Covariant quantization of a relativistic string

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Plan of the talk

From Goto-Nambu to Polyakov

- We have seen that a free string is described by the Nambu-Goto action:

$$S_{NG}(x^\mu) = -T \int d\tau \int d\sigma \sqrt{-\det\left(\partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu}\right)}$$

where $\xi^\alpha \equiv (\tau, \sigma)$ and $\partial_\alpha \equiv \frac{\partial}{\partial \xi^\alpha}$.

- This Lagrangian is very non-linear and not easy to treat if we want to quantize the string using the path integral formalism.
- On the other hand, there exists an alternative to the Nambu-Goto action that was constructed by [Brink, Deser, DV, Howe and Zumino] in 1976.
- It was then used by Polyakov in 1982 for quantizing the string with the path integral formalism.
- For this reason it is called Polyakov action.

- It is given by:

$$S(x^\mu, g_{\alpha\beta}) = -\frac{T}{2}\int d\tau \int_0^\pi d\sigma \sqrt{-g} g^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu}$$

- $x^\mu(\sigma, \tau)$ is the coordinate of the string ($[\mu, \nu = 0, 2, ...d - 1]$).
- $T = \frac{1}{2\pi\alpha'}$ is a dimensional (Energy per unit length) parameter called the string tension.
- $g^{\alpha\beta}(\sigma, \tau)$ is the two-dimensional world-sheet metric tensor with $g = \det(g_{\alpha\beta})$.
- $\eta^{\mu\nu} = (-1, 1...1, 1)$ is the d-dimensional target space metric.
- Viewed as a two dimensional field theory, it describes the interaction of a set of d massless fields with an external gravitational field $g_{\alpha\beta}$.
- From this point of view the d-dimensional Lorentz index plays the role of a flavour index.

- It can be easily shown that the two actions are equivalent.
- We can immediately write the algebraic equation of motion for the world-sheet metric:

$$\theta_{\alpha\beta} \equiv \partial_\alpha x \cdot \partial_\beta x - \frac{1}{2} g_{\alpha\beta} g^{\gamma\delta} \partial_\gamma x \cdot \partial_\delta x = 0$$

where we have used

$$\frac{\delta\sqrt{-g}}{\delta g^{\alpha\beta}} = -\frac{1}{2} g_{\alpha\beta} \sqrt{-g}$$

- From it we get:

$$\det\left(\partial_\alpha x \cdot \partial_\beta x\right) = \frac{g}{4} \left(g^{\alpha\beta} \partial_\alpha x \cdot \partial_\beta x\right)^2$$

- Inserting it in the Polyakov action one gets the Nambu-Goto action $\Longrightarrow$ the two actions are equivalent !!

The bosonic string in the conformal gauge

- Let us start from the Polyakov action:

$$S(x^\mu, g_{\alpha\beta}) = -\frac{T}{2}\int d\tau \int_0^\pi d\sigma \sqrt{-g}\, g^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu}$$

- It is invariant under an arbitrary reparametrization ($\xi \to \xi'(\xi)$) of the world-sheet coordinates $\xi^\alpha \equiv (\tau, \sigma)$:

$$x^\mu(\xi) = x'^\mu(\xi') \;\;,\;\; g_{\alpha\beta}(\xi) = \frac{\partial \xi'^\gamma}{\partial \xi^\alpha}\frac{\partial \xi'^\delta}{\partial \xi^\beta} g'_{\gamma\delta}(\xi')$$

- The second equation implies:

$$d^2\xi \sqrt{-g} = d^2\xi' \sqrt{-g'}$$

- For infinitesimal transformations $\xi' = \xi - \epsilon$ we get:

$$\delta x^\mu = \epsilon^\alpha \partial_\alpha x^\mu \;\;;\;\; \delta g_{\alpha\beta} = \epsilon^\gamma \partial_\gamma g_{\alpha\beta} + \partial_\alpha \epsilon^\gamma g_{\gamma\beta} + \partial_\beta \epsilon^\gamma g_{\alpha\gamma}$$

- It is also invariant under Weyl rescaling of the metric:

$$g_{\alpha\beta}(\xi) \rightarrow \Lambda^2(\xi) g_{\alpha\beta}(\xi) \;\; ; \;\; x^\mu(\xi) \rightarrow x^\mu(\xi)$$

- From the string action we can derive the Euler-Lagrange equations of motion:

$$-\frac{2}{T\sqrt{-g}} \frac{\delta S}{\delta g^{\alpha\beta}} \equiv \theta_{\alpha\beta} = \partial_\alpha x \cdot \partial_\beta x - \frac{1}{2} g_{\alpha\beta} g^{\gamma\delta} \partial_\gamma x \cdot \partial_\delta x = 0$$

  for the two-dimensional world-sheet metric.
- This equation implies that the two-dimensional world-sheet energy-momentum tensor is identically vanishing.
- The eq. of motion for the string coordinate is instead

$$\partial_\alpha \left(\sqrt{-g} g^{\alpha\beta} \partial_\beta x^\mu \right) = 0$$

- It is still non-linear.

- In order to solve the previous equations and find the most general classical motion of a string it is convenient to choose a gauge where the previous equation of motion linearizes.
- A convenient Lorentz covariant gauge is the conformal gauge where the world-sheet metric tensor is taken to be of the form:

$$g_{\alpha\beta} = \rho(\xi)\eta_{\alpha\beta} \qquad \eta_{11} = -\eta_{00} = 1$$

- This gauge choice does not fix completely the gauge.
- We can still perform conformal transformations that leave the metric in the same form, but with a rotated ρ.
- They are characterized by the following equation:

$$\partial^\alpha \epsilon^\beta + \partial^\beta \epsilon^\alpha - \eta^{\alpha\beta}\partial^\gamma \epsilon_\gamma = 0$$

- Under the previous infinitesimal transformation we get:

$$g_{\alpha\beta} + \delta g_{\alpha\beta} = \left(\rho + \partial_\gamma \left(\epsilon^\gamma \rho\right)\right) \eta_{\alpha\beta}$$

- The conditions of the conformal gauge are more transparent if we introduce light-cone coordinates:

$$\xi^{\pm} = \xi^0 \pm \xi^1 \quad , \quad \epsilon^{\pm} = \epsilon^0 \pm \epsilon^1 \quad , \quad \frac{\partial}{\partial \xi^{\pm}} = \frac{1}{2}\left(\frac{\partial}{\partial \xi^0} \pm \frac{\partial}{\partial \xi^1}\right)$$

- In terms of those variables, they reduce to

$$\frac{\partial}{\partial \xi^-}\epsilon^+ = \frac{\partial}{\partial \xi^+}\epsilon^- = 0 \Longrightarrow \epsilon^+(\xi^+) \quad ; \quad \epsilon^-(\xi^-)$$

- In the conformal gauge the equation of motion becomes:

$$\left(\frac{\partial^2}{\partial \sigma^2} - \frac{\partial^2}{\partial \tau^2}\right) x^{\mu}(\sigma, \tau) = 0$$

