WHY THE CP^{N-1} MODEL ?

- THE $1/N$ EXPANSION IS DIFFICULT TO PERFORM IN QCD BECAUSE THE GAUGE FIELD $(A_\mu)^i_j$ IS A MATRIX OF SU(N).
  IT CAN ONLY BE DONE IN D=0,1 BUT NOT IN HIGHER DIMENSIONS.
- IN A VECTOR MODEL (ϕ^i) THE LARGE N EXPANSION CAN BE EXPLICITLY PERFORMED.
- CAN WE FIND A VECTOR MODEL AS CLOSE AS POSSIBLE TO QCD? AND THEN STUDY THE PROPERTIES OF THIS MODEL TO GET A HINT ON WHAT HAPPENS IN QCD?

TWO-DIMENSIONAL σ-MODELS IN TWO SPACE-TIME DIMENSIONS HAVE MANY PROPERTIES IN COMMON WITH QCD + SOLVABLE ~~[illegible]~~ FOR LARGE N.

- IN GENERAL A NON-LINEAR σ-MODEL HAS AN ACTION OF THE TYPE

$$S = \frac{1}{2f} \int d^2x \; g_{ij}(\phi)\, \partial_\mu \phi^i \partial^\mu \phi^j$$

$g_{ij}(\phi)$ IS THE METRIC OF A MANIFOLD DESCRIBED BY THE COORDINATES ϕ^i.

- IN THE CASE OF THE $O(N)$ NON-LINEAR σ MODEL THE MANIFOLD IS THE SPHERE S^{N-1} DESCRIBED BY $(N-1)$ REAL COORDINATES GENERALIZATION OF THE SPHERE S^2 THAT LIVES IN OUR THREE-DIMENSIONAL SPACE.

- THE ~~CORRESPONDING~~ O(N) NON-LINEAR σ-MODEL CAN BE MORE CONVENIENTLY FORMULATED BY INTRODUCING ONE EXTRA COORDINATE

$$\phi^1 \ldots \phi^{N-1} \; \phi^N$$

- WITH THESE COORDINATES ACTION IS

$$S = \frac{1}{2f} \int d^2x \; \partial_\mu \phi^i \partial^\mu \phi^j \delta_{ij} \qquad \left(\begin{array}{c}\text{FLAT}\\ \text{METRIC}\end{array}\right)$$

BUT THERE IS A CONSTRAINT

$$\sum_{i=1}^{N} \phi^i \phi^i = 1$$

TO DESCRIBE THE SPHERE S^{N-1}

- THE MANIFOLD CP^{N-1} IS A COMPLEX PROJECTIVE SPACE IN N COMPLEX DIMENSIONS

$$z^i \sim \alpha z^i$$

α IS A COMPLEX QUANTITY.

- THE MANIFOLD CP^{N-1} IS DESCRIBED BY (N-1) COMPLEX VARIABLES.

- BUT IT IS MORE CONVENIENT TO START FROM A FORMULATION WITH N COMPLEX COORDINATES + ADDING A CONSTRAINT:

$$\sum_{i=1}^{N} |z^i|^2 = 1$$

+ GAUGE INVARIANCE.

- <u>$CP^1 \sim O(3)$ σ-MODEL</u>

$$\phi^i = \bar{z}\,\sigma^i z \quad ; \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\phi^1 = \bar{z}_1 z_2 + \bar{z}_2 z_1 \ ; \ \phi^2 = -i\,\bar{z}_1 z_2 + i\,\bar{z}_2 z_1 \ ; \ \phi^3 = \bar{z}_1 z_1 - \bar{z}_2 z_2$$

$$(\phi^1)^2 + (\phi^2)^2 + (\phi^3)^2 = 1$$

IF $|z_1|^2 + |z_2|^2 = 1$.

CP^{N-1} MODEL IN $D=2$

- ACTION

$$S = \int d^2x \left\{ \overline{D_\mu z}\, D_\mu z + \overline{\psi}\,(\not{D} - M_B)\,\psi - \frac{g}{2N_F}\left[(\overline{\psi}\,\lambda^i\psi)^2 + (\overline{\psi}\,\lambda^i\gamma_5\,\psi)^2 \right] \right\}$$

WITH

$$|z|^2 = \frac{N}{2f} \qquad D_\mu = \partial_\mu + \frac{ie}{\sqrt{N}} A_\mu$$

$e = 1$ FOR z.

