Particules Élémentaires, Gravitation et Cosmologie

Année 2004-2005

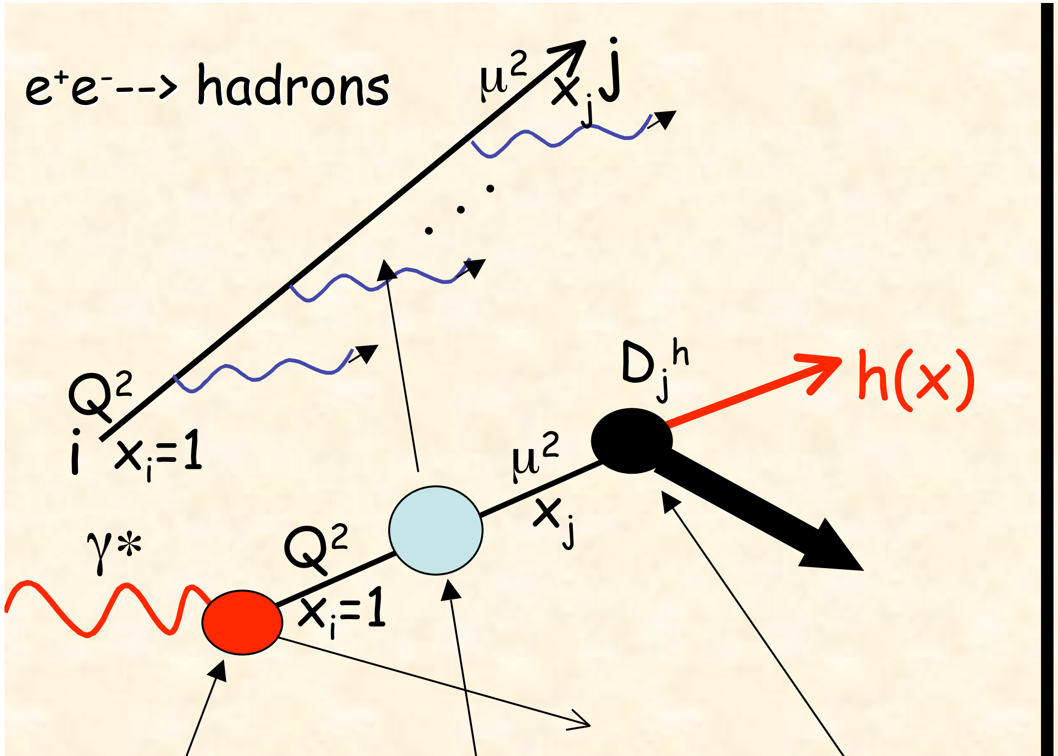
Interactions fortes et chromodynamique quantique I: Aspects perturbatifs

Cours V: 29 mars 2005

1. Summary of previous lecture
2. The Operator-Product Expansion (OPE)
3. Application to e^+e^---> hadrons and to DIS
4. Heavy quarks and their decoupling in QCD

