Lattice QCD with Ginsparg–Wilson fermions

Leonardo Giusti

CERN - Theory Group

Outline

- QCD with an exact chiral symmetry on the lattice
- Ward Identities and chiral anomaly
- The Witten–Veneziano mechanism to solve the $\mathrm{U}(1)_{\mathrm{A}}$ problem
- Non-perturbative definition of the topological susceptibility
- Algorithm for zero-mode counting
- Lattice computation and analysis of systematics

- $K \to \pi\pi$ with Ginsparg–Wilson fermions
- The $\Delta S = 1$ effective Hamiltonian
- Numerical results

- Summary

- Octet compatible with SSB pattern

$$SU(3)_{\rm L} \times SU(3)_{\rm R} \to SU(3)_{\rm L+R}$$

  and soft explicit symmetry breaking

$$m_u, m_d \ll m_s < \Lambda_{\rm QCD}$$

- $m_u, m_d \ll m_s \Longrightarrow m_\pi \ll m_{\rm K}$

- A $9^{\rm th}$ pseudoscalar with $m_{\eta'} \sim \mathcal{O}(\Lambda_{\rm QCD})$

I	I_3	S	Meson	Quark Content	Mass (MeV)
1	1	0	π^+	$u\bar{d}$	140
1	-1	0	π^-	$d\bar{u}$	140
1	0	0	π^0	$(d\bar{d} - u\bar{u})/\sqrt{2}$	135
$\frac{1}{2}$	$\frac{1}{2}$	+1	K^+	$u\bar{s}$	494
$\frac{1}{2}$	-$\frac{1}{2}$	+1	K^0	$d\bar{s}$	498
$\frac{1}{2}$	-$\frac{1}{2}$	-1	K^-	$s\bar{u}$	494
$\frac{1}{2}$	$\frac{1}{2}$	-1	$\overline{\mathrm{K}}^0$	$s\bar{d}$	498
0	0	0	η	$\cos\vartheta\eta_8 + \sin\vartheta\eta_0$	547
0	0	0	η'	$-\sin\vartheta\eta_0 + \cos\vartheta\eta_8$	958

$$\eta_8 = (d\bar{d} + u\bar{u} - 2s\bar{s})/\sqrt{6}$$
$$\eta_0 = (d\bar{d} + u\bar{u} + s\bar{s})/\sqrt{3}$$
$$\vartheta \simeq -11^\circ$$

Quantum Chromo Dynamics (QCD)

- The Euclidean QCD Lagrangian inv. under $SU(3)$ color gauge group (formal level)

$$S_{\rm QCD} = \int d^4x \left\{ -\frac{1}{2g^2} \text{Tr}\left[F_{\mu\nu}F_{\mu\nu}\right] + i\frac{\theta}{16\pi^2}\text{Tr}\left[F_{\mu\nu}\tilde{F}_{\mu\nu}\right] + \bar\psi\left[D+M\right]\psi \right\}$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \qquad \tilde{F}_{\mu\nu} \equiv \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma} \qquad A_\mu = A_\mu^a\, \mathcal{T}^a$$

$$D = \gamma_\mu\{\partial_\mu + A_\mu\} \qquad \psi \equiv \{q_1, \dots, q_{N_{\rm f}}\} \qquad M \equiv \text{diag}\{m_1, \dots, m_{N_{\rm f}}\}$$

- For $M = 0$ the action is invariant under the global group $U(N_{\rm f})_{\rm L} \times U(N_{\rm f})_{\rm R}$

$$\psi_L \to V_L\psi_L \qquad \bar\psi_L \to \bar\psi_L V_L^\dagger \qquad \psi_{L,R} = P_\pm \psi$$

$$\psi_R \to V_R\psi_R \qquad \bar\psi_R \to \bar\psi_R V_R^\dagger \qquad P_\pm = \frac{1\pm\gamma_5}{2}$$

- For the rest of this lecture we will assume that heavy quarks have been integrated out and we will focus on the symmetry group $U(3)_{\rm L} \times U(3)_{\rm R}$