$z^\alpha(x) \qquad \alpha = 1 \ldots N$

$\psi^{A;i} \qquad A = 1 \ldots N_f$ (FLAVOUR)

$\qquad\quad i = 1 \ldots N_F$ (COLOUR)

- DIRAC MATRICES

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad \gamma_5 = \gamma^0\gamma^1 = -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\gamma_\pm = \frac{1}{2}(1 \pm i\gamma_5) ; \quad \gamma_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} ; \quad \gamma_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

- CONFORMAL INVARIANCE AT CLASSICAL LEVEL. f, g ARE DIMENSIONLESS.

- EXISTENCE OF A TOPOLOGICAL CHARGE

$$Q = \frac{1}{2\pi\sqrt{N}} \int d^2x \, \varepsilon_{\mu\nu} \partial_\mu A_\nu$$

THAT IS AN INTEGER.

- EXISTENCE OF INSTANTON CLASSICAL SOLUTIONS:

$$D_\mu z = \pm i \, \varepsilon_{\mu\nu} D_\nu z$$

$$\varepsilon_{\mu\nu} = - \varepsilon_{\nu\mu} \quad ; \quad \varepsilon_{01} = 1$$

- <u>CHIRAL INVARIANCE</u> : $U(N_f) \times U(N_f)$

WE CAN TRANSFORM INDEPENDENTLY ψ_L AND ψ_R: $\psi_R \to A \psi_R$; $\psi_L \to B \psi_L$

THE ACTION IS INVARIANT.

- <u>U(1) AXIAL ANOMALY</u> : $A = B^+ = e^{i\theta}$

$$\partial_\mu \left[\bar{\psi} \gamma_5 \gamma_\mu \psi \right] \sim 2 N_f \, q$$

$$q(x) = \frac{1}{2\pi\sqrt{N}} \varepsilon_{\mu\nu} \partial_\mu A_\nu$$

GENERATING FUNCTIONAL

$$Z(J, \bar{J}; \eta, \bar{\eta}) = \int \mathcal{D}z \, \mathcal{D}\bar{z} \, \mathcal{D}\psi \, \mathcal{D}\bar{\psi} \, \mathcal{D}A_\mu$$

$$\cdot \, \delta\left(|z|^2 - \frac{N}{2f}\right) e^{-S + \int d^2x \{ \bar{J} \cdot z + \bar{z} \cdot J + \bar{\eta} \cdot \psi + \bar{\psi} \eta \}}$$

BY USING

$$\delta\left(|z|^2 - \frac{N}{2f}\right) = \int \mathcal{D}\alpha \, e^{\frac{i\alpha}{\sqrt{N}} \left(|z|^2 - \frac{N}{2f}\right)}$$

$$e^{\int d^2x \, \frac{g}{2NF} (\bar{\psi} \lambda^i \psi)^2} = \int \mathcal{D}\phi^i \, e^{\int d^2x \{ \bar{\psi} \lambda^i \psi \, \phi^i \frac{1}{\sqrt{NF}} - \frac{1}{2g} (\phi^i)^2 \}}$$

WE CAN REWRITE THE ACTION AS:

$$S = \overline{D_\mu z} \, D_\mu z + \bar{\psi} \left(\not{D} - M_B - \frac{\lambda^i}{\sqrt{NF}} (\phi^i + \gamma_5 \phi^i_5) \right) \psi$$

$$+ \frac{1}{2g} \left(\phi_i^2 + \phi_{i5}^2 \right) + \frac{i\alpha\sqrt{N}}{2f} - \frac{i\alpha}{\sqrt{N}} |z|^2$$

ACTION IS NOW QUADRATIC IN THE FUNDAMENTAL FIELDS z AND ψ.