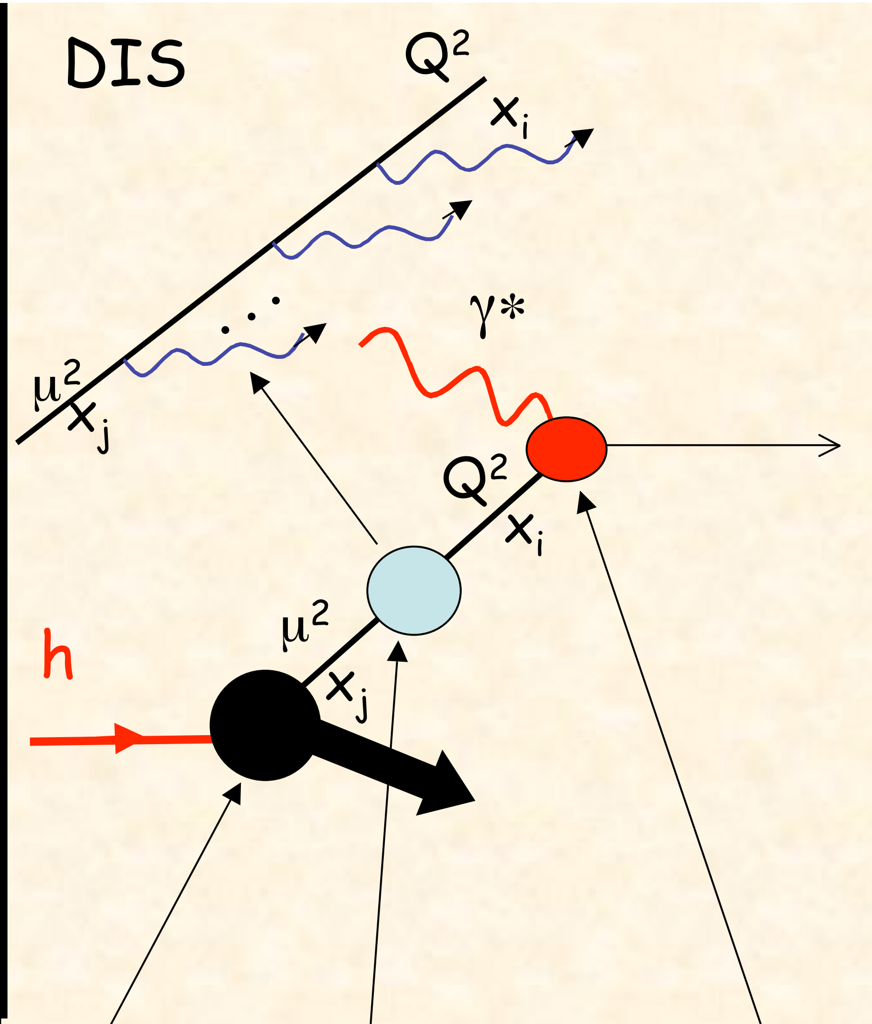
1. Summary of lecture no. 4

- Concept of inclusive cross sections
- Most inclusive: $\sigma_T(e^+e^- \rightarrow$ hadrons) is ICS (see lect.3)
- Next most inclusive in $e^+e^- \rightarrow$ hadrons: $\frac{d\sigma(e^+e^- \rightarrow h(x) + X)}{dx}$
- Turning lines around => $\sigma(e^- p \rightarrow e^- + X)$ = DIS
- The latter two are IS but not CS. However collinear singularities can be indentified, computed and resummed provided we work at partonic level and keep final parton at some off-shellness $\mu > \Lambda$.
- The basic result of this approach is « factorization », best illustrated in a couple of pictures
- It is at the very heart of the QCD parton model



$$\sigma_h = \sigma_i(Q^2) \otimes E_i^j(x_j/x_i, \mu^2, Q^2) \otimes D_j^h(\mu^2)$$

$$D_i^h(x_h, Q^2)$$



$$\sigma_h = F_h^j(\mu^2) \otimes E_j^i(x_i/x_j, \mu^2, Q^2) \otimes \sigma_i(Q^2)$$

$$F_h^i(x_i, Q^2)$$

GLAP

$$Q^2 \frac{\partial}{\partial Q^2} F_h^i(x, Q^2) = \frac{\alpha(Q^2)}{2\pi} \sum_j \int_x^1 dz \frac{P_{j\to i}(z)}{z} F_h^j(x/z, Q^2)$$

2. The Operator-Product Expansion

2.1 General considerations

Hard processes are characterized by some large momentum/energy. By the UP of QM this corresponds to short distance: mathematically, we go from p-space to x-space (and back) by Fourier Transforms (FT).

- Indeed both in e^+e^---> hadrons and in DIS we are dealing (in the x-section) with the product of two currents J(x) and J(y), evaluated @ small z = x-y
- In QM observables such as currents are represented by (hermitian) operators whose matrix elements provide expectation values. A relevant example:

$$J_\mu^{(el)} = \sum_f e_f \bar{\Psi}_f \gamma_\mu \Psi_f$$

In 1969 K. Wilson conjectured that, as x-> y, the product of any two local operators $O_i(x)\, O_j(y)$ can be expanded in a (generally infinite) series of local operators with c-number coefficients that are typically singular for x->y (OPE):

$$O_i(x)\, O_j(y) \dashrightarrow \Sigma_k C_{ij}^{\ k}(z)\, O_k(X);\ z = x-y,\ X = (x+y)/2$$

This is true in free field theory and, with suitable definition of the operators, was also proven in renormalizable QFT.

OPE allows to factorize the Q (canonical variable conjugate to z) dependence from the rest.

