# The STRING BEHIND DRM's

## 1 The Nambu-Goto action

- The harmonic oscillators remind us already of a vibrating string
- The field $Q(x)$ looks like a string (end) position
- Several people had the vague idea of a string lurking behind DRM's (Nielsen, Susskind, Nambu)
- Real connection had to wait till Nambu's proposal of an action & its (Light-cone) quantization by GGRT (1973)
- NG-action

$$S_{NG} = -T \int_{\mathcal{M}} d\sigma\, d\tau \sqrt{-\det \tilde{\gamma}_{\alpha\beta}} \qquad \left(T = \frac{1}{2\pi\alpha'}\right)$$

Inv. under rep. of $\sigma, \tau$

where $\tilde{\gamma}_{\alpha\beta} \equiv \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$ $\qquad \left(\begin{array}{l}\eta_{\mu\nu} = (-1, +1 \cdots +1) \\ \alpha, \beta = 0, 1 \\ \xi_0 = \tau, \xi_1 = \sigma\end{array}\right)$

$$-\det \tilde{\gamma}_{\alpha\beta} = \tilde{\gamma}_{01}^2 - \tilde{\gamma}_{00}\tilde{\gamma}_{11} = (\dot{X} \cdot X')^2 - \dot{X}^2 X'^2$$

Cf. $S_{Point\,P.} = -m \int d\tau \sqrt{(-\dot{X}^\mu \dot{X}_\mu)} \underset{X^0 = \tau}{\Longrightarrow} -m \int d\tau \sqrt{1-\beta^2}$ $= -m \int d\tau_{Prop.}$

A (classically) equivalent form was given by Brink-DiVecchia-Howe-Deser-Zumino and largely exploited by Polyakov:

$$S_P = -\frac{T}{2}\int d\sigma d\tau\, (-\gamma)^{1/2}\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

Class eq. for $\gamma_{\alpha\beta}$ gives

$$\gamma_{\alpha\beta} = c\, \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}\,, \quad c \text{ arbitrary}$$

$$\to 2$$

$$S_P = -\frac{T}{2}\int d\sigma d\tau\, (-\gamma)^{1/2}\, c^{-1}\cdot \gamma^{\alpha\beta}\gamma_{\alpha\beta}$$

$$= -T\int d\sigma d\tau\, \left(-\det \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}\right)^{1/2} = S_{NG}$$

GGRT LC quantization uses $X^+ = (X^0 + X^1) = $
$= p^+\tau$
& then uses Vir. constraints to express $X^-(\sigma,\tau)$ in terms of transverse d.o.f. $X_\perp(\sigma,\tau)$

Only $(D-2)$ sets of oscillators left

Give correct Lorentz algebra only at $D=26$!

$3^{rd}$ time (but not last) we saw this # !!

$\alpha_0 = 1$ comes out as well (massless states have fewer d.o.f.)

# BRST quantization (closed bosonic string)

$$S_p = -\frac{1}{2}\int d^2\xi \left\{ (-\gamma)^{1/2}\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X) + \epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}(X) \right\}$$

describes a string $(X^\mu = X^\mu(\xi))$ moving in a $G_{\mu\nu}$, $B_{\mu\nu}$ background* $(G_{\mu\nu}=G_{\nu\mu},\ B_{\mu\nu}=-B_{\nu\mu})$

Kalb-Ramond field

Define as usual $P_\mu = \dfrac{\delta S}{\delta \dot{X}^\mu}$ $\quad \boxed{\dot{X} \equiv \partial X/\partial\xi^0 = \partial X/\partial\tau}$

$$\Rightarrow P_\mu = -(-\gamma)^{1/2}\gamma^{0\alpha}\partial_\alpha X^\nu G_{\mu\nu} - \epsilon^{0\alpha}\partial_\alpha X^\nu B_{\mu\nu}$$

The following constraints follow (because of 2D general covariance)

$(X' \equiv \partial X/\partial\xi^1 = \partial X/\partial\sigma)$

$$0 = L_\pm \equiv \frac{1}{4}\left(P_\mu \pm X'^\rho G_{\mu\rho} + X'^\rho B_{\mu\rho}\right) G^{\mu\nu}\left(P_\nu \pm X'^\sigma G_{\nu\sigma} + X'^\sigma B_{\nu\sigma}\right)$$

These hold for <u>any</u> gauge choice (of $\gamma_{\alpha\beta}$) e.g.