PERFORMING THE FUNCTIONAL INTEGRAL OVER z AND ψ

$$Z(J,\bar{J};\eta,\bar{\eta}) = \int D\alpha\, D\phi^i\, D\phi_5^i\, DA_\mu$$

$$e^{-S_{eff} + \int d^2x \int d^2y \left[\bar{J}(x)\,\Delta_B^{-1}(x;y)\,J(y) + \bar{\eta}(x)\,\Delta_F^{-1}(x;y)\,\eta(y) \right]}$$

WHERE

$$\Delta_B = -D_\mu D_\mu + m^2 - \frac{i}{\sqrt{N}}\,\alpha$$

$$\Delta_F = \not{D} - M_B - \frac{\lambda^i}{\sqrt{N_F}}\left[\phi^i + \gamma_5 \phi_5^i\right]$$

$$S_{eff} = N\,\mathrm{Tr}\log \Delta_B - N_F\,\mathrm{Tr}\log \Delta_F$$

$$+ \int d^2x \left[i\,\frac{\sqrt{N}}{2f}\,\alpha + \frac{1}{2g}\left(\phi^i\phi^i + \phi_5^i\phi_5^i\right)\right]$$

• LARGE N AND N_F EXPANSION

① BOSONIC $O(\sqrt{N})$

$$N \, \mathrm{Tr} \log\left(-\partial^2 + m^2 - \frac{2i}{\sqrt{N}} A_\mu \partial_\mu + \frac{1}{N} A^2 - \frac{i\alpha}{\sqrt{N}}\right)$$

$$+ \int d^2x \, \frac{i\sqrt{N}}{2f} \alpha = N \, \mathrm{Tr} \log(-\partial^2 + m^2) +$$

$$+ N \, \mathrm{Tr}\left(1 - (-\partial^2 + m^2)^{-1} \frac{i\alpha}{\sqrt{N}} + \dots\right) + \int d^2x \, \frac{i\sqrt{N}}{2f} \alpha$$

$$= i\sqrt{N} \int d^2x \, \alpha(x) \left\{ \frac{1}{2f} - \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2 + m^2} \right\} +$$

$$+ N \, \mathrm{Tr} \log(-\partial^2 + m^2) + O(1)$$

② FERMIONIC $O(\sqrt{N_F})$: $\langle \phi^0 \rangle = M_s \sqrt{N_F N_f}$

$$-N_F \, \mathrm{Tr}(\not{\partial} - M) - N_F \, \mathrm{Tr}\left(1 - \frac{(\not{\partial} - M)^{-1}}{\sqrt{N_F}} \frac{\phi_0}{\sqrt{N_f}} + \dots\right)$$

$$\boxed{M = M_s + M_B}$$

$$+ \frac{M_s}{g} \sqrt{N_F N_f} \int d^2x \, \phi_0 + \dots =$$

$$= -N_F \, \mathrm{Tr}(\not{\partial} - M) + 2\sqrt{N_F N_f} \int d^2x \, \phi_0 \left\{ \frac{M_s}{2g} + \right.$$

$$\left. + \frac{1}{2} (\not{\partial} - M)^{-1}(x, x) \right\} + \dots = -N_F \, \mathrm{Tr}(\not{\partial} - M) +$$

$$+\ 2\sqrt{N_F N_f}\int d^2x\ \phi_0(x)\left\{\frac{M_s}{2g}-M\int\frac{d^2p}{(2\pi)^2}\frac{1}{p^2+M^2}\right\}+\cdots$$

IN CONCLUSION

$$S^{(0)} = N\,\mathrm{Tr}(-\partial^2+m^2) - N_F\,\mathrm{Tr}(\not\partial - M)$$

$$S^{(1)} = i\sqrt{N}\int d^2x\ \alpha(x)\left[\frac{1}{2f}-\int\frac{d^2p}{(2\pi)^2}\frac{1}{p^2+m^2}\right]$$

$$+\ 2\sqrt{N_F N_f}\int d^2x\ \phi_0(x)\left[\frac{M_s}{2g}-M\int\frac{d^2p}{(2\pi)^2}\frac{1}{p^2+M^2}\right]$$