If the $C_{ij}^{\ k}$ are known, Q-dependence is under control even if we do not know how to calculate the matrix elements of O_k

Standard FT considerations tell us that the leading behaviour at large Q corresponds to the most singular $C_{ij}^{\ k}$ as z-->0

2.2 Renormalization and RG equations in OPE

Naive dimensional analysis would suggest that:

$$C_{ij}^{k} \sim z^{(d_k - d_i - d_j)}$$

Remember: $z = x-y$. For fixed i,j, most singular C_{ij}^{k} for smallest d_k

where d_i is the (mass) dimension of O_i . However this is not the true behaviour in an interacting QFT. Indeed the «insertion» of the operator O_i in a Feymann diagram can lead to new UV divergences on top of those we have already removed by renormalizing the elementary fields (Z_2). This is why we have to « renormalize » the composite operators as well by defining*): $O_i^{Bare}(M) = Z_i(\mu, M)\, O_i(\mu)$. Like for the elementary fields we define the anomalous dimensions as:

$$\gamma_i \equiv \partial log Z_i / \partial log \mu$$

*)For simplicity we consider first the case where there is no Op-mixing

However, once more, μ -dependence should cancel out between $Z_i(\mu,M)$ and $O_i(\mu)$ meaning that : $\partial log O_i / \partial log \mu = -\gamma_i$

Now it is easy to find the RG equation obeyed by $C_{ij}^k(z)$:

$$\left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta \frac{\partial}{\partial \alpha} + \frac{1}{2}(\gamma_i + \gamma_j - \gamma_k) \right) C_{ij}^k = 0$$

Going over to Q-space, the solution of this equation is:

$$C_{ij}^k(Q^2, \mu^2, \alpha_\mu) = Q^{(d_i + d_j - d_k)} c_{ij}^k(Q^2/\mu^2, \alpha_\mu)$$

$$c_{ij}^k(Q^2/\mu^2, \alpha_\mu) = c_{ij}^k(1, \alpha(Q^2)) exp \left(\frac{1}{2} \int_{\mu^2}^{Q^2} \frac{dq^2}{q^2} [\gamma_i + \gamma_j - \gamma_k](\alpha(q^2)) \right)$$

For a theory with a non-trivial UV fixed point at $\alpha = \alpha^*$ this would give the « anomalous-dimensions behaviour »:

$$C_{ij}^k(Q^2, \mu^2, \alpha_\mu) = Q^{(d_i + \gamma_i^* + d_j + \gamma_j^* - d_k - \gamma_k^*)}$$

In QCD the situation is slightly more complicated since the UV fixed point is at $\alpha = 0$ (AF).

The factor $c_{ij}{}^{k}(1, \alpha(Q^2))$ has a smooth expansion in $\alpha(Q^2)$ (like our CIS quantities) while the exponential gives rise to powers of log (Q^2) (logarithmic scaling violations) since the γ_i are proportional to α

This already smell a bit like what we have found in the two previous lectures. Let's see why…

3. Applications of OPE

3.1 $\sigma_T(e^+e^- \to$ hadrons)

Unitarity tells us that this quantity is related to (the imaginary part of) the correlator of two em currents

$$Q^4\sigma_T \sim Im\int d^4x e^{iQx}\langle 0|T\left[J_\mu^{(el)}(x)J_\nu^{(el)}(0)\right]|0\rangle$$

$$J_\mu^{(el)} = \sum_f e_f\bar{\Psi}_f\gamma_\mu\Psi_f$$

Large-Q means looking at small x i.e. to OPE

The most singular coefficient is the one related to $O_k = 1$

On naive dimensional grounds it gives a FT ~ Q^2 . In this case all γ are 0 (because $J^{(el)}$ is conserved) and the answer is:

$Q^4\sigma_T(e^+e^- \to$ hadrons) = $Q^2\, c_{JJ}{}^1(1, \alpha(Q^2)) + O(Q^{2-d}\exp(..\gamma))$

The first term corresponds precisely to what we had found more directly in lecture no 3 (e.g. $c_{JJ}^{1}(1,0) \sim N\,\Sigma e_f^2$)

The nice thing about the OPE is that it also gives us an idea about power-suppressed corrections, at least of the power of Q by which they are suppressed. For this we have to identify the next operator which can appear in the OPE and has a non zero vacuum-expectation-value (VEV).