$$L_+ - L_- = X'^\mu P_\mu = -(\gamma^{1/2})\gamma^{0\alpha}\underbrace{\partial_\alpha X^\nu \partial_1 X^\mu G_{\mu\nu}}_{\sim\gamma_{\alpha 1}} - \underbrace{\epsilon^{01}\partial_1 X^\nu B_{\mu\nu}\partial_1 X^\mu}_{0}$$

But $\gamma_{\alpha\beta} \propto \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu} \Rightarrow \sim\gamma_{\alpha 1}$ $\qquad \gamma^{0\alpha}\gamma_{\alpha 1} = \delta^0_1 = 0$

$L_+ + L_-$ is much less trivial (use $\gamma_{11} = \gamma\cdot\gamma^{00}$ etc. etc.)

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*) Note: we have "absorbed" $T$ in our backgrounds $G$, $B$

If by $S_p$ we mean $\hbar^{-1} S_p$, $[G_{\mu\nu}] = [B_{\mu\nu}] = \ell^{-2}$

Relation of $G_{\mu\nu}$ to usual metric $g_{\mu\nu}$? See later

A long, straightforward calculation shows that, at the classical level, the constraints satisfy the following algebra

$$\{ L_\pm(\sigma), L_\pm(\sigma') \}_{P.B.} = \pm \left( L_\pm(\sigma) + L_\pm(\sigma') \right) \partial_\sigma \delta(\sigma - \sigma')$$

while $L_+$ & $L_-$ "commute"

$\int d\sigma\, e^{-in\sigma} \int d\sigma'\, e^{-im\sigma'}$ of above gives

$$\{ L^\pm_n, L^\pm_m \}_{P.B} = i(n-m) L^\pm_{n+m}$$

F.V. 1970's wrong algebra!

if we define $L^+_n = \int d\sigma\, e^{-in\sigma} L_+(\sigma)$ etc.

N.B. Algebra of constraints does not depend on $G, B$ although constraints do!

Furthermore:

i) Constraints close on themselves ($1^{st}$ class)

ii) Structure functions are numerical ($\delta'$, $(n-m)$ ..)

iii) The canonical Hamiltonian vanishes:

$$\mathcal{H}_{can} = P_\mu \cdot \dot{X}^\mu - \mathcal{L} \propto -(-\gamma)^{1/2} \gamma^{0\alpha} \gamma_{\alpha 0} + \frac{1}{2} (-\gamma)^{1/2} \underbrace{\gamma^{\alpha\beta} \gamma_{\alpha\beta}}_{=0}$$

(B-field contribution vanishes by Euler's thrm.)

$\Longrightarrow$ Apply Batalin-Fradkin-Vilkovisky for quantization

# Sketch of BFV procedure

1) Construct $Q \equiv Q_{BRST}$ as

$$Q = \int d\sigma \left( L_+ \eta_+ + L_- \eta_- + \mathcal{P}_+ \eta_+' \eta_+ - \mathcal{P}_- \eta_-' \eta_- \right)$$

where we have associated with each (bosonic) constraint a (Grassmann) pair of ghosts

$(\eta_\pm, \mathcal{P}_\pm) \Rightarrow Q$ is Grassmann

- Coupling of $\eta$ to $L$ is always the same
- Self-coupling of ghosts depends on s.c. of c. algebra
- $\{\eta, \mathcal{P}\}_{P.B.} = \delta(\sigma - \sigma')$ (and $Q^\dagger = Q$)

At classical level, $Q^2 = 0 = \{Q, Q\}$

2) Pick a gauge-fixing fermion $\chi$

then $H_{Tot} = H_{can} + \{\chi, Q\} = \{\chi, Q\}$

$\Longrightarrow \{Q, H\}_{PB} = 0$ (easy to check)

E.G. ON gauge is $\chi = \mathcal{P}_+ + \mathcal{P}_-$
$H = L_+ + L_-$
...

3) Quantize by $i\{\ ,\ \}_{PB} = \hbar^{-1}[\ ,\ ]_\pm$

4) If succeed in keeping $\{\hat{Q}, \hat{Q}\} = 0$

.... bingo!!

