations of motion the energy-momentum tensor is covariantly conserved!

$$T^{\nu}_{\mu;\nu} = 0$$

Proof: set to 0 the variation of S_m wrt infinitesimal GCT

$$0 = \delta_{GCT} S_m = \frac{1}{2}\int d^4x \sqrt{-g}\, T^{\mu\nu} \delta_{GCT} g_{\mu\nu} + \sum_i \int \frac{\delta S_m}{\delta \phi_i} \delta_{GCT} \phi_i$$

Last term vanishes on eom. For $\tilde{x}^{\mu} = x^{\mu} - \epsilon^{\mu}(x)$

$$\delta_{GCT} g_{\mu\nu} = g_{\mu\rho} \epsilon^{\rho}_{,\nu} + g_{\nu\rho} \epsilon^{\rho}_{,\mu} + g_{\mu\nu,\rho} \epsilon^{\rho}$$

Integrating by parts, since ϵ^{ρ} is arbitrary, gives $T^{\nu}_{\mu;\nu} = 0$

The Bianchi identities

We may repeat the argument for S_{gr} itself. Instead of $T_{\mu\nu}$ we get the Einstein tensor and therefore the analogue of

$$T^{\nu}_{\mu;\nu} = 0 \quad \text{becomes} \quad (R^{\nu}_{\mu} - \frac{1}{2}\delta^{\nu}_{\mu} R)_{;\nu} = 0 \quad \text{(trivial for } g_{\mu\nu}\text{)}$$

as it should, because of EEs! The latter equation is part of a larger set of (differential) identities satisfied by the Riemann tensor and known as the Bianchi identities (Cf. homogeneous Maxwell eqns.). Other forms of E.equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - g_{\mu\nu}\Lambda = -8\pi G T_{\mu\nu} \qquad \text{or also}$$

$$R_{\mu\nu} = -8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\lambda}_{\lambda} \right) - g_{\mu\nu}\Lambda$$

10 formidable equations!

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu} \quad ; \quad (\Lambda = 0)$$

Non linear 2nd order PDEs. An enormous literature, often mathematically very sophisticated. Four kinds of results:

1. General results on the existence of solutions and on their qualitative properties (e.g. development of singularities)
2. Explicit analytic solutions in the presence of special symmetries (leading to ODEs or, in some cases, 2D-eqns). Homogeneous isotropic cosmology belongs here (see later)
3. Perturbations of trivial space-times and/or of exact solutions (e.g. gravitational waves, cosm. perturbations)
4. Numerical solutions (much progress recently)

The Schwarzschild solution

One of the first and simplest examples of exact solutions is that of static, spherically symmetric gravitational field (Schwarzschild, 1916).

A very interesting property of the Sch. metric is its uniqueness outside the region of space where the source of the gravitational field is. Like in the Newtonian case, the solution is fully determined by a single parameter, M.

The Sch. metric can be written in different coordinate systems. In a particularly convenient one the most general solution has the explicitly static and spherically symmetric form:

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\Omega^2 \quad ; \quad d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$$

The Schwarzschild solution

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\Omega^2 \quad ; \quad d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$$

Assume that there is no matter for r > r* ($T_{\mu\nu}$(r > r*)=0)

Then also $R_{\mu\nu}$(r > r*)=0. Also A(r), B(r) -->1 at very large r

Only R_{rr}, R_{tt}, $R_{\theta\theta}$ give non trivial equations (and R_{rr} =0 follows from the other two). We get the following two equations:

$$(AB)' = 0 \Rightarrow A = 1/B \; ; \; (rB)' = 1 \Rightarrow B = 1 + \frac{C}{r}$$

We have already found (cours 2) the meaning of the constant of integration

$$-g_{00} \sim 1 - \frac{2GM}{r}$$

=> C = -2GM

Finally we have (r >r*):

$$ds^2_{Schw.} = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{1}{(1 - \frac{2GM}{r})} dr^2 + r^2 d\Omega^2$$