- Boundary conditions for open string

$$\frac{\partial}{\partial \sigma} x^{\mu}(\tau, \sigma)|_{\sigma=0,\pi} = 0$$

- Boundary condition for closed string:

$$x^{\mu}(\tau, \sigma) = x^{\mu}(\tau, \sigma + \pi)$$

- The most general solution for open string:

$$x^\mu(\tau,\sigma) = q^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-in\tau} \cos n\sigma$$

- and for closed string

$$x^\mu(\tau,\sigma) = q^\mu + 2\alpha' p^\mu \tau + \frac{i}{2}\sqrt{2\alpha'} \sum_{n\neq 0} \left[\frac{\alpha_n^\mu}{n} e^{-2in(\tau+\sigma)} + \frac{\tilde{\alpha}_n^\mu}{n} e^{-2in(\tau-\sigma)} \right]$$

- Here α_n and $\tilde{\alpha}_n$ are just constant parameters.
- We must impose the vanishing of the two independent components of the world-sheet energy-momentum tensor:

$$\theta^{00} \pm \theta^{01} \sim \frac{1}{2} \left(\dot{x} \pm x' \right)^2 = 0$$

$$\dot{x} \equiv \partial_\tau x \ \ ; \ \ x' \equiv \partial_\sigma x$$

Notations

- The operators α_n and $\tilde{\alpha}_n$ are related to the harmonic oscillators and the center of mass variables by:

$$\alpha_n^\mu = \begin{cases} \sqrt{n} a_n^\mu & \text{if } n > 0 \\ \sqrt{2\alpha'} \hat{p}^\mu & \text{if } n = 0 \\ \sqrt{|n|} a_{|n|}^{+\mu} & \text{if } n < 0 \end{cases}$$

  for the open string,

- and by

$$\alpha_n^\mu = \begin{cases} \sqrt{n} a_n^\mu & \text{if } n > 0 \\ \sqrt{2\alpha'} \frac{\hat{p}^\mu}{2} & \text{if } n = 0 \\ \sqrt{|n|} a_{|n|}^{+\mu} & \text{if } n < 0 \end{cases} \quad ; \quad \tilde{\alpha}_n^\mu = \begin{cases} \sqrt{n} \tilde{a}_n^\mu & \text{if } n > 0 \\ \sqrt{2\alpha'} \frac{\hat{p}^\mu}{2} & \text{if } n = 0 \\ \sqrt{|n|} \tilde{a}_{|n|}^{+\mu} & \text{if } n < 0 \end{cases}$$

  for the closed string.

- In the case of the open string they give the same condition, namely:

$$L_n = \frac{1}{4\pi\alpha'} \int_0^\pi d\sigma \, \mathrm{e}^{in(\tau\pm\sigma)} (\dot{x} \pm x')^2 = \frac{1}{2} \sum_{m=-\infty}^{\infty} \alpha_{n-m} \cdot \alpha_m = 0$$

- where $\alpha_0 \equiv \sqrt{2\alpha'} p$
- In the case of a closed string we get instead:

$$L_n = \frac{1}{4\pi\alpha'} \int_0^\pi d\sigma \, \mathrm{e}^{in(\tau+\sigma)} (\frac{\dot{x} + x'}{2})^2 = \frac{1}{2} \sum_{m=-\infty}^{\infty} \alpha_m \cdot \alpha_{n-m} = 0$$

$$\tilde{L}_n = \frac{1}{4\pi\alpha'} \int_0^\pi d\sigma \, \mathrm{e}^{in(\tau-\sigma)} (\frac{\dot{x} - x'}{2})^2 = \frac{1}{2} \sum_{m=-\infty}^{\infty} \tilde{\alpha}_m \cdot \tilde{\alpha}_{n-m} = 0$$

$$\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{2\alpha'} \, \frac{p^\mu}{2}$$

Old covariant quantization

- The theory is quantized by imposing the following commutation relations:

$$[\alpha_n^\mu, \alpha_m^\nu] = n\,\eta^{\mu\nu}\delta_{n+m;0} \quad ; \quad [\hat{q}^\mu, \hat{p}^\nu] = i\eta^{\mu\nu}$$

  for an open string.
- In the case of a closed string, one must also imposes the commutation relations for the other infinite set of oscillators:

$$[\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu] = n\,\eta^{\mu\nu}\delta_{n+m;0}$$

  that commute with the oscillators of the previous set.
- In the quantum theory, the operators L_n are defined with the normal ordering:

$$L_n = \frac{1}{2}\sum_{m=-\infty}^{\infty} :\alpha_{n-m}\cdot\alpha_m:$$

  that, however, regards only $L_0 = \alpha' \hat{p}^2 + \sum_{n=1}^{\infty} n a_n^\dagger \cdot a_n$.

- In the quantum theory, the vanishing of L_n for all n is too restrictive.
- One can only impose their vanishing between physical states.
- In other words one can define a physical subspace where:

$$\langle Phys., P|(L_n - \alpha_0 \delta_{n0})|Phys.', P\rangle = 0 \quad ; \quad -\infty < n < +\infty$$

  α_0 is a constant to be determined.
- They are satisfied if

$$(L_0 - \alpha_0)|Phys., P\rangle = L_n|Phys., P\rangle = 0 \quad ; \quad n = 1, 2 \dots$$

- Those conditions are exactly those obtained from the analysis of the residues of the poles in the N-point dual amplitude.
- except that there and in the light-cone gauge $\alpha_0 = 1$, while here there is no obvious way to compute it.

- In the present covariant way of quantizing the string, we cannot reproduce two properties of the string that we have obtained in the light-cone gauge, namely
- the fact that the intercept of the Regge trajectory $\alpha_0 = 1$
- and the critical dimension $d = 26$ that in the light-cone was essential to have a Lorentz invariant theory.
- On the other hand, one expects that, quantizing the theory in two different gauges, one would get the same result.
- Here conformal invariance is a gauge symmetry because it comes from the invariance under reparametrizations.
- Therefore, we expect the energy momentum tensor to transform as a two-index tensor without an anomaly term:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{d}{12}n(n^2-1)\delta_{n+m;0}$$

corresponding to the c-number of the Virasoro algebra.
- What is wrong in our present treatment of the conformal gauge?