# Physical states, operators

$$\hat{Q}\,|Phys.\rangle = 0 \quad , \quad [\hat{Q}, O_{phys}] = 0$$

$\Rightarrow H_{Tot}$ is physical

$$e^{-iH_T t}\,|Phys, t=0\rangle = |Phys, t\rangle$$

Spurious states:

$$|sp\rangle = \hat{Q}\,|X\rangle \quad \text{for some } X$$

$$\Rightarrow \quad \langle Phys|sp\rangle = 0 \quad , \quad \langle Phys|O_{phys}|s\rangle = 0$$

$\Rightarrow$ BRST cohomology:

$$|Phys\rangle \sim |Phys\rangle + \hat{Q}\,|X\rangle$$

F.V. theorem: $\langle Phys|\,O_{phys}\,|\widetilde{Phys}\rangle$ indep. of $\chi$
(guaranteed by $\hat{Q}^2=0$ ?)

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Problem is now clear: can we maintain $\hat{Q}^2=0$ while making all operators finite?
In $D=2$ all we need is normal ordering, but even that can give anomalies (Not in $D=1$!)
Let us check this for the trivial bkgnd, $G_{\mu\nu} \sim \eta_{\mu\nu}$, $B_{\mu\nu}=0$

$$Q = \int (L_+ \eta_+ + \mathcal{P}_+ \eta'_+ \eta_+ ) d\sigma \qquad L_+ = (P_+ X')^2$$

(same procedure works for $(+) \to (-)$)

In computing $[Q, Q]_+$ single commutators give the classical contributions (adding up to 0)

Anomalies come from "double contractions" e.g.

$$(P_+ X')^2 \eta_+(\sigma) \; (P_+ X')^2 \eta_+(\sigma') \sim \frac{\eta_{\mu\nu}\eta^{\mu\nu}}{(z-z')^4} \eta_+(\sigma) \eta_+(\sigma')$$

$$\sim D \, \delta'''(\sigma - \sigma') \, \eta_+(\sigma) \eta_+(\sigma')$$

$z = e^{i\sigma}$

- Mixed terms $( L_+ \eta_+ \times \mathcal{P}_+ \eta'_+ \eta_+ )$ are harmless (only $\eta_+ \mathcal{P}_+$ non-trivial)

- Ghost × Ghost is essential!

$$\mathcal{P}_+ \; \eta'_+ \sim \left( \frac{1}{z - z'} \right)^2$$

same!!

$$\mathcal{P}_+ \eta'_+ \eta_+(\sigma) \times \mathcal{P}_+ \eta'_+ \eta_+(\sigma') \sim \frac{1}{(z-z')^4} \eta_+ \eta_+$$

Counting: (but also terms $\sim (z-z')^{-2} \Rightarrow \alpha_0 = 1$)

$$\left( \frac{1}{z - z'} \right)^2 \eta' \eta' + 2 \frac{1}{(z-z')^3} \eta' \eta + \left( \frac{1}{z - z'} \right)^4 \eta \eta$$

Int. by parts: $2 \times 3 + 2 \times 3 + 1 = 13$

Cancels for $D = 2 \times 13 = 26$ !!

# From the world-sheet to space-time

In general, we may ask: which conditions should $G_{\mu\nu}$, $B_{\mu\nu}$, ... satisfy in order to keep $\hat{Q}^2=0$ ?

For general $G, B$ the theory is an interacting one ($G, B, ..$ being like generalized coupling constants) and the problem is not an easy one...

Something can be done, however, for slowly-varying fields (weak "coupling")

The possible sl.v. fields are the massless modes $G_{\mu\nu}$, $B_{\mu\nu}$ ... and $\phi$ w/

$[\phi]=\ell^0$

$$S = -\frac{1}{2}\int d^2\xi \left\{ (-\gamma)^{1/2}\left[\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu} - \frac{1}{4\pi}{}^{(2)}R(\gamma)\phi(x)\right] + \epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}\right\}$$

The new (Fradkin-Tseytlin) term is very interesting

It becomes a surface term if $\phi(x)=\phi_0=\text{const}$

since $\frac{1}{4\pi}\int d^2\xi\,(-\gamma)^{1/2}R(\gamma) = 2(1-g)$ ($g$ = genus of R-surface)

$$\Rightarrow \int_g e^{-S} \approx e^{-(1-g)\phi} \times (\text{function of } \nabla\phi) \Rightarrow e^{\phi} \text{ counts string loops!}$$

- Consider then the lowest order in the loop expansion (sphere topology, $g=0$)