Since only *gauge-invariant* operators can appear one would think immediately about fermion bilinears ($d_k=3$) or *gluon* composites ($d_k=4$). However, unless the vacuum breaks L.I., the former have to be scalars under Lorentz and thus must involve two l.h. or two r.h. spinors. Vector currents involve only mixed pairs. Hence these operators can only appear with a coefficient ~ m (chiral symmetry). The *gluon* condensate is OK

Hence these corrections are either $O(m\Lambda^3/Q^4)$ or $O(\Lambda^4/Q^4)$

3.2 DIS

At first sight the case of DIS looks very similar from the point of view of the OPE (while it looked closer to the one-part. inclusive x-section from the IR-sing. point of view). This is because, again from unitarity, we can write

$$Q^4 \sigma(eN \to e + X) \sim Im \int d^4x e^{iQx} \langle N | T \left[J_\mu^{(el)}(x) J_\nu^{(el)}(0) \right] | N \rangle$$

The same product of currents as in $\sigma_T(e^+e^- \to \text{hadrons})$ appears. Only difference is that VEV becomes NEV…

It turns out that this makes a big difference. Why?

1. The lowest operator (1) gives a « disconnected » diagram and consequently no contribution to σ

2. The fermion bilinears are now important even for m->0. Indeed only vector (and axial vector) currents appear but now they have perfectly finite matrix elements in the nucleon state prop. to p_μ . Example (γ are all zero again!)

$$T[J_\mu(x)J_\nu(0)] \sim \sigma_{\mu\lambda\nu\rho}\frac{x^\lambda}{(x^2-i\varepsilon)^2}J^\rho(0)$$

where $\sigma_{\mu\lambda\nu\rho} = \eta_{\mu\lambda}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\lambda} - \eta_{\mu\nu}\eta_{\lambda\rho}$

gives a contribution proportional to $(2p_\rho Q^\rho)/Q^2 = 1/x$ (scaling!)

3. Higher-dimension operators are not really suppressed. For $J_{\mu\nu} = \bar{\Psi}\gamma_\mu D_\nu \Psi$ the coeff. will give and extra Q^σ/Q^2 and the NEV an extra p_σ ,thus together, just another factor $1/x = O(1)$ (modulo anomalous dimension effects!)

It is quite easy to classify all the operators that contribute leading terms (same Q^2 , different x-dependence, usually called leading twist). They fall in three classes:

1. Quark bilinears (n= 1, 2, …; S = symmetrization over μ's)

$$O_f^{(n)} = S\left[\bar{\Psi}_f \gamma_{\mu_1} D_{\mu_2} \ldots D_{\mu_n} \Psi_f\right] , f = 1, 2 \ldots N_f$$

2. Purely gluonic (n=2, 3, …)

$$O_g^{(n)} = S\left[F_{\alpha\mu_1} D_{\mu_3} \ldots D_{\mu_n} F^{\alpha}_{\mu_2}\right]$$

3. Axial operators that only contribute to polarized DIS

It is easy to check that all these operators can contribute at leading order to DIS with an x-dependence given by x^{-n}

Using dispersion relations we can invert this relation and connect the matrix elements of the operators $O^{(n)}$ to the moments of the structure functions $F_N^i(n, \mu)$ ($i = q_f, g$).

One has to fold in the effect of the coefficient functions in particular their non-trivial RG behaviour due to the anomalous dimensions $\gamma^{(n)}{}_i$ (actually $\gamma^{(n)}{}_{ij}$) which gives a non-trivial Q dependence to the actual structure functions $F_N{}^i(n, Q)$

$$Q^2 \frac{\partial}{\partial Q^2} F^i(n, Q^2) = \sum_j \gamma_j^{(n)i}(\alpha) F^j(n, Q^2)$$