- Before this, let us consider shortly the case of the closed string.
- In this case we have two sets of Virasoro operators L_n and $\tilde{L}_n$.
- The equations that characterize the on-shell physical states are:

$$(L_0 - 1)|Phys.\rangle = (\tilde{L}_0 - 1)|Phys.\rangle = 0$$
$$L_n|Phys.\rangle = \tilde{L}_n|Phys.\rangle = 0 \ ; \ \ n = 1, 2 \ldots$$

- with

$$L_0 = \alpha' \left(\frac{\hat{p}}{2}\right)^2 + \sum_{n=1}^{\infty} n a_n^\dagger \cdot a_n \ ; \ \ \tilde{L}_0 = \alpha' \left(\frac{\hat{p}}{2}\right)^2 + \sum_{n=1}^{\infty} n \tilde{a}_n^\dagger \cdot \tilde{a}_n$$

- The mass spectrum is given by ($\hat{p}^2 = -m^2$):

$$\frac{\alpha'}{2} m^2 = \sum_{n=1}^{\infty} n \left(a_n^\dagger \cdot a_n + \tilde{a}_n^\dagger \cdot \tilde{a}_n\right) - 2 \ ; \ \ \sum_{n=1}^{\infty} n a_n^\dagger \cdot a_n = \sum_{n=1}^{\infty} n \tilde{a}_n^\dagger \cdot \tilde{a}_n$$

- The lowest state is the ground state $|0, P\rangle$ with mass $-P^2 = m^2 = -\frac{4}{\alpha'} \Longrightarrow$ it is a tachyon.
- The state contributing to the next massless level is the following:

$$a^{\dagger}_{1\mu}\tilde{a}^{\dagger}_{1\nu}|0, P\rangle$$

- The symmetric and traceless part corresponds to a massless spin 2 $\Longrightarrow$ graviton $G_{\mu\nu}$
- The trace part corresponds to a scalar particle called dilaton ϕ.
- The antisymmetric part corresponds to a 2-index antisymmetric tensor $B_{\mu\nu}$.
- In the open string we have a massless gauge boson, while in the closed string we have a massless graviton together with a massless dilaton and a massless $B_{\mu\nu}$.

- The physical states are a subset of the previous states that satisfy the conditions:

$$L_n|Phys.\rangle = \tilde{L}_n|Phys.\rangle = 0$$

- The analysis at this level proceeds as at the massless level of the open string.
- In the reference frame where the momentum of the state is $P_\mu = (P, \dots, P)$, after the elimination of the zero norm states, the only physical states are:

$$a^\dagger_{1,i}\tilde{a}^\dagger_{1,j}|0, P\rangle \quad ; \quad i, j = 1 \dots (d-2)$$

- In conclusion, one gets $\frac{(d-2)(d-1)}{2} - 1$ physical states for the graviton, $\frac{(d-2)(d-3)}{2}$ physical states for the two-index antisymmetric tensor and one state associated to the dilaton.
- The total number of physical states at this level is therefore $(d-2)^2$.

The Polyakov path integral

- The most convenient way to find what is lacking in the old covariant quantization is to compute the string partition function using the string path integral formalism:

$$\int Dx^{\mu} Dg_{\alpha\beta} \, \mathrm{e}^{-S(x^{\mu}, g_{\alpha\beta})}$$

- The string action in Euclidean space is equal to

$$S(x^{\mu}, g_{\alpha\beta}) \equiv \frac{T}{2} \int d^2\xi \sqrt{g} \, g^{\alpha\beta} \partial_{\alpha} x \cdot \partial_{\beta} x$$

- It is invariant under world-sheet reparametrizations that act on the world-sheet metric and on the string coordinates as follows:

$$x^{\mu}(\xi) = (x')^{\mu}(\xi') \quad ; \quad g_{\alpha\beta}(\xi) = \frac{\partial \xi'^{\gamma}}{\partial \xi^{\alpha}} \frac{\partial \xi'^{\delta}}{\partial \xi^{\beta}} g'_{\alpha\beta}(\xi')$$

- For infinitesimal transformations $((\xi'^{\alpha} = \xi^{\alpha} - \epsilon^{\alpha}(\xi))$ they become

$$\delta x^{\mu} = \epsilon^{\alpha} \partial_{\alpha} x^{\mu} \, ; \, \delta g_{\alpha\beta} = \epsilon^{\gamma} \partial_{\gamma} g_{\alpha\beta} + g_{\alpha\gamma} \partial_{\beta} \epsilon^{\gamma} + g_{\beta\gamma} \partial_{\alpha} \epsilon^{\gamma} = \nabla_{\alpha} \epsilon_{\beta} + \nabla_{\beta} \epsilon_{\alpha}$$

- It is also invariant under Weyl transformations (rescaling of the metric):

$$x^\mu(\xi) \to x^\mu(\xi) \ \ ; \quad g_{\alpha\beta}(\xi) \to \Lambda^2(\xi) g_{\alpha\beta}(\xi)$$

- These two invariances involve three arbitrary functions $\epsilon^\alpha(\xi)$ with $\alpha = 1, 2$ and $\Lambda(\xi)$.
- The metric tensor has also three independent components.
- Locally, one can always choose a suitable reparametrization and a Weyl transformation that lead to a flat metric or to the one in the conformal gauge where

$$\hat{g}_{\alpha\beta} = \delta_{\alpha\beta} \quad ; \quad \hat{g}_{\alpha\beta} = \rho(\xi)\delta_{\alpha\beta}$$

  if reparametrization and Weyl invariances are mantained at the quantum level.
- Because of these two local invariances, the path integral is ill defined being the volume of the reparametrizations and Weyl transformations infinite.

- We can define the functional integral by dividing by the volume of the reparametrizations and Weyl rescalings:

$$\int \frac{Dg_{\alpha\beta}\ Dx^{\mu}}{V_{rep.} \times V_{Weyl}} e^{-S(x,g)}$$

- In order to extract from Dg the two volumes, we perform the Faddeev and Popov procedure that can be applied to any theory with local gauge invariance.
- Starting from a fixed fiducial metric $\hat{g}_{\alpha\beta}(\xi)$ we can obtain the most general metric by transforming it by a reparametrization and a Weyl transformation:

$$\hat{g}^{\zeta}_{\alpha\beta}(\xi') = e^{2\omega(\xi)} \frac{\partial \xi^{\gamma}}{\partial \xi'^{\alpha}} \frac{\partial \xi^{\delta}}{\partial \xi'^{\beta}} \hat{g}_{\gamma\delta}(\xi) \quad ; \quad \zeta \equiv (\xi'(\xi), \omega(\xi))$$

- In order to extract the volume of the reparametrization and Weyl transformations, we change integration variables from the original $g_{\alpha\beta}$ to the parameters of those transformations $\omega(\xi)$ and $\xi'^{\alpha}(\xi)$.
- The integral over the parameters of the reparametrization and Weyl transformations gives the volume $V_{rep.} \times V_{Weyl}$ that cancels the volume in the denominator.
- One is left with the jacobian of the transformation from $g_{\alpha\beta}$ to the parameters of the invariance group, called the determinant of Faddeev-Popov.
- This procedure is explained in detail in a section at the end of this lecture.
- Here we only give the final result:

$$\int Dx^{\mu} \; \Delta_{FP}(\hat{g}) \; e^{-S(x,\hat{g})}$$

- The determinant of the Faddev-Popov can be expressed it terms of a functional integral over the ghost fields $b^{\alpha\beta}$ (traceless) and c^{α} obtaining:

$$Z(\hat{g}) = \int Dx\, Db\, Dc\, \mathrm{e}^{-S_x - S_{gh}}$$

- where

$$S_{gh} = \frac{1}{2\pi} \int d^2\xi \sqrt{\hat{g}}\, b_{\alpha\beta} \hat{\nabla}^{\alpha} c^{\beta} \;\; ; \;\; S_x = \frac{1}{2\pi} \int d^2\xi \sqrt{\hat{g}}\, \hat{g}^{\alpha\beta} \partial_\alpha x \cdot \partial_\beta x$$

- We call them ghosts because they anti-commute (they are Grassmann variables), but not Dirac fermions.
- We have made x dimensionless by dividing it by $\sqrt{2\alpha'}$.