- For "slowly"-varying (to be defined below) $G_{\mu\nu}$, $B_{\mu\nu}$, $\phi$ the conditions $\hat{Q}^2=0$ coincide with the equs. of motion that follow from the space-time action:

$$\Gamma_{eff} = \int d^{10}x \sqrt{-G}\, e^{-\phi}\left[R(G) + \partial_\mu\phi\,\partial_\nu\phi\, G^{\mu\nu} + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho}\right]$$

where $H_{\mu\nu\rho} = \left[\partial_\mu B_{\nu\rho} + \text{cyclic}\right]$

& indices are raised & lowered through $G_{\mu\nu}$

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Note that:
- $\Gamma_{eff}$ is automatically dimensionless without having to introduce any dimensionful constant (e.g. $G_N$)
- Dimensionless constants are fixed
- $\Gamma_{eff}$ is general-covariant in space-time & is also inv. under $B_{\mu\nu} \to \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$
- $\Gamma_{eff}$ is rescaled under $\phi \to \phi + \text{const}$.

Q: Can we understand the origin of those local space-time symmetries? After all we did not impose them!

A: One approach is based on can. transf.$^{\text{ons}}$ on the w. sheet

Two examples of C.T. $(X,P) \to (\tilde{X}, \tilde{P})$

1) $X^\mu \to X^\mu + \xi^\mu(x)$ , $P_\mu \to P_\mu - \xi^\nu{}_{,\mu} P_\nu$

2) $X^\mu \to X^\mu$ , $P_\mu \to P_\mu + X'^\nu (\hat{\xi}_{\mu,\nu} - \tilde{\xi}_{\nu,\mu})$

Both are classical can. transf.$^{\text{ns}}$. Formally,

$$\int dX^\mu dP_\mu \cdots \exp\left( i \int P\dot{X} - H(P_\mu, X^\mu; G, B, \phi) \right) \equiv Z(G,B,\phi)$$

$$= \int d\tilde{X}^\mu d\tilde{P}_\mu \exp\left( i \int \tilde{P}\dot{\tilde{X}} - H(\tilde{P}, \tilde{X}; \tilde{G}, \tilde{B}, \tilde{\phi}) \right) \equiv Z(\tilde{G}, \tilde{B}, \tilde{\phi})$$

In ON gauge:

$$2H = P_\mu G^{\mu\nu} P_\nu + X'^\mu (G - BG^{-1}B)_{\mu\nu} X'^\nu + 2X'^\mu (BG^{-1})_\mu{}^\nu P_\nu$$

Under 1): $\tilde{G}_{\mu\nu} = G_{\mu\nu} + \xi^\rho{}_{,\mu} G_{\rho\nu} + \xi^\rho{}_{,\nu} G_{\rho\mu} + \xi^\rho G_{\mu\nu,\rho}$ etc

Under 2): $\tilde{G}_{\mu\nu} = G_{\mu\nu}$ , $\tilde{B}_{\mu\nu} = B_{\mu\nu} + (\xi_{\mu,\nu} - \xi_{\nu,\mu})$ , $\phi \to \phi$

We have "proven" inv. under G.C.T.'s 4 gauge transf.$^{\text{ons}}$ of $B_{\mu\nu}$ !! Too formal? Watch anomalies (Cf Fujikawa)

This is an unfinished program:

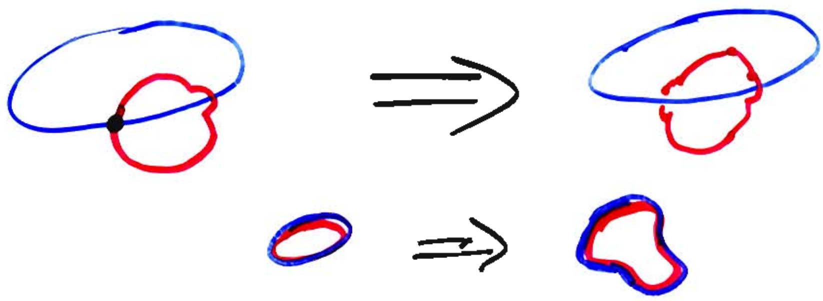
- Gauge invariance can be "proven" as well
- Stringy symmetries (e.g. T-dualities) also follow from C.T.'s (see next lecture)

..but

Group of all canonical transformations huge
Many of them do <u>not</u> become unitary transformation @ quantum level

Q: Which subgroup of C.T.'s survives quantization?
What is the symmetry group underlying string theory?
Could it include, for instance:

$$\delta X^\mu(\sigma,\tau) = \xi^\mu[X(\sigma,\tau)](\sigma,\tau) = \xi^\mu(X(\sigma,\tau))$$
$$+\,\gamma^{\alpha\beta}\,\partial_\alpha X^\rho\,\partial_\beta X^\sigma\,\xi^\mu_{\rho\sigma}(X) + \dots$$



A: ???

