Particules Élémentaires, Gravitation et Cosmologie
Année 2010-'11

Théorie des cordes: quelques applications

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Classical and semiclassical black holes

Black holes in General Relativity

The universal attractive character of gravity can lead, according to GR, to a phenomenon known as gravitational collapse.
An isolated system can become so compact to be completely surrounded by a surface, called the horizon, out of which nothing, even light, can escape.

While the collapse process and the solution inside the horizon are complicated problems, the "late-time" stationary solutions outside the horizon are simple and basically "unique": they depend on a small number of parameters each one with a precise physical meaning.
Hereafter we shall use units in which $G = c = 1$
($M \sim$ length, $J \sim$ length^2).

The Schwarzschild solution

The simplest black-hole solution was given very soon after the final formulation of GR (Schwarzschild, 1916). It is given in terms of a single parameter, the BH mass M.

It can be written in different coordinate systems. In the original Schwarzschild coordinates it has the explicitly static and spherically symmetric form (G = c = 1):

$$ds^2_{Schw.} = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{(1 - \frac{2M}{r})} dr^2 + r^2 d\Omega^2$$

$$d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$$

$$ds^2_{Schw.} = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{(1 - \frac{2M}{r})} dr^2 + r^2 d\Omega^2$$

As in the gravitational field of the earth clocks tick slower and slower as one approaches the source of gravity (see gravitational redshift, 2009 course):

$$\frac{dt(\infty)}{dt(r)} = \frac{\sqrt{-g_{00}}(r)}{\sqrt{-g_{00}}(\infty)} = \sqrt{1 - 2M/r}$$

This means that light coming from r = 2M (+ε) suffers an infinite (very large) redshift (for the earth r= 2M is deep inside the earth where solution is not valid).

Nevertheless r = 2M (the horizon H) is a regular submanifold of the Schw. spacetime. This can be easily seen by going to different coordinates.

Ingoing Eddington-Finkelstein coordinates

$$v = t + r_* \qquad r_* = \int^r \frac{dr' \ r'}{r' - 2M} = r + 2M \log\left(\frac{r}{2M} - 1\right)$$

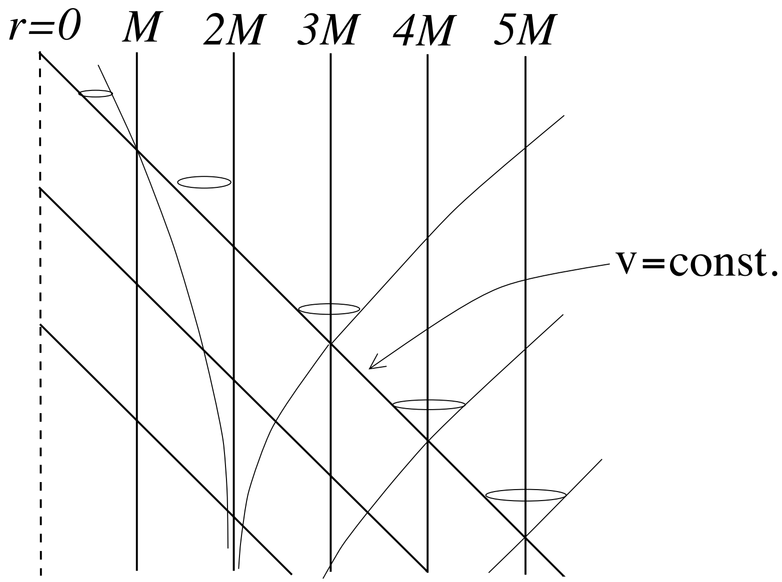
$$ds^2_{Schw.} = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2$$

This goes to a perfectly smooth metric at r = 2M.

The hypersurface (r - 2M) = 0 is lightlike (null) i.e.:

$$g^{\mu\nu}\partial_\mu(r - 2M)\ \partial_\nu(r - 2M) = g^{rr} = \left(1 - \frac{2M}{r}\right) \to 0$$

The light cone gets more and more tilted as one approaches H. On H the light cone is tangential to H so that light emitted near H can barely escape to infinity.