- In the conformal gauge and world-sheet light-cone coordinates $z = \xi^1 + i\xi^2$ and $\bar{z} = \xi^1 - i\xi^2$ where

$$g_{\alpha\beta} = \rho(\xi)\delta_{\alpha\beta} \Longrightarrow g_{z\bar{z}} = g_{\bar{z}z} = \frac{\rho}{2} \;\; ; \;\; g_{zz} = g_{\bar{z}\bar{z}} = 0$$

- the ghost action becomes:

$$S_{gh} = \frac{1}{2\pi} \int d^2\xi \sqrt{\hat{g}}\, b_{\alpha\beta} \hat{g}^{\alpha\gamma} \hat{\nabla}_\gamma c^\beta = \frac{1}{2\pi} \int d^2\xi \left[b_{zz} \partial_{\bar{z}} c^z + b_{\bar{z}\bar{z}} \partial_z c^{\hat{\bar{z}}} \right]$$

- In the present derivation we have ignored the possibility of anomalies.
- It can be shown that, in general, we can have a Weyl anomaly that disappears, however, if the space-time dimension $d = 26$.

- One can quantize the theory preserving reparametrization invariance.
- But then, in general, one cannot preserve Weyl invariance.
- How does a quantum violation of Weyl invariance manifest itself?
- On the fact that the functional integral over x^μ, b, c will depend on $\rho \Longrightarrow$ one does not get anymore the volume of the Weyl group.
- It turns out that the contribution of the functional integral over x^μ, b, c gives:

$$e^{\frac{1}{12\pi}\left(\frac{d}{2}-13\right)\int d^2\xi\left[\frac{1}{2}\partial_\alpha\varphi\partial^\alpha\varphi+\mu^2 e^\varphi\right]} \quad ; \quad \rho \equiv e^\varphi$$

- The dependence on φ disappears only if $d = 26$.
- Only for $d = 26$ one has a Weyl invariant quantum theory.

Conformal invariance

- Introducing the simpler notation:

$$b \equiv b_{zz} \;\; ; \;\; \bar{b} \equiv b_{\bar{z}\bar{z}} \;\; ; \;\; \bar{c} \equiv c^{\bar{z}} \;\; ; \;\; c \equiv c^{z} \; ; \; \partial \equiv \partial_z \;\; ; \;\; \bar{\partial} \equiv \partial_{\bar{z}}$$

- the action becomes:

$$S = \frac{1}{\pi} \int d^2\xi \, [\frac{1}{2} \partial x \cdot \bar{\partial} x + b \bar{\partial} c + \bar{b} \partial \bar{c}]$$

- This action is conformal invariant if we assume that x, b, c transform as conformal fields with dimension respectively equal to $0, 2, -1$, namely:

$$\delta x = \epsilon \partial x + \bar{\epsilon} \bar{\partial} x$$
$$\delta b = \epsilon \partial b + 2 \partial \epsilon \; b \;\; ; \;\; \delta c = \epsilon \partial c - \partial \epsilon \; c$$
$$\delta \bar{b} = \bar{\epsilon} \bar{\partial} \bar{b} + 2 \bar{\partial} \bar{\epsilon} \; \bar{b} \;\; ; \;\; \delta \bar{c} = \bar{\epsilon} \bar{\partial} \bar{c} - \bar{\partial} \bar{\epsilon} \; \bar{c}$$

- Each of the three pieces of the previous Lagrangian transforms as a total derivative (it is a conformal tensor with dimension $\Delta = 1$) under the conformal transformations with parameters ϵ and $\bar{\epsilon}$:

$$
\begin{aligned}
&\delta\left(\frac{1}{2}\partial x \cdot \bar{\partial} x\right) = \partial\left(\epsilon \frac{1}{2}\partial x \cdot \bar{\partial} x\right) + \bar{\partial}\left(\bar{\epsilon}\frac{1}{2}\partial x \cdot \bar{\partial} x\right) \\
&\delta\left(b\bar{\partial}c\right) = \partial\left(\epsilon\, b\,\bar{\partial} c\right) \\
&\delta\left(\bar{b}\partial\bar{c}\right) = \bar{\partial}\left(\bar{\epsilon}\,\bar{b}\,\partial\bar{c}\right)
\end{aligned}
$$

- But now the energy-momentum tensor and the corresponding operators L_n get also a contribution from the ghosts!!
- In particular, one get:

$$
L_n = \oint_0 dz\, z^{n+1}\, T(z) = \oint_0 dz\, z^{n+1}\left(T^x(z) + T^{gh}(z)\right)
$$

- where

$$
T^x(z) = -\frac{1}{2} : (\frac{\partial x}{\partial z})^2 : \quad ; \quad T^{gh}(z) =: cb' + 2c'b :
$$

- It can be shown that the new operators L_n satisfy the following algebra:

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{d - 26}{12}\delta_{n+m;0}n(n^2 - 1)$$

- The c-number of the Virasoro algebra is vanishing at the critical dimension D=26.
- as it must happen in any theory where the conformal symmetry is a gauge symmetry obtained after a partial fixing of the reparametrization invariance.
- This is the first sign that also in the covariant quantization we need to have $d = 26$ as in the light-cone gauge.