# 2-parameter expansion of $\Gamma_{eff}$



We have already seen one: the topological expansion w/ exp. parameter $e^{\phi}$:

tree — one loop

$$\Gamma_{eff} = \int dx \sqrt{-G}\, e^{-\phi} [\ldots \partial\phi \ldots] + \int dx \sqrt{-G} [\ldots \partial\phi] + \int dx \sqrt{-G}\, e^{\phi} [\ldots \partial\phi \ldots] + \ldots$$

two loops

Expansion valid (asympt.$^{ic}$) for $e^{\phi} \ll 1$ ($\phi \ll 0$)

What about the slowly-varying-field approx? Corrections have been computed (Tseytlin...) particularly at $g=0,1$ – At $g=0$ level they are local, gauge inv. w/ 4 or more derivatives:

$$\Gamma_{eff}^{(g=0)} = \int dx \sqrt{-G}\, e^{-\phi} \left[ R + (\partial\phi)^2 + H^2 + R^2 + R_{\mu\nu}^2 + R_{\mu\nu\rho\sigma}^2 + (\partial\phi)^2 R + (\partial\phi)^4 + \ldots \right]$$

There is no obvious expansion parameter! The dimensionless par. is $G^{\mu\nu}\partial_\mu\partial_\nu$ !

Actually, for $D < D_c$ (26, 10) there is another (leading) term in $\Gamma$: $\int dx \sqrt{-G}\, e^{-\phi} \Lambda$ w/ $\Lambda = \frac{D-D_c}{3}$

For $D \neq D_c$ $\Lambda$ is $O(1)$ & the derivative expansion breaks down! (no s.v. solutions!).

# Constants of Nature as V.E.V.'s

Q: If $\Gamma_{eff}$ has no free parameters & no dimensionful constants where do the constants of Nature come from?

A: They come as parameters in the solutions i.e. as vacuum parameters

Example of trivial Mink. vacuum in $D=D_c$:

$$\boxed{G_{\mu\nu} = \ell_s^{-2}\eta_{\mu\nu}\ ,\ \phi=\phi_0\ ,\ H_{\mu\nu\rho}=0}$$

- $\ell_s$ is arbitrary (but non-vanishing) and can be used to define (new) units of length
- $\phi_0$ is arbitrary at tree level but loops depend non-trivially from $\phi_0$
- Expansion in $G^{\mu\nu}\partial_\mu\partial_\nu$ becomes, near flat space-time, exp. in $\ell_s^2\partial^2 \equiv \alpha'\hbar\,\partial^2$. Low-E eff. action valid when $\partial \ll \ell_s^{-1}$
- Comparison w/ Einstein-Hilbert action $\times\hbar^{-1}$

$$\hbar^{-1}\Gamma^{(EH)} = \frac{1}{16\pi\,\ell_P^{D-2}}\int dx\sqrt{-G}\,R \implies \ell_P^{D-2} = e^{\phi_0}\,\ell_s^{D-2}$$

$\Rightarrow$ $\ell_P$, $G_N$ appear as phenomenological par's (cf. $G_F$ vs $M_{W,Z}$)

- At this point I should mention another virtue of $\Gamma_{eff}$: it can be used as a standard QFT eff. action to construct 1PI vertices & to compute the S-matrix: one recovers the low-E limit of DRM/ST amplitudes
- A well-known theorem says that the S-matrix is invariant under local redefinition of the fields
- One such local redef. can bring $\Gamma_{eff}$ to a form closer to EH's

$$G_{\mu\nu} = l_s^{-2}\, g_{\mu\nu}\, e^{\frac{2}{D-2}(\phi-\phi_0)}$$

$$\sqrt{-G}\, e^{-\phi} R(G) = l_s^{-(D-2)} e^{\frac{D}{D-2}(\phi-\phi_0)} \sqrt{-g}\, e^{-\phi} R(g)\, e^{-\frac{2}{D-2}(\phi-\phi_0)} + \partial\phi$$