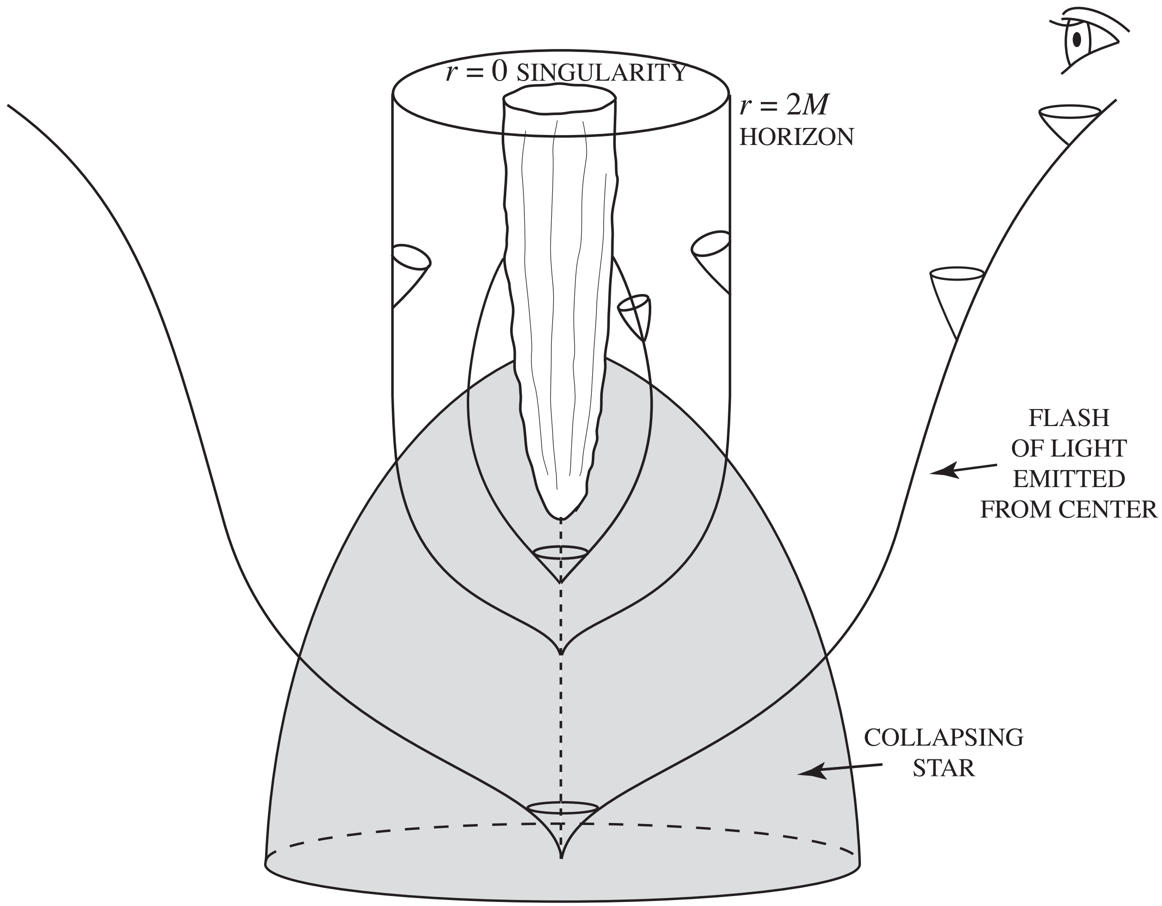


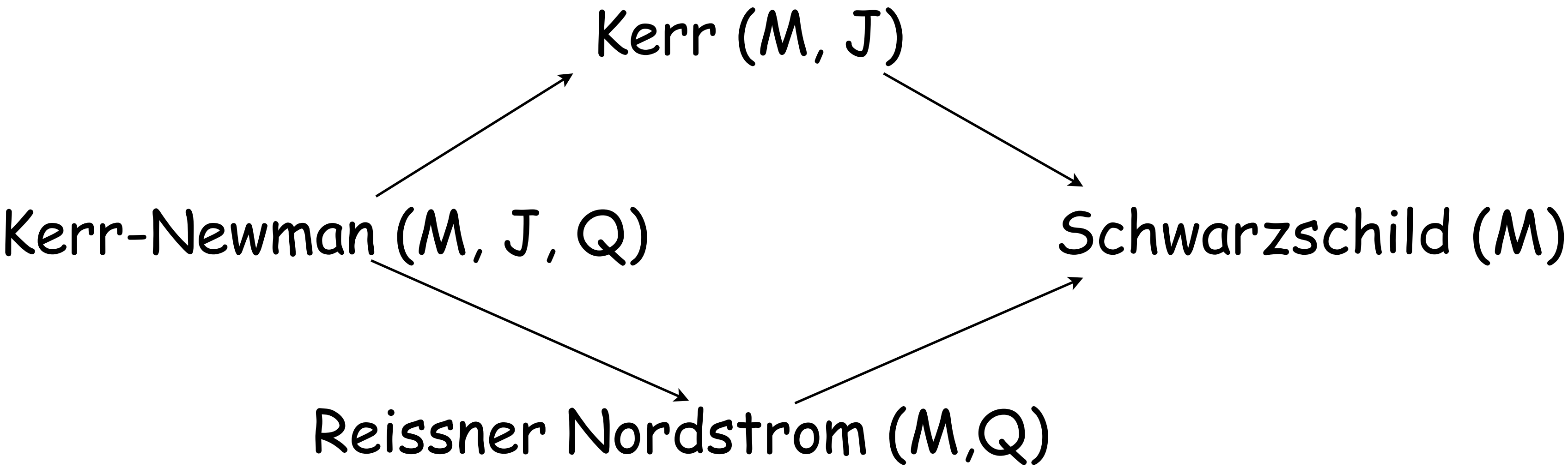
Instead, r=0 is a true (Riemann) curvature singularity (Ricci is zero except at r=0). The hypersurface r=0 is spacelike (meaning that its normal is timelike) so that r=0 is to be thought of as an instant in time (rather than as a point in space). The metric close to r=0 corresponds to an anisotropically collapsing Universe (a big crunch).

In reality we should solve the time-dependent EEs for a collapsing object made of realistic matter and the problem becomes much harder.

A lot of work strongly suggests that, under certain conditions, a horizon forms and the (outside vacuum) solution settles to a stationary (not necessarily static) one characterized by just a few conserved quantum numbers (associated with the large-distance fields of the theory): the mass M, the angular momentum J, and the electric charge Q.

This general solution is known as the Kerr-Newman metric the other black holes solutions being special cases of it.


r = 0 SINGULARITY
r = 2M
HORIZON
FLASH
OF LIGHT
EMITTED
FROM CENTER
COLLAPSING
STAR



Black hole solutions with angular momentum (K & KN) correspond to rather complicated metrics (see below).

Instead, the RN solution is a rather trivial generalization of Sch. (note that, in our units, Q is also a length):

$$ds^2_{RN} = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \frac{dr^2}{(1 - \frac{2M}{r} + \frac{Q^2}{r^2})} + r^2 d\Omega^2$$