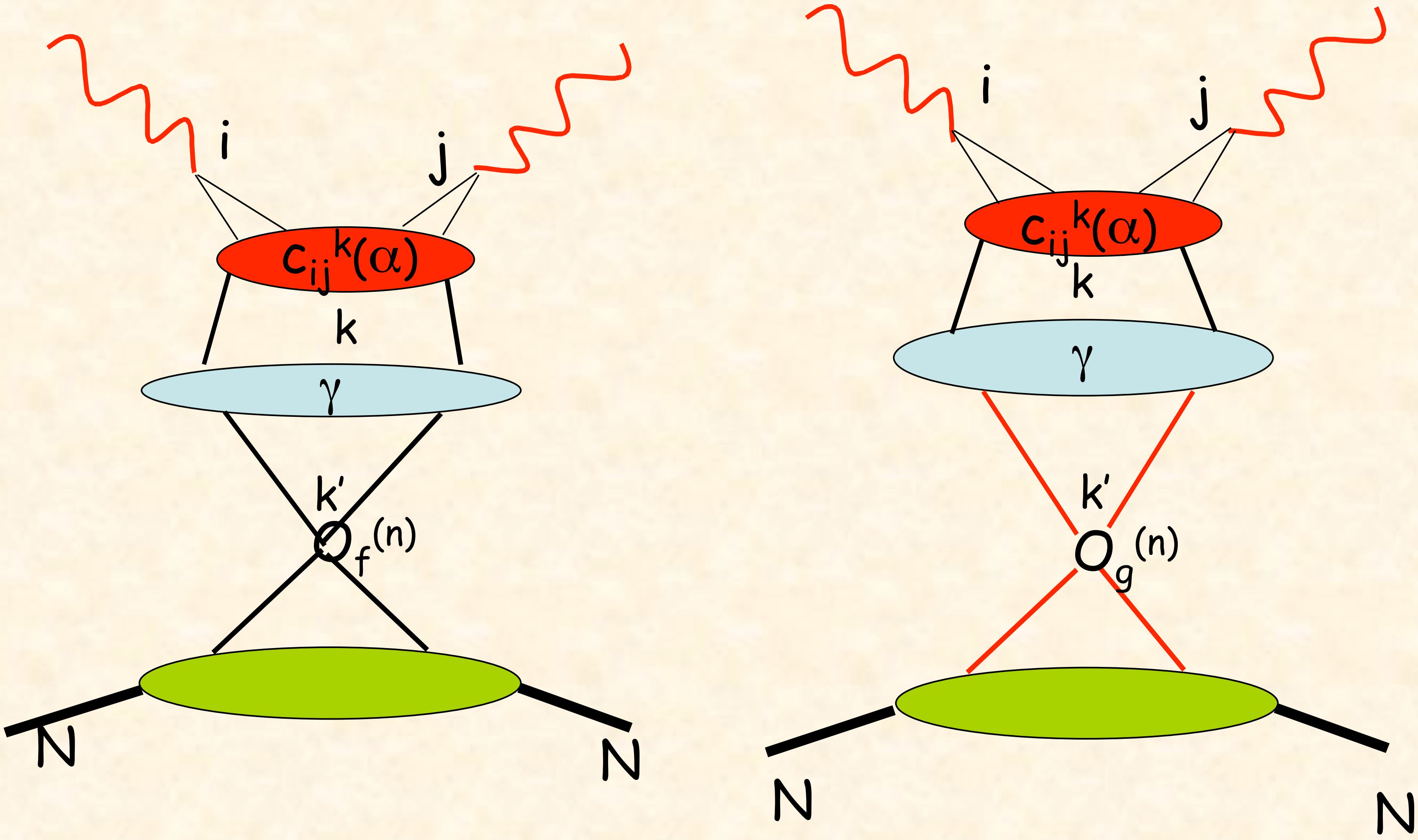
This is the same as the equations we arrived at last week

$$Q^2 \frac{\partial}{\partial Q^2} F_n(Q^2) = \frac{\alpha(Q^2)}{2\pi} P_n F_n(Q^2) \equiv \gamma_n(Q^2) F_n(Q^2)$$

$$Q^2 \frac{\partial}{\partial Q^2} F_h^i(x, Q^2) = \frac{\alpha(Q^2)}{2\pi} \sum_j \int_x^1 dz \frac{P_{j\to i}(z)}{z} F_h^j(x/z, Q^2)$$

Historically, the OPE derivation came first and was then reinterpreted as a parton splitting process by DGLAP…

Graphical illustration of OPE description of DIS



3.3 Applications of OPE description

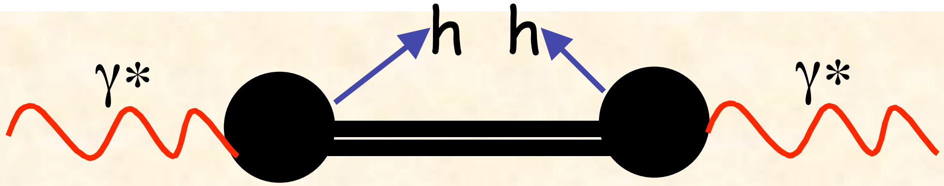
Since moments of the PDF's get related via OPE to matrix elements of operators this allows:

1. To obtain some interesting sum rules if we know (we measure or we estimate theoretically) that matrix element. Examples:
   - Adler SR: absolute prediction in $\sigma(\nu\ p)$- $\sigma(\nu^*p)$ (F_2)
   - Gross-L. Smith SR in $\sigma(\nu\ p)$ + $\sigma(\nu^*p)$ (F_3)
   - Bjorken SR relating polarized DIS in $\sigma(ep)$- $\sigma(en)$ to $\langle N|J_{\mu 5}|N\rangle$ (which is known from n beta decay)
   - Momentum SR leading to gluonic component
2. To compute moments of partonic distributions by non-perturbative methods, such as lattice QCD, that can give those matrix elements (next year?)