Motion in a Schwarzschild metric

$$ds^2_{Schw.} = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{1}{(1 - \frac{2GM}{r})} dr^2 + r^2 d\Omega^2$$

Easier to get eom by varying the action for a particle in that specific metric. For r >r* and for a motion on the equatorial plane θ= π/2, the generic action (c=1)

$$S = -m \int d\lambda \sqrt{-g_{\mu\nu}(x) \frac{dx^\mu(\lambda)}{d\lambda} \frac{dx^\nu(\lambda)}{d\lambda}}$$ gives $$\left(B(r) = 1 - \frac{2GM}{r}\right)$$

$$S = -m \int d\lambda \sqrt{B(r) \left(\frac{dt}{d\lambda}\right)^2 - B^{-1}(r) \left(\frac{dr}{d\lambda}\right)^2 - r^2 \left(\frac{d\phi}{d\lambda}\right)^2}$$

The actual motion is the one minimizing S wrt variations of t(λ), ϕ(λ), r(λ). First two variations are quite trivial:

Motion in a Schwarzschild metric

$$S = -m \int d\lambda \sqrt{B(r) \left(\frac{dt}{d\lambda}\right)^2 - B^{-1}(r) \left(\frac{dr}{d\lambda}\right)^2 - r^2 \left(\frac{d\phi}{d\lambda}\right)^2} \qquad \left(B(r) = 1 - \frac{2GM}{r}\right)$$

$$\frac{d}{d\lambda}\left(B \frac{dt}{d\lambda}\right) = 0 \quad \Rightarrow \quad \frac{dt}{d\lambda} = B^{-1}$$

(normalizing λ)

$$\frac{d}{d\lambda}\left(r^2 \frac{d\phi}{d\lambda}\right) = 0 \quad \Rightarrow \quad \frac{d\phi}{d\lambda} = \frac{J}{Er^2}$$

(J and E are int. constants)

The third equation corresponds to conservation of energy

$$\left(\frac{dr}{d\lambda}\right)^2 = 1 - \frac{B(r)}{E^2}\left(m^2 + \frac{J^2}{r^2}\right) \equiv \Delta(r; E, J)$$

This is the (implicitly given) analytic solution of the problem!

Motion in a Schwarzschild metric

$$\frac{dt}{d\lambda} = B^{-1} \;;\; \frac{d\phi}{d\lambda} = \frac{J}{Er^2} \;;\; \left(B(r) \equiv 1 - \frac{2GM}{r}\right)$$

$$\left(\frac{dr}{d\lambda}\right)^2 = 1 - \frac{B(r)}{E^2}\left(m^2 + \frac{J^2}{r^2}\right) \equiv \Delta(r; E, J)$$

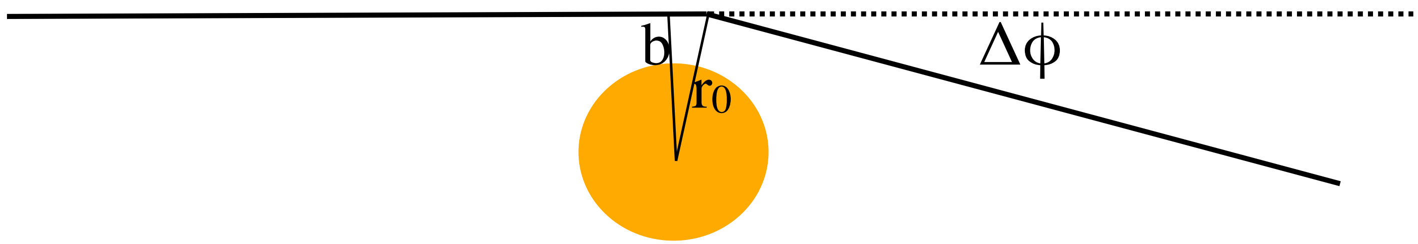
reduces to quadratures (elliptic integrals)

$$d\lambda = \frac{dr}{\sqrt{\Delta(r)}} \;;\; dt = \frac{dr}{B(r)\sqrt{\Delta(r)}} \;;\; d\phi = \frac{J\,dr}{Er^2\sqrt{\Delta(r)}}$$

$\Delta=0$ corresponds to closest point for unbound trajectories or to perihelion and aphelion for bound trajectories.