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The RN metric possesses a horizon H if and only if the quadratic form $(1 - 2M/r + Q^2/r^2)$ can vanish at some real value of r. This happens if $Q < M$ (i.e. sufficiently small Coulomb repulsion) in which case the horizon is at:

$$r = r_+ = M + \sqrt{M^2 - Q^2}$$

The area of (a time section of) H is

$$\mathcal{A}_{RN} = 4\pi r_+^2 = 4\pi\left(2M^2 - Q^2 + 2M\sqrt{M^2 - Q^2}\right)$$

For Q = M the BH is said to be "extremal".

The Kerr Newman solution

The general KN solution is considerably more complicated:

$$ds^2_{KN} = -\frac{\Delta}{\Sigma}(dt - a\sin^2\theta d\phi)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma\, d\theta^2 + \frac{\sin^2\theta}{\Sigma}\left[(r^2+a^2)d\phi - adt\right]^2$$

$$a = \frac{J}{M} \; ; \; \Delta = r^2 - 2Mr + a^2 + Q^2 \; ; \; \Sigma = r^2 + a^2\cos^2\theta$$

There is also, of course, an electromagnetic field.

The KN metric has a horizon if Δ vanishes at some real value of r. It does if $M^2 > a^2 + Q^2$.

KN is extremal for $M^2 = a^2 + Q^2$ while Kerr is extremal for $M = a$ (i.e. for $J = GM^2$).

The area of the KN horizon is:

$$\mathcal{A}_{KN} = 4\pi(r_+^2 + a^2) = 4\pi\left(2M^2 - Q^2 + 2M\sqrt{M^2 - a^2 - Q^2}\right)$$

$$r_+ = M + \sqrt{M^2 - a^2 - Q^2}$$

Area theorem and entropy hints

In 1972, extending work by Christodoulou & Ruffini, Hawking proved that in any conceivable process the area A of a BH horizon cannot decrease. This started a feeling in the GR community that A had something to do with an entropy (Cf. 2nd law of thermodynamics):

$S = A/\lambda^2$ with λ a length (in units where $k_B = 1$).