3.4 Limitations of OPE approach

While the OPE approach is straightforward for $e^+e^- \to$ hadrons and DIS, its extension to other hard processes is quite non-trivial. For instance, the one-particle inclusive x-section can expressed, via unitarity, as

$$\frac{d\sigma(e^+e^- \to h+X)}{d^3k_q/E_q} \sim Disc \int d^4x e^{iQx} \langle 0|T\left[J_\mu^{(el)}(x)h(y)h(z)J_\nu^{(el)}(0)\right]|0\rangle$$



We see that we do not quite have to do with a product of currents. However agreement with the other approach can be obtained by appealing to an extension of OPE that goes under the name of « cut vertices ». In this case the IR approach looks much more straightforward.

4. Heavy quarks

Up to here we have not paid much attention to quark masses. We have implicitely assumed that they can be set to zero when we go to high energies (meaning $E \gg \Lambda \sim 1$ GeV). This is certainly a very good approximation for the u and d quarks (masses of order a few MeV) and also a good one for the s-quark (mass of order 100 MeV).

However, we now know that quarks heavier than Λ do exist in (at least) three flavours:

$$c(m_c \sim 1.5 \text{ GeV}),\ b(m_b \sim 5 \text{ GeV})\ \&\ t(m_t \sim 175 \text{ GeV})$$

We will see in MC's seminar that heavy quarks offer new opportunities for testing QCD, for measuring some parton densities of otherwise difficult access, etc.

Here we would just like to discuss how they may affect the predictions at energy scales much lower than $m_{Heavy\text{-}q}$ (m_H).

- There are clear analogies with OPE. Having a very large momentum flowing in a diagram forces points in space to be very close to one another
- Similarly, having a virtual heavy quark propagating in a diagram forces it to move over very short space-time distances
- In other words, at $E \ll m_H$ the effect of virtual heavy-quark amonts to adding some local terms to the effective action (plus corrections that vanish like powers of E/m_H).
- If removing the heavy quark keeps the theory consistent and sharing the symmetries with the original one, the new local terms must still respect those symmetries.
- One can take into account heavy-quark effects through a further local correction to the effective action.

- Let us recall from Lecture 3:

$$S_{eff} = -\frac{1}{16\pi}\int d^4x \left(\frac{1}{\alpha_0} + L_3(-\frac{\Box}{M^2}, \frac{m}{M}, \alpha_0)\right) \hat{F}^a_{\mu\nu}\hat{F}^{a,\mu\nu}$$

$$+ \int d^4x \left(1 + L_2(-\frac{\Box}{M^2}, \frac{m}{M}, \alpha_0)\right) \bar{\Psi} i\gamma^\mu \hat{D}_\mu \Psi$$

$$- \int d^4x \left(1 + L_m(-\frac{\Box}{M^2}, \frac{m}{M}, \alpha_0)\right) m_0 \bar{\Psi}\Psi + S'_{eff}$$

$$L_i(-\frac{\Box}{M^2}, \frac{m}{M}, \alpha_0) = L_i(\frac{\mu^2}{M^2}, \frac{m}{M}, \alpha_0) + L'_i$$

The heavy-q contribution to L_i is Q-independent for Q << m_H. At those energies effect is accounted for by an additional (unobservable) contribution to $L_i(m^2)$. A Q-dependent L_i' only comes up when we go above the heavy-quark threshold. The full contribution is obtained by matching at Q ~ m_H

- This works fine in QCD since the number of quark flavors is arbitrary: QCD is *gauge* invariant for any N_f
- On the other hand, in the EW theory this is not the case since eliminating a single quark upsets a doublet (with respect to the $SU(2)_L$ *gauge group*). There is no decoupling of the t-quark in the EW theory and indeed the mass of the t-quark was first successfully estimated through its virtual effects and their comparison with precision data.
- One would then suspect that in the EW theory decoupling works if we *give* a large mass to a full multiplet. However things are not so simple since quark-masses in the EW theory cannot be *given* as such (they break the *gauge* symmetry) but through the Higgs mechanism i.e. through Yukawa couplings of the quarks to the Higgs-boson and the latter's VEV. Large masses means large YC's and a large coupling is not as easy to « decouple » as a large mass…