If the Schwarzschild radius R_S= 2GM lies outside the body we have a black hole with a horizon at r = R_S.

I. Deflection of light



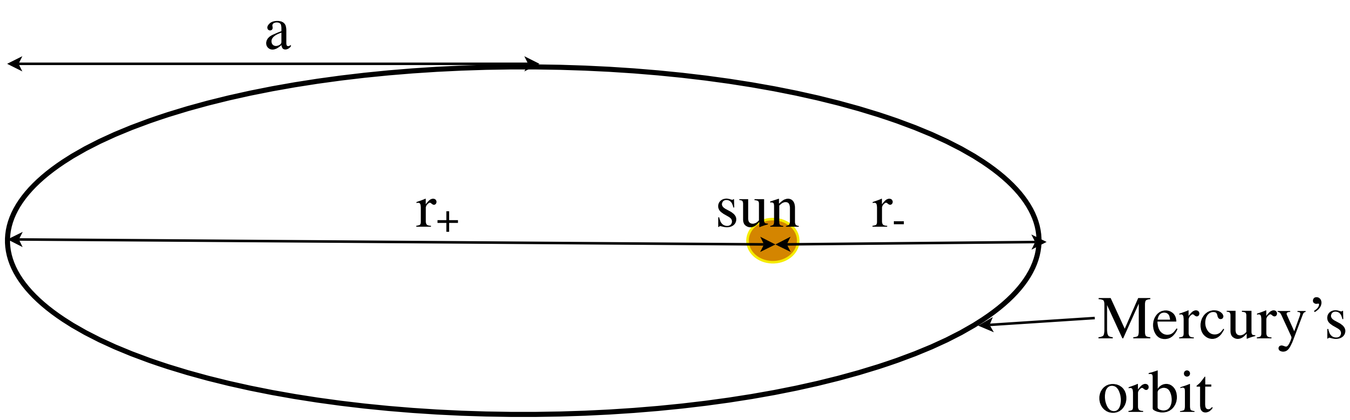
$$\Delta\phi = 2\int_{r_0}^{\infty} \frac{dr(J/E)}{r^2\sqrt{\Delta(r)}} - \pi \quad ; \quad J/E = b \sim r_0$$

The approximate calculation is long but straightforward and gives:

$$\Delta\phi \sim \frac{2R_S}{b} = \frac{4GM_{sun}}{R_{sun}}\frac{R_{sun}}{b} \sim 1.75'' \frac{R_{sun}}{b}$$

Confirmed during sun eclipses (first time in 1919!)

II. Precession of Perihelia



$$\Delta\phi = 2\int_{r_-}^{r_+} \frac{dr(J/E)}{r^2\sqrt{\Delta(r)}} - 2\pi$$ Another long calculation gives:

$$\Delta\phi \sim \frac{3\pi R_S}{2}\left(\frac{1}{r_+} + \frac{1}{r_-}\right) = \frac{3\pi R_S}{(1-e^2)a} \; ; \; r_\pm = (1 \pm e)a$$

For Mercury this is ~ 0.1038" /revolution or ~ 43.03"/century (100y~ 415 revolutions). Data since 1765 confirm it (Clemence 1943) with high accuracy (better than from light deflection)

Addendum about GR tests

Actually, the perihelion precession test is affected by uncertainties. These are due to the existence of other competing contributions having nothing to do with GR:

1) planetary perturbations, 2) the Earth's spin precession, and 3) a possible quadrupole moment of the sun. In particular, the latter effect is small but poorly known

As we shall see in T. Damour's seminars, there are by now better tests of GR. Furthermore, it has become standard practice to test GR against alternative theories by establishing experimental bounds on some "GR-deformation" parameters that characterize them. No evidence for significant deviations from GR has been found so far.