One could also prove the analogue of the 1st law and extract a temperature: $T = \dfrac{\partial M}{\partial S}$

$$dM = \Omega\, dJ + \Phi\, dQ + T_{\mathrm{BH}}\, dS_{\mathrm{BH}}$$

$$\Omega = \frac{a}{(r_+^2 + a^2)} \quad ; \quad \Phi = \frac{Q r_+}{(r_+^2 + a^2)}$$

= angular velocity and electric potential

After a straightforward calculation we find ($G = 1$):

$$T_{BH} = \frac{\lambda^2}{8\pi} \frac{\sqrt{M^2 - a^2 - Q^2}}{r_+^2 + a^2}$$

Note that the temperature drops to zero in the extremal case while the area of the horizon does not:

$$\mathcal{A}_{KN} = 4\pi \left(2M^2 - Q^2 + 2M\sqrt{M^2 - a^2 - Q^2}\right) \to 4\pi(M^2 + a^2) > 0$$

Furthermore, one can prove a general relation between T_{BH} and the "surface gravity" κ of the BH:

$$T_{BH} = \frac{\lambda^2 \kappa}{8\pi} \quad ; \quad \kappa = \frac{\sqrt{M^2 - a^2 - Q^2}}{r_+^2 + a^2}$$

Puzzles

There were still some puzzles when associating an entropy and a temperature with a black hole:

1. The area of the BH did always increase even (or particularly) if the BH was not an isolated system. Instead, entropy can decrease for a subsystem.
2. Entropy is a pure number (k_B =1). An area is not.
3. Hot bodies radiate (Cf. black-body radiation) while the black hole did not.

Bekenstein's educated guess (1972)

Bekenstein was the first to suspect that the length λ relating entropy to area had to do with QM (GR has no fundamental length!). He guessed that λ had to be proportional to the only length that one can construct out of the fundamental constants of quantum gravity: c, h, and G. It had been introduced in physics by Max Planck at the beginning of the 20th century and carries his name:

$$l_p = \sqrt{\frac{G\hbar}{c^3}} \sim 1.616\ 10^{-33} cm$$

Bekenstein proposed to rewrite the previous formulae as

$$S_{BH} = \alpha\ \frac{\mathcal{A}}{l_P^2} \quad ; \quad T_{BH} = \frac{l_P^2\ \kappa}{8\pi G\alpha} = \frac{\hbar\kappa}{8\pi\alpha}$$

and even tried to estimate the dim.less constant α.

Hawking's 1974 breakthrough

In a ground-breaking paper Steven Hawking found that BH aren't black after all. They behave like black bodies whose temperature is given by Bekenstein's expression but now with a precise coefficient $\alpha = 1/4$. Thus:

$$S_{BH} = \frac{\mathcal{A}}{4l_P^2} \quad ; \quad T_{BH} = \frac{\hbar\kappa}{2\pi}$$

He also argued in favour of a generalized 2nd law (GSL) stating that the sum of the entropy of the BH and that of the surrounding matter/radiation can never decrease.

This appears to solve the previous puzzles but...

Poor man's derivation of T_{BH}

A rigorous derivation of Hawking's formula is quite involved and even subject to possible criticism (e.g. the "transplanckian problem"). A much simplified "proof" goes as follows:

Consider an observer staying at a fixed r near the horizon of a Schwarzschild BH and write the metric he/she sees in terms of new coordinates τ and u:

$$r = 2M + \frac{u^2}{8M} \quad , \quad \tau = t \quad \Rightarrow \quad ds^2 \sim -\frac{u^2}{16M^2} d\tau^2 + du^2$$

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This metric is known as Rindler's and (as a 2-D metric) is just flat spacetime as seen by an accelerated observer.
This can be seen by the change of coordinates:

$$t' = u \ \sinh\left(\frac{\tau}{4M}\right) \ ; \ x = u \ \cosh\left(\frac{\tau}{4M}\right) \Rightarrow ds^2 = -dt'^2 + dx^2$$

This map only covers a wedge in the (t',x) plane in which an observer sitting at fixed u (r) describes a hyperbola along which 1/u is the constant acceleration of the observer.