- When the theory is quantized the chiral anomaly breaks explicitly the subgroup $U(1)_A$

- The non-singlet AWIs in the chiral limit

$$\partial_\mu \left\langle A^a_\mu(x) P^a(0) \right\rangle = -\frac{1}{3}\delta(x) \left\langle \bar\psi\psi \right\rangle$$

imply that if $\langle\bar\psi\psi\rangle \neq 0$

$$\langle \partial_\mu A^a_\mu(x) P^a(0) \rangle = -\frac{3}{4\pi^2} \langle\bar\psi\psi\rangle \frac{x_\mu}{(x^2)^2} \qquad x \neq 0$$

i.e. the current-density correlation function is long-ranged

- The energy spectrum does not have a gap and the correlation function has a particle pole at zero momentum (Goldstone theorem)

- In the chiral limit $\langle\bar\psi\psi\rangle \neq 0$ implies the presence of 8 Goldstone bosons identified with the 8 pseudoscalar light mesons $[\pi, \ldots, , K, \ldots, \eta]$

The breaking of the $U(1)_A$ symmetry

- In the chiral limit the singlet AWI reads

$$\langle \partial_\mu A^0_\mu(x) P^0(0)\rangle = 2N_f \langle Q(x) P^0(0)\rangle - 2\,\delta(x)\langle\bar\psi\psi\rangle$$

with

$$Q(x) = -\frac{g^2}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\,[F_{\mu\nu}(x)F_{\rho\sigma}(x)]$$

- In perturbation theory

$$Q(x) = \partial_\mu K_\mu(x)$$

- If the argument valid non-perturbatively the same argumentation would lead to a SSB pattern $U(3)_L \times U(3)_R \to U(3)_{R+L}$, with a 9^{th} pseudo GB in QCD

$$m_{\eta'-\eta} < \sqrt{3}\, m_\pi$$

[Weinberg 75]

Light pseudoscalar meson spectrum

- Octet compatible with SSB pattern

$$SU(3)_{\rm L} \times SU(3)_{\rm R} \rightarrow SU(3)_{\rm L+R}$$

  and soft explicit symmetry breaking

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The Adler–Bell–Jackiw $U(1)_A$ anomaly does play a rôle

- Despite of $Q(x) = \partial_\mu K_\mu(x)$, for topologically non-trivial configurations [Belavin et al. 75]

$$\int d^4x\, Q(x) = Q \neq 0$$

- A semiclassical approximation shows that instantons of radius R contribute $\propto \exp\{-8\pi^2/g^2(1/R)\}$ ['t Hooft 76, Jackiw Rebbi 76, Callan Dashen Gross 76]

- No IR bound on the radius of instantons: a non-perturbative approach needed

- Witten–Veneziano mechanism: The anomaly gives a mass to the η' boson thanks to the non-perturbative quantum fluctuations of the topological charge

Ginsparg–Wilson fermions

- The "mildest way" of breaking standard chiral symmetry [Ginsparg Wilson 82]

$$\gamma_5 D + D\gamma_5 = \bar{a} D \gamma_5 D$$

- An exact symmetry at finite cut-off implied [Lüscher 98]

$$\delta q = \epsilon \hat{\gamma}_5 q \qquad \delta \bar{q} = \epsilon \bar{q} \gamma_5 \qquad \hat{\gamma}_5 = \gamma_5 (1 - \bar{a} D)$$

- $U(1)_{\mathrm{A}}$ anomaly from the Jacobian

$$J = \exp\{\epsilon \bar{a} \sum_x \mathrm{Tr}\, [\gamma_5 D(x,x)]\}$$

- The topological charge density defined as [Neuberger 97, Hasenfratz et al. 98, Lüscher 98]

$$a^4 Q(x) = -\frac{\bar{a}}{2} \mathrm{Tr}[\gamma_5 D(x,x)] \qquad n_+ - n_- = \mathrm{index}(D) = \sum_x Q(x)$$

and for smooth gauge configurations $Q(x) \to -\frac{g^2}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\, [F_{\mu\nu}(x) F_{\rho\sigma}(x)]$