Some details of the previous calculation

- Using the following contraction rules:

$$< x^{\mu}(z) x^{\nu}(\zeta) >= -\eta^{\mu\nu} \log(z-\zeta) \quad ; \quad < b(z) c(\zeta) >= \frac{1}{z-\zeta}$$

- it can be shown that the transformation properties of a conformal tensor with dimension Δ are completely equivalent to the following singular terms in the OPE of the energy-momentum tensor with the conformal field:

$$T(z)\phi(w) \sim \frac{\frac{\partial \phi}{\partial w}}{z-w} + \Delta \frac{\phi(w)}{(z-w)^2} + \ldots$$

- In fact, from it we get:

$$\delta\phi \sim [L_n, \phi(w)] = \oint_w dz\, z^{n+1} T(z)\phi(w)$$
$$= w^{n+1} \frac{\partial \phi(w)}{\partial w} + \Delta(n+1) w^n \phi(w)$$

- In particular, we can compute the OPE between two energy-momentum tensors (conformal fields with $\Delta = 2$):

$$T(z)T(\zeta) = \frac{\frac{\partial}{\partial \zeta} T(\zeta)}{(z-\zeta)} + 2\frac{T(\zeta)}{(z-\zeta)^2} + \frac{\frac{D-26}{2}}{(z-\zeta)^4} + \ldots$$

- and from it we get:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{d-26}{12}\delta_{n+m;0} n(n^2-1)$$

- Remember:

$$L_n = \oint_0 dz\, z^{n+1} T(z)$$

BRST invariance

- By fixing the gauge, we have lost the invariance under reparametrizations and Weyl transformations.
- But we are left with BRST invariance.
- It is straightforward to show that, under the following transformations:

$$\delta x = \lambda c \partial x \qquad \delta c = \lambda c \partial c \qquad \delta b = -\frac{1}{2}\lambda(\partial x)^2 + \lambda[c\partial b + 2\partial c b]$$

- the gauge fixed Lagrangian transforms as a total derivative:

$$\delta L = \partial[\lambda c L]$$

- λ is a constant Grassmann parameter.
- It is generated by the following operator:

$$Q = \oint_0 dz : c(z)[T^x(z) + \frac{1}{2}T^{gh}(z)] :$$

- Because of its Grassmann character, in the classical theory the product of two BRST transformations is identically vanishing.
- In the quantum theory the square of the BRST charge is given by:

$$\{Q, Q\} = \frac{1}{12}(d - 26) \oint_0 d\zeta c'''(\zeta) c(\zeta)$$

- The square of the BRST charge is vanishing only if d=26.
- This is another sign that our covariant quantization is consistent only for the critical dimension d=26.

Physical states

- In terms of the oscillators the BRST charge in given by:

$$Q = \sum_{n=1}^{\infty} [c_n L^x_{-n} + c_n^\dagger L^x_n] + c_0 [L^x_0 + L^g_0] + \tilde{Q}$$

where

$$\tilde{Q} = \sum_{n,m=1}^{\infty} m[c_n^\dagger c_m^\dagger b_{n+m} - c_n c_m b^\dagger_{n+m}] - 2b_0 \sum_{n=1}^{\infty} n c_n^\dagger c_n +$$

$$+ \sum_{n,m=1}^{\infty} (n+2m)[c_m^\dagger c_{n+m} b_n^\dagger + c^\dagger_{n+m} c_m b_n]$$

- The ghost fields have the following expansion in terms of the harmonic oscillators:

$$b(z) = \sum_{n=-\infty}^{\infty} b_n z^{-n-2} \quad c(z) = \sum_{n=-\infty}^{\infty} c_n z^{-n+1}$$

- The oscillators satisfy the algebra:

$$\{c_n, b_m\} = \delta_{n+m,0} \quad ; \quad \{c_n, c_m\} = \{b_n, b_m\} = 0$$

- In the BRST quantization the physical states are defined as those annihilated by the BRST charge:

$$Q|Phys.\rangle = 0$$

- This is the residual invariance left from having fixed the gauge.
- The generators of this invariance must annihilate the physical states.
- What are the states that satisfy this equation?
- In order to answer this question we have to introduce and discuss the ghost number current.

- The ghost Lagrangian is invariant under a $U(1)$ current that acts on the ghost fields as follows:

$$\delta b = i\alpha b \quad \delta c = -i\alpha c$$

- The generator corresponding to this invariance can be constructed in terms of the the ghost number density:

$$j(z) =: c(z)b(z) :$$

- The ghost number is given by

$$q = \oint_0 dz\, j(z) = \sum_{n=-\infty}^{\infty} : c_n b_{-n} :$$

$$= c_0 b_0 + c_1 b_{-1} + c_{-1} b_1 + \sum_{n=2}^{\infty} (c_{-n} b_n - b_{-n} c_n)$$

- It turns out that the ghost number current is anomalous and requires the following unconventional normal ordering for the ghost oscillators:

$$: c_n b_{-n} := \begin{cases} c_n b_{-n} & \text{if} \quad n \leq 1 \\ -b_{-n} c_n & \text{if} \quad n \geq 2 \end{cases}$$

- or equivalently
  $b_{-1}, b_0, b_1, b_2 \ldots c_2, c_3 \ldots$ are "annihilation operators"
  $b_{-2}, b_{-3}, b_{-4} \ldots c_1, c_0, c_{-1} \ldots$ are "creation operators".
- In particular, a state that satisfies the following equations:

$$(\ldots b_2, b_1, b_0, b_{-1})|q = 0\rangle = (\ldots c_3, c_2)|q = 0\rangle = 0$$

  has ghost number zero.
- It plays the role of the vacuum because it is annihilated by all "annihilation operators".
- The state with $q = 1$ is what "one would normally call a vacuum":

$$|q = 1\rangle \equiv c_1|q = 0\rangle \Longrightarrow$$

$$(\ldots b_2, b_1, b_0)|q = 1\rangle = (\ldots c_3, c_2, c_1)|q = 1\rangle = 0$$

- A detailed analysis shows that the on-shell physical states must have the following form [Freeman and Olive, 1986]:

$$|Phys.\rangle = |q = 1; \psi_a\rangle$$

where the state $|\psi_a\rangle$ is constructed only in terms of the oscillators of the string coordinate x.

- Remembering the form of Q in terms of the oscillators we see that

$$\tilde{Q}|q = 1\rangle = 0$$

- and the action of Q on the physical state is then given by:

$$Q|q = 1; \psi_a\rangle = \left[\sum_{n=1}^{\infty} [c_n L^x_{-n} + c_n^\dagger L^x_n] + c_0 [L^x_0 + L^g_0] \right] |q = 1; \psi_a\rangle$$

$$= \left[\sum_{n=1}^{\infty} c_n^\dagger L^x_n + c_0 (L^x_0 - 1) \right] |q = 1; \psi_a\rangle = 0$$

- We have used the two identities:

$$c_n|q=1\rangle = 0 \;\; ; \;\; n = 1, 2 \ldots \quad ; \quad L_0^{gh}|q=1\rangle = -|q=1\rangle$$

- The second equation follows from the following expression for L_0^g:

$$L_0^g = \sum_{n=-\infty}^{\infty} n : b_{-n}c_n := \sum_{n=2}^{\infty} n(b_{-n}c_n + c_{-n}b_n) + c_{-1}b_1 - c_1 b_{-1}$$

- In conclusion, we correctly reproduce the conditions for on physical states:

$$L_n^x|\psi_a\rangle = (L_0^x - 1)|\psi_a\rangle = 0$$

- The most general physical state has therefore the following form:

$$|Phys.\rangle = |q=1, \psi_a\rangle + Q|\lambda\rangle$$

where $|\lambda\rangle$ is an arbitrary state.