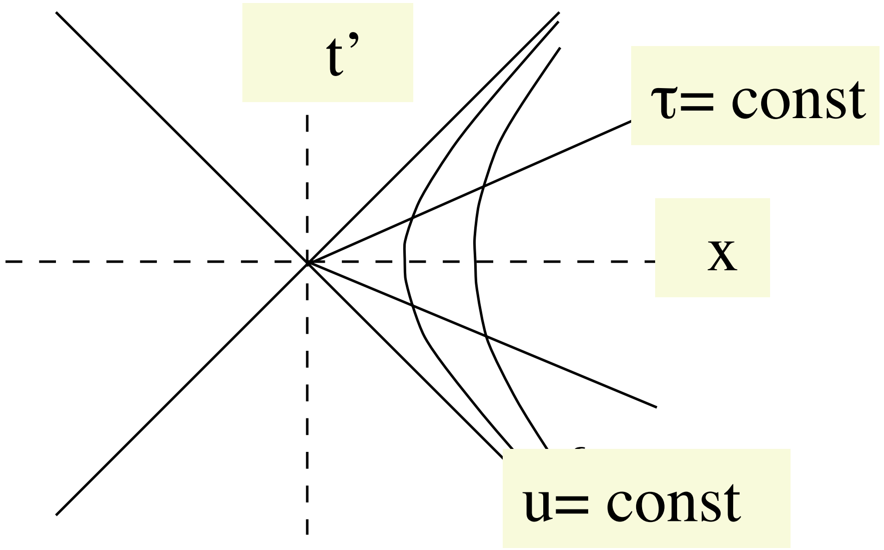
The above transformation is periodic with an imaginary period $\Delta_\tau = 8 \pi G M i$. Periodicity in imaginary time is naturally associated with finite temperature:

$$\beta_\tau = \frac{1}{T_\tau} = -i\frac{\Delta_\tau}{\hbar} = \frac{8\pi G M}{\hbar} = \beta_{BH}$$

Unruh has shown that

an accelerated observer measures a temperature given by

$$T_U = \frac{\hbar a}{2\pi}$$



Actually, the physical $\beta = T^{-1}$ at some fixed u (or r) is:

$$\beta(r) = \beta_\tau \frac{u}{4M} = \frac{2\pi u}{\hbar} = \frac{2\pi}{\hbar a} = \frac{2\pi}{\hbar}\sqrt{8M(r-2M)}$$

while the temperature at infinity is redshifted by a factor

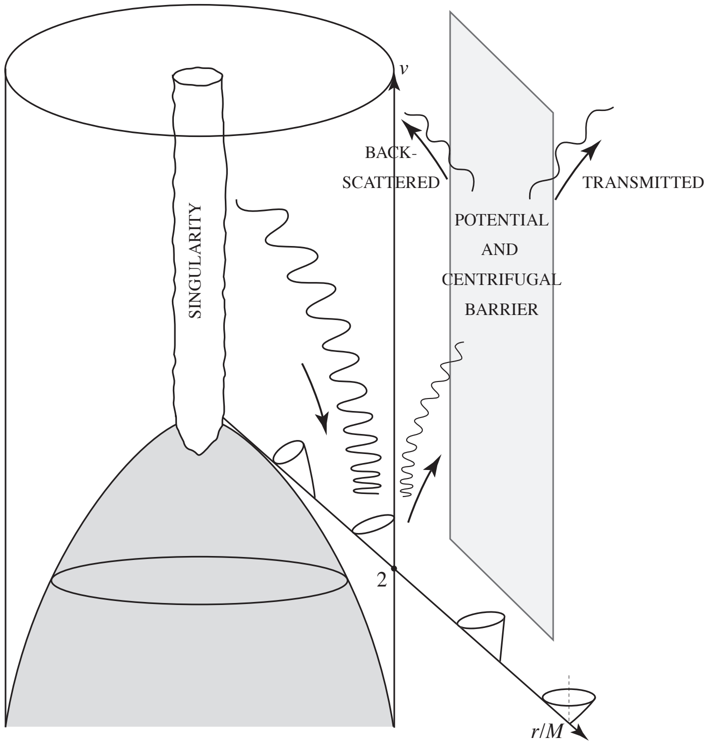
$$1+z = \sqrt{1-\frac{2M}{r}} \Rightarrow \beta(r=\infty) = \frac{2\pi}{\hbar}\frac{\sqrt{8M(r-2M)}}{\sqrt{1-\frac{2M}{r}}} = \frac{8\pi M}{\hbar}$$