PAULI-VILLARS REGULARIZATION

$$\int\frac{d^2p}{(2\pi)^2}\frac{1}{p^2+m^2}\longrightarrow\int\frac{d^2p}{(2\pi)^2}\left(\frac{1}{p^2+m^2}-\frac{1}{p^2+\Lambda^2}\right)$$

$$=\frac{1}{4\pi}\log\frac{\Lambda^2}{m^2}$$

SADDLE POINT EQS.

$$\frac{1}{2f(\Lambda)}-\frac{1}{4\pi}\log\frac{\Lambda^2}{m^2}=\frac{M_s}{2g(\Lambda)}-\frac{M}{4\pi}\log\frac{\Lambda^2}{M^2}=0$$

FERMION EQ. IS SATISFIED IF

$$M_B(\Lambda) = \varepsilon \frac{M_S \, 2\pi}{\log \Lambda^2/M^2} \quad ; \; M = M_B + M_S$$

IN THIS CASE 2ND EQ. BECOMES:

$$0 = \frac{2\pi}{g(\Lambda)} - \frac{M}{M_S} \log \frac{\Lambda^2}{M^2} = \frac{2\pi}{g(\Lambda)} - \log \frac{\Lambda^2}{M^2}$$

$$- \frac{M_B}{M_S} \log \frac{\Lambda^2}{M^2} = \boxed{\frac{2\pi}{g(\Lambda)} - \log \frac{\Lambda^2}{M^2} - 2\pi\varepsilon = 0}$$

ASYMPTOTIC FREEDOM FOR BOTH $f(\Lambda)$ AND $g(\Lambda)$!!

DIMENSIONAL TRANSMUTATION

$S^{(2)}$ IS $O(1)$

$$S^{(2)} = \frac{1}{2}\int d^2x \int d^2y \Big\{ \alpha(x)\, \Gamma^{\alpha}(x-y)\, \alpha(y) + $$

$$+ A_\mu(x)\, \Gamma^{A}_{\mu\nu}(x-y)\, A_\nu(y) + \phi^i(x)\, \Gamma^{\phi}_{ij}(x-y)\, \phi^j(y)$$

$$+ \phi_5^i(x)\, \Gamma^{\phi_5}_{ij}(x-y)\, \phi_5^j(y) + 2 A_\mu(x)\, \Gamma^{A\phi}_\mu\, \phi_5^0 \Big\}$$

WHERE

- $$\tilde{\Gamma}^{\alpha}(p) = A(p;m^2) = \frac{1}{2\pi\sqrt{p^2(p^2+4m^2)}} \log\frac{\sqrt{p^2}+\sqrt{p^2+4m^2}}{\sqrt{p^2+4m^2}-\sqrt{p^2}}$$

$$\xrightarrow[p\to 0]{} \frac{1}{4\pi m^2}\left[1 - \frac{2}{3}\frac{p^2}{4m^2} + \cdots\right]$$

- $$\tilde{\Gamma}^{A}_{\mu\nu} = \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\Big\{(p^2+4m^2)A(p;m^2) - \frac{1}{\pi} - \frac{N_f N_F}{N} e^2 \left[4M^2 A(p;M^2) - \frac{1}{\pi}\right]\Big\}$$

- $$\tilde{\Gamma}^{\phi}_{ij} = \delta_{ij}\left[\varepsilon + (p^2+4M^2)A(p;M^2)\right]$$

- $$\tilde{\Gamma}^{\phi_5}_{ij} = \delta_{ij}\left[\varepsilon + p^2 A(p;M^2)\right]; \quad \tilde{\Gamma}^{A\phi}_\mu(p) = -\varepsilon_{\mu\nu}\, p_\nu\, 2e\sqrt{N_f}\cdot M\sqrt{\frac{N_F}{N}}\, A(p;M^2)$$