Conclusions

- Quantizing correctly the bosonic string in a covariant gauge we have obtained the same results as in the light-cone gauge !
- namely the correct values for the Regge intercept and the critical dimensions:

$$\alpha_0 = 1 \qquad d = 26$$

- It turns out the equations characterizing the on-shell physical states are precisely those obtained in 1970 from factorizing the N-point amplitude without knowing that there was an underlying string theory !!
- The new feature is the presence in the covariant gauge of the reparametrization ghosts b and c.
- They are, however, in practice not relevant if we limit ourselves to the computation of the spectrum and of tree diagrams.

- They are, instead, essential for computing one-loop and especially multiloop diagrams.
- If one computes loop diagrams in the light-cone gauge one has only the physical transverse states circulating in the loop.
- In a covariant formulation one must keep all string oscillators and not just the physical transverse ones.
- One has then too many states circulating in the loops.
- The ghost degrees of freedom that are fermions, are there to cancel the contribution of the non-physical states kept in order to have a manifest Lorentz invariant formulation of the string theory.

The material that follows is for helping those interested in understanding some of the more technical details.

Faddeev-Popov procedure

- We define the functional integral by dividing by the volume of the reparametrizations and Weyl rescalings:

$$\int \frac{Dg\ Dx}{V_{rep.} \times V_{Weyl}} e^{-S(x,g)}$$

- In order to extract from Dg the two volumes, we perform the Faddeev and Popov procedure that can be applied to any theory with local gauge invariance.
- Starting from a fiducial metric $\hat{g}_{\alpha\beta}(\xi)$ we can transform it by a reparametrization and a Weyl transformation:

$$\hat{g}^{\zeta}_{\alpha\beta}(\xi') = e^{2\omega(\xi)} \frac{\partial \xi^{\gamma}}{\partial \xi'^{\alpha}} \frac{\partial \xi^{\delta}}{\partial \xi'^{\beta}} \hat{g}_{\gamma\delta}(\xi) \quad ; \quad \zeta \equiv (\epsilon, \omega)$$

- We define the Faddeev-Popov measure by

$$1 = \Delta_{FP}(g) \int D\zeta \delta(g - \hat{g}^{\zeta})$$

- $D\zeta$ is the invariant measure of the reparametrizations plus Weyl transformations.
- We can insert 1 in the functional integral, integrate over h and rename the dummy variable $x \to x^\zeta$:

$$\int \frac{D\zeta \; Dx^\zeta}{V_{rep.} \times V_{Weyl}} \Delta_{FP}(\hat{g}^\zeta) \; \mathrm{e}^{-S(x^\zeta, \hat{g}^\zeta)}$$

- Using the gauge invariance of the action, of the measure and of Δ_{FP} one gets:

$$\int \frac{D\zeta \; Dx}{V_{rep.} \times V_{Weyl}} \Delta_{FP}(\hat{g}) \; \mathrm{e}^{-S(x, \hat{g})}$$

- Nothing depends on ζ and therefore we can integrate on it producing the volume of the invariance groups that cancels the volume in the denominator:

$$\int Dx \; \Delta_{FP}(\hat{g}) \; \mathrm{e}^{-S(x, \hat{g})}$$

- Δ_{FP} can be computed for ζ near the identity where:

$$\hat{g}_{\alpha\beta} - \hat{g}^{\zeta}_{\alpha\beta} \sim 2\delta\omega g_{\alpha\beta} - \nabla_\alpha \epsilon_\beta - \nabla_\beta \epsilon_\alpha$$
$$= (2\delta\omega - \nabla_\gamma \epsilon^\gamma) g_{\alpha\beta} - 2(P_1\epsilon)_{\alpha\beta}$$

and

$$(P_1\epsilon)_{\alpha\beta} = \frac{1}{2}\left(\nabla_\alpha \epsilon_\beta + \nabla_\beta \epsilon_\alpha - g_{\alpha\beta}\nabla_\gamma \epsilon^\gamma\right)$$

- Near the identity we can compute the Faddeev-Popov determinant:

$$\Delta^{-1}_{FP}(\hat{g}) = \int D\epsilon\, D\delta\omega\, \delta\left(-2(\delta\omega - \hat{\nabla}\cdot\epsilon)\hat{g} + 2\hat{P}_1\epsilon\right)$$
$$= \int D\epsilon\, D\delta\omega D\,\beta\; \mathrm{e}^{2\pi i \int d^2\xi \sqrt{\hat{g}}\beta^{\alpha\beta}\left(-2(\delta\omega - \hat{\nabla}\cdot\epsilon)\hat{g} + 2\hat{P}_1\epsilon\right)_{\alpha\beta}}$$

- The integration over $\delta\omega$ forces $\beta^{\alpha\beta}$ to be traceless and one gets:

$$\Delta^{-1}_{FP}(\hat{g}) = \int D\epsilon D\beta\; \mathrm{e}^{4\pi i \int d^2\xi \sqrt{\hat{g}}\beta^{\alpha\beta}(\hat{P}_1\epsilon)_{\alpha\beta}}$$

- In this way we have computed the inverse determinant.

- In order to obtain directly the Faddeev-Popov determinant we have to replace any bosonic with a fermionic field:

$$\beta^{\alpha\beta} \rightarrow b^{\alpha\beta} \quad ; \quad \epsilon^{\alpha} \rightarrow c^{\alpha}$$

obtaining

$$\Delta_{FP}(\hat{g}) = \int DcDb\, e^{4\pi i \int d^2\xi \sqrt{\hat{g}} b^{\alpha\beta} (\hat{P}_1 c)_{\alpha\beta}}$$

where b is traceless.
- We call them ghosts because they are Grassmann, but not Dirac fermions.
- In conclusion, with a convenient normalization of the two ghost fields we obtain the following gauge fixed partition functiion:

$$Z(\hat{g}) = \int Dx\, Db\, Dc\, e^{-S_x - S_{gh}}$$

- where

$$S_{gh} = \frac{1}{2\pi} \int d^2\xi \sqrt{\hat{g}}\, b_{\alpha\beta} \hat{\nabla}^{\alpha} c^{\beta} \;\; ; \;\; S_x = \frac{1}{2\pi} \int d^2\xi \sqrt{\hat{g}}\, \hat{g}^{\alpha\beta} \partial_{\alpha} x \cdot \partial_{\beta} x$$

Determinants in the numerator or in denominator

- If we have a gaussian integral with bosonic complex variables we get:

$$\int \prod_i d^2 z_i \mathrm{e}^{-\sum_{i,j} \bar{z}_i M_{ij} z_j} = \frac{1}{\det M}$$

- Instead, if we have a gaussian integral involving fermionic (Grassmann) complex variables we get:

$$\int \prod_i d^2 \psi_i \mathrm{e}^{-\sum_{i,j} \bar{\psi}_i M_{ij} \psi_j} = \det M$$

- Remember that Grassmann variables anticommute:

$$\psi_i \psi_j = -\psi_j \psi_i \quad ; \quad \psi_i \bar{\psi}_j = -\bar{\psi}_j \psi_i \quad \Longrightarrow \quad \psi_i^2 = 0$$