Spectrum of Hawking's radiation

Hawking's calculation gives essentially a Planck spectrum for the radiation emitted by a black hole as if the black hole were a black body of temperature T_{BH} with a surrounding barrier giving a transmission (or grey-body) factor:

$$\frac{d\langle N\rangle}{dt\,d\omega} = \frac{1}{2\pi}\frac{\Gamma_l(\omega)}{e^{8\pi GM\omega} - 1}$$

$$8\pi GM\omega = \frac{\hbar\omega}{T_{BH}} \quad ; \quad T_{BH} = \frac{\hbar}{8\pi GM}$$



Numerically:

$$T_{BH} = 10^{-6} K \frac{M_{sun}}{M}$$

Generalizations

For more general black holes the result is similar the temperature being directly related to the gravitational acceleration at the horizon:

$$S_{BH} = \frac{\mathcal{A}}{4l_P^2} \quad ; \quad T_{BH} = \frac{\hbar\kappa}{2\pi}$$

Furthermore, the Boltzmann factor gets replaced by a more general one for the emission of a quantum of charge q and angular momentum j:

$$e^{\frac{\hbar\omega}{T_{BH}}} \to e^{\frac{(\hbar\omega - j\Omega - q\Phi)}{T_{BH}}}$$

where Ω is the angular velocity of the BH and Φ its electric potential (signs favour loss of Q, J).

Black hole lifetime

Let us make a (rough) estimate of the lifetime of a (Schwarzschild) BH of mass M if we simply let it evaporate.

(the BH emits O(1) Hawking quanta per R/c time)

$$\frac{d\langle N\rangle}{dt d\omega} = -\frac{1}{2\pi}\frac{\Gamma_l(\omega)}{e^{8\pi GM\omega}-1} \to \frac{dM}{dt} \sim -\int \frac{\hbar\omega d\omega}{e^{8\pi GM\omega}-1} \sim \frac{\hbar}{(GM)^2}$$

$$\frac{d(GM)}{dt} \sim -\frac{l_P^2}{(GM)^2} \Rightarrow (GM)_t^3 = (GM)_0^3 - l_P^2 t \Rightarrow \tau_{ev} \sim t_P \left(\frac{M}{M_P}\right)^3$$

In numbers:

$$t_P = \frac{l_P}{c} \sim 5.4\ 10^{-44} s\ ;\ M_P = \sqrt{\hbar c/G} \sim 2.2\ 10^{-5} gr \Rightarrow \tau_{ev} \sim 10^{10} yr \left(\frac{M}{10^{14} gr}\right)^3$$

An information puzzle?

The BH evaporation is slow, except towards its end, justifying a-posteriori the static approximation used in the calculation.

However, the finite evaporation time raises some interesting conceptual issues that go under the name of information puzzle (or paradox).

They have to do with the fact that the Hawking radiation appears to depend only from the quantum numbers that characterize the black hole (M, J, Q in our case) and NOT from what made the BH in the first place.

Information about the initial state appears to be lost.

At the quantum level we can imagine to start from a pure state (of zero entropy) prepared in such a way that it should produce a BH. As long as some information is hidden forever beyond the horizon there is really no contradiction.

However, if the BH evaporates and eventually disappears, information about the initial state should be recovered. This is not possible if the final state is a (superposition of) thermal state(s) that does not care about what went into the BH. Even if the semiclassical treatment will break down in the latest stages of evaporation, possible "remnants" will not be able to carry all the missing information.

In 1976 Hawking went as far as saying that, in the presence of BHs, the laws of QM have to be modified and there is no longer a unitary description of the quantum system's evolution.

Hawking's claim raised a big controversy. Abandoning one of the very basic principles of QM, the unitary evolution of a system which is at the basis of quantum coherence and all that, was hard to swallow particularly for the particle physics community.

On the other hand, in the absence of a really consistent theory of gravity, the problem itself was not completely well-posed: the BH metric was taken to be classical while the matter fields to be radiated away were quantized. The back reaction of the emission on the geometry was also treated in an approximate way.

With the advent of QST as a candidate theory of quantum gravity the issue became of central interest.The string theory community became increasingly convinced that there had to be a way to save QM from the threat of BHs.

Can string theory solve the puzzle?

There are good reasons to believe that QST offers a solution to the puzzle:

1. At least in some favourable cases QST can give a microscopic (stat. mech.) interpretation of BH entropy (see today's seminar and next week's lecture);

2. In QST we can hope to study the quantum process of BH formation and decay from a simple pure initial state and to construct an explicitly unitary S-matrix (see next few lectures);

3. In some cases QST can map the collapse problem into an explicitly unitary QFT problem via holography (AdS/CFT correspondence).