LOW-ENERGY EFFECTIVE ACTION

$$\mathcal{L}_{eff} = \frac{1}{2}\left[(\partial_\mu \pi^i)^2 + m_\pi^2 (\pi^i)^2\right] +$$

$$+ \frac{1}{2}\left[(\partial_\mu \sigma^i)^2 + (m_\pi^2 + 4M^2)(\sigma^i)^2\right] + \frac{\alpha^2}{8\pi m^2} +$$

$$+ \frac{F^2}{24\pi m^2} + i\sqrt{\frac{2N_f}{N}}\, F_\pi \; F \; S$$

WHERE

$$F = \varepsilon_{\mu\nu}\partial_\mu A_\nu \; ; \quad \pi^i = \frac{\phi_5^i}{2\sqrt{\pi}M} \; ;$$

$$\sigma^i = \frac{\phi^i}{2\sqrt{\pi}M} \; ; \quad \pi^0 = S \; ; \quad F_\pi = \frac{1}{\sqrt{2\pi}}$$

$$m_\pi^2 = 4\pi \varepsilon M^2$$

CONFINEMENT

- AT THE CLASSICAL LEVEL THERE IS NO KINETIC TERM FOR THE GAUGE FIELD $A_\mu \Longrightarrow$ GENERATED IN THE QUANTUM THEORY $\Longrightarrow$ CONFINEMENT

- ONE GETS

$$\langle A_\mu(p)\, A_\nu(-p)\rangle = \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{12\pi m^2}{p^2}$$

- THIS MEANS THAT THERE IS A LINEAR POTENTIAL GENERATED IN THE QUANTUM THEORY:

$$V(R) = \frac{12\pi m^2}{N} R \quad ; \quad \sigma = \frac{12\pi m^2}{N}$$

- REMEMBER THAT

$$\int_{-\infty}^{\infty} \frac{dk}{k^2} e^{iRk} \sim R$$

- BUT CONFINEMENT IN TWO DIMENSION DOES NOT TEACH MUCH ABOUT CONFINEMENT IN FOUR DIMENSIONS: NO USE FOR QCD.

U(1) ANOMALY AND U(1) PROBLEM

- THE U(1) ANOMALY IMPLIES A MASS (NON VANISHING IN THE CHIRAL LIMIT: $m_\pi \to 0$) FOR THE PSEUDOSCALAR SINGLET (THIS SOLVES THE U(1) PROBLEM).
- FOR THIS THE MECHANISM IS THE SAME IN CP^{N-1} MODEL AND IN QCD.
- REWRITE

$$\mathcal{L}_{eff} = \frac{1}{2} \sum_{i \neq 0} \left[(\partial_\mu \pi^i)^2 + m_\pi^2 (\pi^i)^2 \right] +$$

$$+ \frac{1}{2} \sum_i \left[(\partial_\mu \sigma^i)^2 + (m_\pi^2 + 4M^2)(\sigma^i)^2 \right] + \frac{\alpha^2}{8\pi m^2}$$

$$+ \frac{1}{2}\left[(\partial_\mu S)^2 + m_\pi^2 S^2 \right] + \frac{1}{2\chi} q^2 + i \frac{\sqrt{2N_f}}{F_\pi} q S$$

WHERE

$$\chi = \frac{3m^2}{\pi N} \quad ; \quad F_\pi = \frac{1}{\sqrt{2\pi}}$$

$$\hookrightarrow \frac{1}{2}\left[(\partial_\mu S)^2 + M_S^2 S^2 \right] + \frac{1}{2\chi}\left(q + \frac{i\chi\sqrt{2N_f}}{F_\pi} S \right)^2$$

$$M_S^2 = m_\pi^2 + \frac{2\chi N_f}{F_\pi^2}$$

- θ - VACUA (NO FERMIONS)

$$\mathcal{L}_{eff} = \frac{\pi N}{6 m^2} q^2 + i\theta q + q J$$

$$q(x) = \frac{F(x)}{2\pi\sqrt{N}} = \text{TOPOLOGICAL CHARGE DENSITY}$$

J IS AN EXTERNAL SOURCE.

