Particules Elémentaires, Gravitation et Cosmologie
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Gravitation et Cosmologie: le Modèle Standard

Cours II: 9 janvier 2009

Du principe d'équivalence à la Covariance Générale

- The Equivalence Principle
- Invariant line element and the metric tensor
- Motion in a gravitational field, Newtonian limit
- The principle of General Covariance

The Equivalence Principle

- Einstein's Equivalence Principle (EP) is a remarkable (and daring) extension of the Galilean/Newtonian universality of free fall (implied by $m^{(in)} = m^{(gr)}$ and verified w/ very high precision).
- In a uniform gravitational field the effects of gravity are completely eliminated if one is in a freely falling system ("Einstein's elevator")
- Einstein then postulated that something similar should be true in any gravitational field

The Equivalence Principle

In a sufficiently small neighbourhood (to be better defined later) of every space-time point it is possible to choose a "locally inertial" (or "free-falling") coordinate system in which the laws of Nature take the same (special relativistic) form as if there were no gravitational field.

In an arbitrary coordinate system a space-time point X is given by $x^\mu=(x^0= ct, x^i)$. Let us denote the locally inertial coordinates in the neighbourhood of x^μ by $\xi^a{}_X(x^\mu)$

According to the EP, the Lorentz-invariant infinitesimal distance between two points will be:

$$ds^2 = \eta_{ab} d\xi^a_X d\xi^b_X = \eta_{ab} \frac{\partial \xi^a_X}{\partial x^\mu} \frac{\partial \xi^b_X}{\partial x^\nu} dx^\mu dx^\nu$$

$$ds^2 = \eta_{ab} d\xi_X^a d\xi_X^b = \eta_{ab} \frac{\partial \xi_X^a}{\partial x^\mu} \frac{\partial \xi_X^b}{\partial x^\nu} dx^\mu dx^\nu$$

Introducing:

$$g_{\mu\nu}(x) \equiv \eta_{ab} \frac{\partial \xi_X^a}{\partial x^\mu} \frac{\partial \xi_X^b}{\partial x^\nu} = g_{\nu\mu}(x)$$

we can write one of the basic formulae of GR

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$$

Associated with any coordinate system there is a “metric tensor” field $g_{\mu\nu}(x)$ such that the infinitesimal distance ds is the same in all coordinate systems (as one can easily check) i.e. is invariant under General Coordinate Transformations. The metric tensor is supposed to encode all the information about the gravitational fields that are present.

As a second application of the EP let us derive the equations of motion of a particle in an arbitrary gravitational field as described by the associated metric tensor $g_{\mu\nu}(x)$.

Indeed, in the L.I.C. we must have no acceleration:

$$\frac{d^2\xi_X^a}{d\tau^2} = 0 \quad ; \quad d\tau^2 = -ds^2$$

which can be written in arbitrary coordinates as follows:

$$0 = \frac{d}{d\tau}\left(\frac{\partial \xi_X^a}{\partial x^\mu}\frac{dx^\mu}{d\tau}\right) = \left(\frac{\partial \xi_X^a}{\partial x^\mu}\frac{d^2 x^\mu}{d\tau^2} + \frac{\partial^2 \xi_X^a}{\partial x^\mu \partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right)$$

and then $$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0 \quad ; \quad \Gamma^\mu_{\rho\sigma} = \frac{\partial x^\mu}{\partial \xi^a}\frac{\partial^2 \xi^a}{\partial x^\rho \partial x^\sigma}$$

NB: the relation between ξ and x can be inverted

$\Gamma^\mu_{\rho\sigma}$ is called the affine connection or Christoffel symbol

Relation between $g_{\mu\nu}$ and $\Gamma^{\mu}{}_{\rho\sigma}$

$g_{\mu\nu}$ and $\Gamma^{\mu}{}_{\rho\sigma}$ are two very basic objects of GR. However, they are not independent: we can express $\Gamma^{\mu}{}_{\rho\sigma}$ in terms of $g_{\mu\nu}$. The calculation is simple but lengthy...

One writes $\xi^{a}{}_{X}(x^{\mu})$ around the point X as a 2nd order Taylor expansion in $(x^{\mu}-X^{\mu})$ (should eliminate local acceleration!) and then computes both $g_{\mu\nu}$ and $\Gamma^{\mu}{}_{\rho\sigma}$.