- The determinant is computed using the following integration rules:

$$\int d\psi = 0 \quad ; \quad \int d\psi \psi = 1$$

- We have made x dimensionless by dividing it by $\sqrt{2\alpha'}$.
- In the conformal gauge and world-sheet light-cone coordinates $z = \xi^1 + i\xi^2$ and $\bar{z} = \xi^1 - i\xi^2$ where

$$g_{\alpha\beta} = \rho(\xi)\delta_{\alpha\beta} \Longrightarrow g_{z\bar{z}} = g_{\bar{z}z} = \frac{\rho}{2} \;\; ; \;\; g_{zz} = g_{\bar{z}\bar{z}} = 0$$

- the ghost action becomes:

$$S_{gh} = \frac{1}{2\pi}\int d^2\xi \sqrt{\hat{g}}\, b_{\alpha\beta}\hat{g}^{\alpha\gamma}\hat{\nabla}_\gamma c^\beta = \frac{1}{2\pi}\int d^2\xi \left[b_{zz}\partial_{\bar{z}}c^z + b_{\bar{z}\bar{z}}\partial_z c^{\hat{z}}\right]$$

- In the present derivation we have ignored the possibility of anomalies.
- It can be shown that, in general, we can have a Weyl anomaly that disappears, however, if the space-time dimension $d = 26$.

Conformal tensors

- Consider string theory in the conformal gauge, characterized by the following choice of the Euclidean world-sheet metric tensor:

$$g_{\alpha\beta} = \rho(\xi)\, \delta_{\alpha\beta} \;\; ; \;\; \rho = e^{\varphi(\xi)}$$

- We have seen that the conformal transformations leave in the conformal gauge.
- It is convenient to work with light-cone coordinates:

$$z = \xi^1 + i\xi^2 \quad ; \quad \bar{z} = \xi^1 - i\xi^2$$

- In these coordinates the invariant length is defined by:

$$(ds)^2 = g_{\alpha\beta} d\xi^\alpha d\xi^\beta = \frac{\rho}{2} \left[dz d\bar{z} + d\bar{z} dz \right]$$

- implying the following light-cone coordinates for the metric tensor:

$$g^{zz} = g^{\bar{z}\bar{z}} = g_{zz} = g_{\bar{z}\bar{z}} = 0$$
$$g^{z\bar{z}} = g^{\bar{z}z} = 2/\rho \qquad g_{z\bar{z}} = g_{\bar{z}z} = \rho/2$$

- In terms of the light-cone components of a vector:

$$\epsilon^z = \epsilon^1 + i\epsilon^2 \quad \epsilon^{\bar{z}} = \epsilon^1 - i\epsilon^2 \quad \epsilon_z = \frac{1}{2}(\epsilon_1 - i\epsilon_2) \quad \epsilon_{\bar{z}} = \frac{1}{2}(\epsilon_1 + i\epsilon_2)$$

- one can define the scalar product between two vectors:

$$A^\alpha B_\alpha = [A^z B_z + A^{\bar{z}} B_{\bar{z}}] = [A^z B_z + A_z B^z] = A_z B^z + A_{\bar{z}} B^{\bar{z}}$$

where the indices are lowered and raised by means of the metric tensor as follows:

$$A^z = g^{z\bar{z}} A_{\bar{z}} \qquad A_z = g_{z\bar{z}} A^{\bar{z}} \qquad A^{\bar{z}} = g^{\bar{z}z} A_z \qquad A_{\bar{z}} = g_{\bar{z}z} A^z$$

- The covariant derivatives are given by:

$$\nabla_\alpha \epsilon^\beta = \partial_\alpha \epsilon^\beta + \Gamma^\beta_{\alpha\gamma} \epsilon^\gamma \quad , \quad \nabla_\alpha \epsilon_\beta = \partial_\alpha \epsilon_\beta - \Gamma^\gamma_{\alpha\beta} \epsilon_\gamma$$

where the Christoffel symbols are given in the conformal gauge by:

$$\Gamma^\gamma_{\alpha\beta} = \frac{g^{\gamma\delta}}{2}[\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\alpha\delta} - \partial_\delta g_{\alpha\beta}] = \left[\partial_\alpha \delta^\gamma{}_\beta + \partial_\beta \delta^\gamma{}_\alpha - \partial^\gamma \delta_{\alpha\beta}\right] \frac{\log \rho}{2}$$

- Only two non-vanishing components:

$$\Gamma^{z}_{zz} = \rho^{-1}\partial_z \rho \quad , \quad \Gamma^{\bar z}_{\bar z \bar z} = \rho^{-1}\partial_{\bar z}\rho$$

- One gets:

$$\nabla_{\bar z}\epsilon^{z} = \partial_{\bar z}\epsilon^{z} \quad ; \quad \nabla_{\bar z}\epsilon^{\bar z} = \rho^{-1}\partial_{\bar z}\rho\epsilon^{\bar z}$$
$$\nabla_{z}\epsilon^{\bar z} = \partial_{z}\epsilon^{\bar z} \quad ; \quad \nabla_{z}\epsilon^{z} = \rho^{-1}\partial_{z}\rho\epsilon^{z}$$

- Raising the index of the covariant derivative with the metric tensor one gets:

$$\nabla^{z}\epsilon^{z} = \frac{2}{\rho}\partial_{\bar z}\epsilon^{z} \quad ; \quad \nabla^{\bar z}\epsilon^{z} = \frac{2}{\rho^2}\partial_{z}\rho\epsilon^{z}$$
$$\nabla^{\bar z}\epsilon^{\bar z} = \frac{2}{\rho}\partial_{z}\epsilon^{\bar z} \quad ; \quad \nabla^{z}\epsilon^{\bar z} = \frac{2}{\rho^2}\partial_{\bar z}\rho\epsilon^{\bar z}$$

- The action of the covariant derivative on a conformal tensor $T^{z\cdots z}$ with rank n is given by:

$$\nabla^n_{\bar z} T^{z\cdots z} = \partial_{\bar z} T^{z\cdots z} \qquad \nabla^n_{z} T^{z\cdots z} = \rho^{-n}\partial_z \rho^n T^{z\cdots z}$$
$$\nabla^z_n T^{z\cdots z} = \frac{2}{\rho}\partial_{\bar z} T^{z\cdots z} \qquad \nabla^{\bar z}_n T^{z\cdots z} = 2\rho^{-1-n}\partial_z \rho^n T^{z\cdots z}$$

- Under a general relativity transformation a vector transforms as follows:

$$\epsilon^\mu(\xi) \to \frac{\partial \xi^\mu}{\partial \xi'^\nu}\epsilon^\nu(\xi')$$