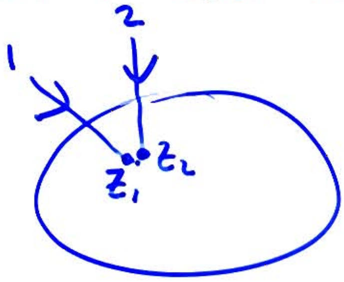
$$= l_s^{-(D-2)} e^{-\phi_0} \sqrt{-g}\left[R(g) + \partial\phi\cdot\right] = l_P^{-(D-2)} \sqrt{-g}\, R(g)$$

However, simplification only works for $\sqrt{-G}R$. Higher-der. terms become more complicated to show that the scale of new phys. is $l_s$ and not $l_P$

Also, for non-constant $\phi$, physics is $\neq$ from GR & one has a JBD effective theory w/ $\omega_{BD} = -1$

Can we understand this mysterious relation between conformal invariance on the w. sheet and eqns. of motion (S-matrix) in space-time?

Qualitatively it goes as follows:
Violations of C.I. on the w.s. come from the necessity of regularizing the 2-D CFT at short (w.s.) distances

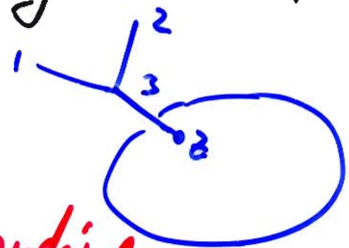



$$\sim \int dz_1\, dz_2\, V_1(z_1)\, V_2(z_2)$$

OPE : $V_1(z_1) V_2(z_2) \sim g_{123} (z_1 - z_2)^{-\delta} V_3(z)$, $e^{2\sigma(z)}$

Short-dist. cutoff depends on conf. factor $g_{z\bar{z}} \Delta z \Delta \bar{z} > \epsilon^2$

Same short dist. limit gives coupling of three particles $\sim g_{123}$




Full quantitative understanding is still absent

For some attempts see: A. Polyakov (book)
T. Kubota & G.V. Going from S-matrix functional to 1PI functional $\Gamma$ converts free equations for vertex operators into non l. eqns $\delta\Gamma / \delta\phi_i = 0$

One final remark is that, so far, we do not have a non-pert$^{ve}$ def. of string theory (say the equiv. of lattice QCD)

The 1st quantization approach looks tied up to a sum over surfaces of fixed, increasing genus corresponding to a loop expansion: can we do better?

I do not see why not! String field theory does not look to be the way!

For NP phenomena, QFT methods are used exp. for extended SUSY cases..

# T-duality for closed & open strings

## Closed strings

Flat space time but w/ compact subspace $T^n$: $X^i = X^i + 2\pi R^i$ $(i = 1, 2, \ldots n)$ $\left(n \leq d = D-1\right)$

Allowing arbitrary, constant $G_{ij}$, $B_{ij}$ we can take $R^i = R$

$$G_{\mu\nu} = \begin{array}{c} \left(\begin{array}{c|c} G_{ab} & 0 \\ \hline 0 & G_{ij} \end{array}\right) \begin{array}{l} D-n \\ n \end{array} \end{array}$$

Constraints:

$$\begin{cases} L_+ - L_- = P_\mu X'^\mu = Z^T \eta Z + \ldots \\ L_+ + L_- = Z^T M Z + \ldots \end{cases} = 2\mathcal{H}$$

where $Z_a = \begin{pmatrix} P_i \\ X'^i \end{pmatrix}$; $\eta_{ab} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$

$$M = \left(\begin{array}{c|c} G^{-1} & -G^{-1}B \\ \hline BG^{-1} & G - BG^{-1}B \end{array}\right); \quad \{Z_a(\sigma), Z_b(\sigma')\}_{PB} = \eta_{ab}\,\delta'(\sigma - \sigma')$$

$a, b = 1 \ldots 2n$

Canonical Transformation:

$$Z \to \tilde{Z} = \Omega^{-1} Z \quad \text{w/} \quad \Omega^T \eta \Omega = \eta \quad (\Omega \in O(n,n))$$

$L_+ - L_- \to$ same ; $L_+ + L_- \to L_+ + L_- \,(\tilde{M})$

w/ $\tilde{M} = \Omega^T M \Omega$ $\quad \left(H(Z, G, B) = H(\tilde{Z}, \tilde{G}, \tilde{M})\right)$

Arguing as for G.C.T. we would conclude that there is inv. under $M \to \tilde{M}$. (4.2)

Too naive! In general these C.T.'s do not become U.T.'s at quantum level!