The following relation emerges (w/ $g^{\mu\nu}$ the inverse of $g_{\mu\nu}$)

$$\Gamma^{\mu}_{\rho\sigma} = \frac{1}{2} g^{\mu\nu} \left(g_{\rho\nu,\sigma} + g_{\sigma\nu,\rho} - g_{\rho\sigma,\nu} \right) = \Gamma^{\mu}_{\sigma\rho} \quad , \quad A_{,\mu} \equiv \frac{\partial A}{\partial x^{\mu}}$$

- $\Gamma^{\mu}{}_{\rho\sigma}$ represents the "gravitational force" in some generic coordinate system. Since $g_{\mu\nu}$ represents the gravitational potential, it is quite natural for $\Gamma^{\mu}{}_{\rho\sigma}$ to be linear in the gradients (derivatives) of $g_{\mu\nu}$

The equation of motion for a particle in a generic gravitational field:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0 \quad , \quad \Gamma^\mu_{\rho\sigma} = \frac{1}{2}g^{\mu\nu}\left(g_{\rho\nu,\sigma} + g_{\sigma\nu,\rho} - g_{\rho\sigma,\nu}\right)$$

has an interesting physical and geometric interpretation: it corresponds to geodesic (i.e. distance minimizing) motion.

$$\tau_{AB} = \int_A^B d\lambda\sqrt{-ds^2/d\lambda^2} = \int_A^B d\lambda\sqrt{-g_{\mu\nu}(x)\frac{dx^\mu(\lambda)}{d\lambda}\frac{dx^\nu(\lambda)}{d\lambda}}$$

Let's vary τ_{AB} wrt changes of $x^\mu(\lambda)$ which vanish at A and B

$$\delta\tau_{AB} \sim \int_A^B \frac{d\lambda}{\sqrt{\cdots}}\left[2\frac{d}{d\lambda}\left(g_{\mu\nu}(x)\frac{d}{d\lambda}x^\nu(\lambda)\right) - g_{\rho\sigma,\mu}(x)\frac{d}{d\lambda}x^\rho(\lambda)\frac{d}{d\lambda}x^\sigma(\lambda)\right]\delta x^\mu(\lambda)$$

Setting this to zero gives back the equation of motion: a geometric interpretation of motion in a grav. field!

Newtonial limit (weak stationary field)

Consider the equation of motion for a slowly moving body in a weak and stationary field. In that case in:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0 \quad , \quad \Gamma^\mu_{\rho\sigma} = \frac{1}{2}g^{\mu\nu}\left(g_{\rho\nu,\sigma} + g_{\sigma\nu,\rho} - g_{\rho\sigma,\nu}\right)$$

we can neglect $\Gamma^\mu{}_{ij}$ and keep just $\Gamma^\mu{}_{00}$ to give:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} = 0$$

which give:

$$\frac{d^2x^0}{d\tau^2} = 0 \quad \Rightarrow x^0 = ct = c\tau \quad ; \quad \frac{dx^i}{dt^2} = -\Gamma^i_{00} = \frac{1}{2}g^{ij}g_{00,j} \sim \frac{1}{2}g_{00,i}$$

reproduces Newton if: $\quad g_{00} = -(1+2\phi^N) \rightarrow -1 + \dfrac{2GM}{r}$

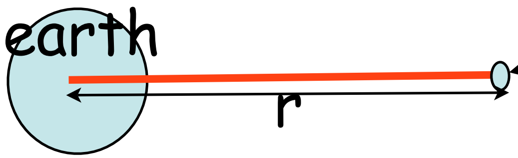
The gravitational red-shift

Consider a clock at rest in a gravitational field: According to the EP the time interval dt between two ticks is given by

$$-g_{00}dt^2 = d\tau^2 = \text{independent of gravitational field}$$

$$-g_{00} = 1 - \frac{2GM}{r}$$

is larger at some altitude than on earth



For two identical clocks, dt(altitude) < dt (earth)

If we synchronize two clocks on earth and take one of them at some altitude for a long time, when we bring it back on earth it will show a later time wrt the other clock.

In practice the experiment is done with a satellite and there is a competing effect due to the motion of the satellite, but the net effect is calculable and agrees with GR.

From the EP to General Covariance

In principle one could derive all equations of GR by first going to L.I. coordinates and by then transforming to arbitrary ones. This is quite cumbersome.

There is an equivalent, easier procedure. Impose that all equations of GR be covariant under GCT (i.e. should look the same in all coordinate systems) and should reduce to those of SR if we replace $g_{\mu\nu}$ by $\eta_{\mu\nu}$

In order to use this latter method we have to get first acquainted with some mathematical tools for working in four-dimensional spaces with a general metric (and even topological) structure. This will be done next week.