- In terms of light-cone coordinates one gets:

$$\epsilon^z(z,\bar z) \to \frac{\partial z}{\partial w}\epsilon^w = \frac{1}{w'(z)}\epsilon^w \qquad \epsilon^{\bar z}(z,\bar z) \to \frac{\partial \bar z}{\partial \bar w}\epsilon^{\bar w} = \frac{1}{\bar w'(\bar z)}\epsilon^{\bar w}$$

- We have restricted us to conformal transformations for which:

$$\frac{\partial w}{\partial \bar z} = \frac{\partial \bar w}{\partial z} = 0$$

- A conformal tensor of rank n transforms as follows under a conformal transformation:

$$T^{z\cdots z}(z) \to \frac{1}{[w'(z)]^n} T^{w\cdots w}(w) \;\; ; \;\; T^{\bar{z}\cdots\bar{z}}(\bar{z}) \to \frac{1}{[\bar{w}'(\bar{z})]^n} T^{\bar{w}\cdots\bar{w}}(\bar{w})$$

$$T_{z\cdots z}(z) \to [w'(z)]^n T_{w\cdots w}(z) \;\; ; \;\; T_{\bar{z}\cdots\bar{z}}(\bar{z}) \to [\bar{w}'(\bar{z})]^n T_{\bar{w}\cdots\bar{w}}(\bar{w})$$

- We have lowered the indices with the metric tensor and we have used the transformation of ρ under a conformal transformation:

$$\rho(z,\bar{z}) \to w'(z)\bar{w}'(\bar{z})\rho(w,\bar{w})$$

- The covariant derivative ∇_n^z applied to a conformal tensor of rank n gives a conformal tensor of rank $n+1$:

$$\nabla_n^z T_{(n)}^{z\ldots z}(z) \equiv \frac{2}{\rho}\partial_{\bar{z}} T_{(n)}^{z\ldots z}(z) \to [w'(z)]^{-n-1}\nabla^w T_{(n)}^{w\ldots w}(w)$$

- The covariant derivative ∇_z^n applied to a conformal tensor of rank n gives a conformal tensor with rank n-1:

$$\nabla_z^n T_{(n)}^{z\ldots z}(z) \equiv \rho^{-n}(z)\partial_z \rho^n(z) T_{(n)}^{z\ldots z}(z) \rightarrow [w'(z)]^{1-n} \nabla_w T_{(n)}^{w\ldots w}(w)$$

- In conclusion, the action of the covariant derivative on a conformal tensor of rank n gives the following tensors:

$$T^{(n)} \xrightarrow{\nabla_n^z} T^{(n+1)} \xrightarrow{\nabla_z^{n+1}} T^{(n)}$$
$$T^{(n)} \xrightarrow{\nabla_z^n} T^{(n-1)} \xrightarrow{\nabla_{n-1}^z} T^{(n)}$$

- In terms of the covariant derivatives we can define the following Laplacians:

$$\Delta_n^{(+)} = -\nabla_z^{n+1}\nabla_n^z \qquad \Delta_n^{(-)} = -\nabla_{n-1}^z \nabla_z^n$$

- They satisfy the relation:

$$\Delta_n^{(+)} - \Delta_n^{(-)} = \frac{n}{2} R$$

where R is the scalar curvature:

$$R = \frac{4}{\rho} \frac{\partial^2 \varphi}{\partial z \partial \bar{z}}$$

The ghost number current

- The ghost Lagrangian is invariant under a $U(1)$ current that acts on the ghost fields as follows:

$$\delta b = i\alpha b \quad \delta c = -i\alpha c$$

- The generator corresponding to this invariance can be constructed in terms of the the ghost number density:

$$j(z) =: c(z)b(z) :$$

- Using the $b - c$ contraction one can to compute the following OPE's:

$$j(z)j(\zeta) = \frac{1}{(z-\zeta)^2}$$

$$T^g(z)j(\zeta) = \frac{\frac{\partial j(\zeta)}{\partial \zeta}}{z-\zeta} + \frac{j(z)}{(z-\zeta)^2} - \frac{3}{(z-\zeta)^3}$$

- j(z) is a conformal field with dimension $\Delta = 1$, but there is an extra term that makes the analysis more complicated.

- The ghost coordinates b(z) and c(z) are conformal fields with conformal dimension $\Delta = 2$ and -1 respectively.
- Their expansion in term of the harmonic oscillators is given by:

$$b(z) = \sum_{-\infty}^{\infty} b_n z^{-n-2} \quad c(z) = \sum_{-\infty}^{\infty} c_n z^{-n+1}$$

- The oscillators satisfy the algebra:

$$\{c_n, b_m\} = \delta_{n+m,0} \quad ; \quad \{c_n, c_m\} = \{b_n, b_m\} = 0$$

- Introduce the Fourier components of $j(z)$ and $T^g(z)$

$$j_n = \oint_0 dz z^n j(z) = \sum_m : c_{n-m} b_m :$$

$$L_n^g = \oint_0 dz z^{n+1} T^g(z) = \sum_m (m+n) : b_{n-m} c_m :$$

- They satisfy the algebra:

$$[j_n, j_m] = n\delta_{n+m,0} \quad ; \quad [L^g_n, j_m] = -mj_{n+m} - \frac{3}{2}n(n+1)\delta_{n+m,0}$$

$$[L^g_n, L^g_m] = (n-m)L^g_{n+m} - \frac{26}{12}n(n^2-1)\delta_{n+M,0}$$

- It can be reproduced in terms of the oscillators only if the normal ordering is defined in the following non-conventional way:

$$: c_n b_{-n} := \begin{cases} c_n b_{-n} & \text{if} \quad n \leq 1 \\ -b_{-n} c_n & \text{if} \quad n \geq 2 \end{cases}$$

- From the algebra it turns out that j_0 is not anti-hermitian as j_n for $n \neq 0$, but it satisfies the more complicated relation:

$$j_0 + j_0^\dagger - 3 = 0$$

- Therefore if $|q>$ is an eigenstate of the ghost number

$$j_0|q>= q|q>$$

the previous equation implies that

$$< q'|q>\sim \delta_{q,3-q'}$$

- It can be checked that the state defined by

$$b_n|q>= 0 \quad \text{if} \quad n > q-2$$

$$c_n|q>= 0 \quad \text{if} \quad n \geq -q+2$$

is an eigenstate of the ghost number operator with ghost number equal to q.
- It satisfies also the equation:

$$L_0|q>= \frac{1}{2}q(q-3)|q>$$

- Among those eigenstates of j_0 the only one, that is annihilated by the three generators of the projective group is $|q = 0>$:

$$L_0|q = 0 >= L_1|q = 0 >= L_{-1}|q = 0 >= 0$$

  $|q = 0 >$ is therefore projective invariant.
- The non-anti-hermicity of j_0 implies that, if we compute any matrix element containing objects with a definite ghost number, we will get zero unless the total ghost charge is equal to 3.
- In particular, in order to compute $b - c$ the contraction, we must compute the following matrix element:

$$< q = 3|b(z)c(\zeta)|q = 0 >= \frac{1}{z - \zeta}$$