Example (n=1) $\Omega = \left(\begin{array}{c|c} \alpha^{-2} & 0 \\ \hline 0 & \alpha^2 \end{array}\right)$ ; $G_{11} \neq 0$, $B = 0$

$\mathcal{H} = P^2 + X'^2 \Rightarrow \alpha^{-2} P^2 + \alpha^2 X'^2$

which has a different spectrum

In general $O(n,n)$ transformation changes the spectrum. Only the subgroup $O(n) \otimes O(n)$ w/ $\Omega^T M \Omega = M$ preserves the spectrum (Narain)

Moduli space is $O(n,n)/O(n) \otimes O(n) = G/H$

& $\dim(G/H) = n(2n-1) - n(n-1) = n^2$ $(= G, B)$

Yet, the canonical transf.$^{on}$ has mapped a CFT into another (generally inequivalent) one. There is, however, a non-trivial exception

# T-duality

$$\mathcal{H} = G^{ij} P_i P_j + X'^i X'^j G_{ij}$$

$$= n^2 G^{ii}/R^2 + m^2 G_{ii} \cdot R^2$$

is invariant under $G_{ii} R^2 \to \frac{1}{G_{ii} R^2}$

and $n \leftrightarrow m$

This is essentially $\Omega = \gamma$

(if we use units in which $R = 1$)

For $R=1$ it is $G \to G^{-1}$

For $G=1$ it is $R \to 1/R$

If we also turn on $B_{ij}$, T-duality becomes (at $R=1$)

$(G \pm B) \to (G \pm B)^{-1}$ i.e.

$$G \to (G - BG^{-1}B)^{-1}, \quad B \to (B - GB^{-1}G)^{-1}$$

This $\mathbb{Z}_2$ transformation can be extended to the discrete group

$$O(n,n;\mathbb{Z})$$

- Actually, all duality transformations must be accompanied by a shift of the dilaton
- Not too easy to understand from CFT point of view
- Easier from $\Gamma_{eff}$

$$\Gamma_{eff} = \int d^D x \sqrt{-G}\, e^{-\phi} \left[ \Lambda + R + \cdots \right]$$

$$\Longrightarrow \underbrace{\int d^n x \sqrt{-G^{(n)}}\, e^{-\phi}}_{\equiv e^{-\phi_{D-n}} \simeq 1/g^2_{eff,D-n}} \cdot \int d^{D-n} x \sqrt{-g} \cdots$$

When performing T-duality transformation we change $\int d^n x \sqrt{-G^{(n)}}$, & we should make a compensating shift in $\phi$ in order to keep $g_{eff, D-n}$ constant.

Modulo this subtelty, T-duality looks to be an exact symmetry (like a gauge symmetry). At fixed points under T enlarged gauge symmetries appear.

- There is an interesting extension of these ideas to the case of space-time dependent $G$ & $B$
- If $G_{ij}$, $B_{ij}$ $(i,j=1\ldots n)$ do not depend on $X^i$ but only on $X^\mu$ $(\mu=0,1\ldots d-n)$ then the C.T.'s we made go through and the $O(n,n;R)$ group still connects solutions to (generally inequivalent) solutions
- $\sqrt{G^{(n)}}\,e^{-\phi}$ should again be held fixed $(\phi\to\phi+\cdots)$
- $O(n,n;R)$ now acts non-trivially even if $n$-dim. subspace is not compact

Example of Scale-factor-duality in string cosmology:

$$\begin{cases} ds^2 = -dt^2 + a^2(t)\,d\vec{x}^2 \longrightarrow -dt^2 + a^{-2}(t)\,d\vec{x}^2 \\ \phi \to \phi + 2n\ln a \quad \text{gives a new cosmology} \end{cases}$$

inequivalent

(See Lectures 5 & 6)

# OPEN STRINGS



More subtle, more interesting

- At first sight there is no analog of winding for open strings, hence no $R \to 1/R$ duality
- Puzzle: for $R \to 0$ open strings move in one less dim.$^{\text{on}}$ than closed strings (for them $R \to 0$ same as $R \to \infty$)
- If the same "stuff" makes open & closed strings it must be only the ends of open strings that live in the subspace... .. but that is not so w/ Neumann b.c. !?
- Lets go back to Polyakov's action:

$$S = \frac{1}{2} \int d\tau \int_0^\pi d\sigma \sqrt{-\gamma} \left[ \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu} \right]$$

$$\delta S = \delta S_{\text{Bulk}} + \int d\tau \int_0^\pi d\sigma \, \partial_\beta \left[ \sqrt{-\gamma} \gamma^{\alpha\beta} \partial_\alpha X^\mu \delta X^\nu G_{\mu\nu} \right]$$