EQ. OF MOTION FOR $q(x)$:

$$q = -\frac{3m^2}{\pi N}(J + i\theta)$$

INSERTING IT INTO THE LAGRANGIAN:

$$\mathcal{L}_{eff} = -\frac{3m^2}{2\pi N}(J+i\theta)^2$$

GENERATING FUNCTIONAL

$$Z(J,\theta) = e^{\frac{3m^2}{2\pi N}\int d^2x\,(J+i\theta)^2} = e^{-W(J,\theta)}$$

- VACUUM ENERGY

$$E(\theta) = + \frac{W(J=0;\theta)}{V_2} = \frac{3m^2}{2\pi N}\theta^2 =$$

$$= N\left(\frac{3\,m^2}{2\pi}\right)\left(\frac{\theta}{N}\right)^2 \quad ; \quad E(\theta) = N\,C\,F\!\left(\frac{\theta}{N}\right)$$

$$F(x) = x^2$$

- ONE-POINT FUNCTION

$$\langle q(x) \rangle_\theta = \left.\frac{\delta Z}{\delta J}\right|_{J=0} = i\,\frac{3m^2}{\pi N}\,\theta$$

- TWO-POINT FUNCTION

$$\langle q(x)\, q(y) \rangle = \frac{3m^2}{\pi N}\,\delta^{(2)}(x-y)$$

- TOPOLOGICAL SUSCEPTIBILITY

$$\langle q(x) \int d^2y\, q(x) \rangle = \frac{3m^2}{\pi N} \equiv \chi$$

WITTEN-VENEZIANO FORMULA

LIKE

$$\left.\frac{d^2E(\theta)}{d\theta^2}\right|_{\theta=0} = \frac{3m^2}{N\pi} = \chi = \frac{F_\pi^2}{2N_f} M_S^2$$

THIS IMPLIES

$$M_S^2 = \frac{2N_f}{F_\pi^2}\chi = \frac{2N_f}{F_\pi^2} \left.\frac{d^2E(\theta)}{d\theta^2}\right|_{\theta=0}.$$

NOTICE THAT IF WE HAD CONSIDERED THE THEORY WITH FERMIONS WE WOULD HAVE OBTAINED AT LOW ENERGY

$$\langle S \quad (q + i M_S \sqrt{\chi}\, S) \rangle = 0$$

$$\Downarrow$$

$$\langle S \quad q \rangle = -i \frac{\sqrt{\chi}}{M_S}$$

AND

$$\langle (q + i M_S \sqrt{\chi}\, S)\,(q + i M_S \sqrt{\chi}\, S) \rangle = \chi$$

$$= \langle q \; q \rangle + 2 i M_S \sqrt{\chi} \langle q \, S \rangle - M_S^2 \chi \langle S \, S \rangle =$$

$$= \langle q \; q \rangle + 2\chi - \chi = \chi$$

$$\Downarrow \quad \langle q \; q \rangle = 0$$ (AT LOW ENERGY)

WHY IS THE AXIAL U(1) ANOMALY IMPORTANT?

- THE U(1) AXIAL ANOMALY HAS BEEN DERIVED IN THE FUNDAMENTAL LAGRANGIAN CONTAINING THE FUNDAMENTAL FIELDS $(z, \psi) \Leftrightarrow$ (GLUON + QUARKS IN QCD)

- IN THE QUANTUM THEORY "COLOUR" IS CONFINED AND THOSE FUNDAMENTAL FIELDS DO NOT APPEAR IN THE SPECTRUM.
- THE SPECTRUM CONSISTS OF COMPOSITE COLOURLESS STATES.
- WE HAVE DERIVED AN EFFECTIVE LAGRANGIAN FOR THEM.
- THE EFFECTIVE LAGRANGIAN MUST HAVE THE SAME SYMMETRIES AND ANOMALIES OF THE BASIC FUNDAMENTAL LAGRANGIAN.
- HOW DO WE SEE THE ANOMALY IN THE EFFECTIVE LAGRANGIAN ?
- THE EFFECTIVE LAGRANGIAN MUST <u>NOT</u> BE INVARIANT UNDER THE ANOMALOUS TRANSFORMATIONS, BUT IT MUST TRANSFORM ACCORDING TO THE ANOMALY:

$$\delta S_{eff} = \int d^2x \, \delta \mathcal{L}_{eff} \sim 2N_f \int q(x) \, d^2x$$