0 on e.o.m.

$$= \int d\tau \left[ \sqrt{-\gamma} \gamma^{1\alpha} \partial_\alpha X^\mu \delta X^\nu G_{\mu\nu} \right] \Big|_0^\pi$$

$$\left( G_{\mu\nu} \partial' X^\mu \right) \cdot \left( \delta X^\nu \right) = 0 \quad \text{at } \sigma = 0, \pi$$

($\sigma = 0, \pi$ contributions may cancel $\Rightarrow$ closed string!)

$$(\partial' X)_\mu \, \delta X^\mu = 0 \text{, at } \sigma = 0, \pi$$

If $\delta X^\mu$ is arbitrary there is only one solution: $\partial' X^\mu = 0 \;\; \forall \mu \Rightarrow$ N.B.C.

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However we may try to constrain $\delta X^\mu$ by forcing the ends of the open string to stay on a lower dimensional subspace, i.e. on a p-brane. Two equiv. definitions:

$$X^\mu = X^\mu(\xi^0, \xi^1, \ldots \xi^p) \quad \text{or} \quad \left(\begin{array}{c} p+1 \text{ dim} \\ \text{world-volume} \end{array}\right)$$

$$\phi_a(X^\mu) = 0 \;, \quad a = 1, 2 \ldots d-p$$

If $X^\mu(\sigma = 0, \pi)$ is stuck on the brane

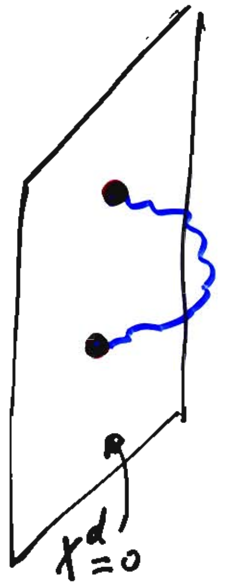
$$\partial_\mu \phi_a \, \delta X^\mu = 0 \quad \forall a$$

But then instead of $\partial' X^\mu = 0 \;\; \forall \mu$ we just need

$$(\partial' X)_\mu = \sum_a c_a \, \phi_{a,\mu}$$

E.g. $\phi_a = X^d$; $\quad X'^\mu = 0$ for $\mu \neq d$ (N.B.C)

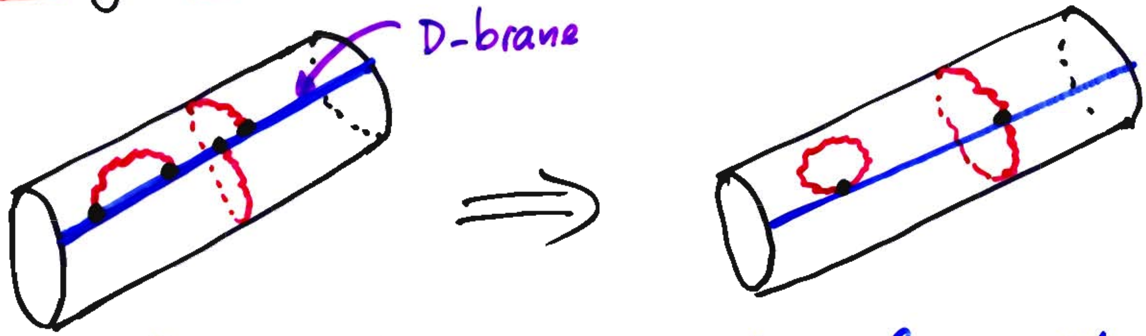
$$\dot{X}^d = 0 \quad \text{(D.B.C.)}$$




Branes on which open strings are forced to end are called D-branes

This does already smell of duality. Indeed, we can go from N.B.C. to D.B.C. by the same canonical transf. ($X' \longleftrightarrow P$) that we used for the closed string
(try! Not completely trivial!)

D-strings can wind and convert in winding closed strings




... the closed string may then leave the brane...

- D branes become dynamical objects carrying energy (tension) & charges
- Since gauge quantum numbers lie at the end of open strings (Chan-Paton factors) there are gauge fields stuck on D-branes ⇒ the brane world!