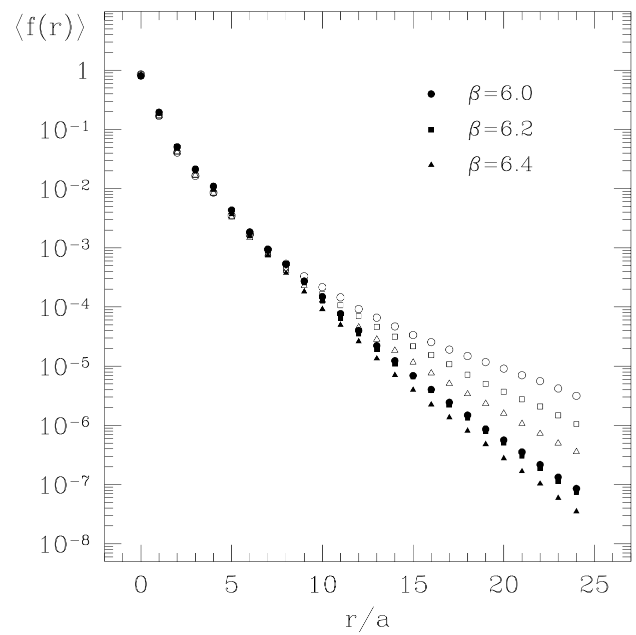
- After 15 years from the GW relation, a Dirac operator that satisfies the GW relation, is local and leads to the correct continuum limit was found [Neuberger 97]

$$D = \frac{1}{\bar{a}}\left(1 + \frac{X}{\sqrt{X^\dagger X}}\right)$$

with

$$X = D_W - 1/\bar{a} \qquad \bar{a} = a/(1+s)$$

Hernández, Jansen, Lüscher 99



$$f(r) = \max\left\{ \|\frac{X}{\sqrt{X^\dagger X}}(x,y)\| \;\Big|\; \|x-y\| = r \right\}$$

$$\langle f(r)\rangle \propto e^{-\mu r/a} \qquad r/a \gg 1$$

- A family of GW regularizations for $0 < s < 2$

- Numerical treatment challenging

- In the chiral limit for a given string of renormalized fundamental fields $\hat{O}$

$$\langle \partial^*_\mu A^0_\mu(x) \hat{O} \rangle = 2N_{\rm f} \langle Q(x) \hat{O} \rangle - \langle \delta^x_A \hat{O} \rangle$$

- Neither $Q(x)$ or $A^0_\mu(x)$ are finite operators:
  1. $\delta^x_A \hat{O}$ is finite
  2. $A^0_\mu(x)$ is multiplicatively renormalizable
  3. $Q(x)$ can mix with $d \leq 4$ operators $\Longrightarrow$ only with $\partial^*_\mu A^0_\mu(x)$

$$\hat{Q}(x) = Q(x) - \frac{Z}{2N_{\rm f}} \partial^*_\mu A^0_\mu(x) \qquad \hat{A}^0_\mu(x) = (1-Z) A^0_\mu(x)$$

Singlet axial Ward identities in the chiral limit

- In the chiral limit for a given string of renormalized fundamental fields $\hat{O}$

$$\langle \partial^*_\mu A^0_\mu(x) \hat{O} \rangle = 2N_\mathrm{f} \langle Q(x) \hat{O} \rangle - \langle \delta^x_A \hat{O} \rangle$$

- Distinctive features of GW fermions [L. G., Rossi, Testa, Veneziano 01]

1. No further multiplicative renormalization required for $\hat{Q}(x)$
2. No mixing of $Q(x)$ with $P^0(x)$
3. No extra contact terms in the AWIs

$$\hat{Q}(x) = Q(x) - \frac{Z}{2N_\mathrm{f}} \partial^*_\mu A^0_\mu(x) \qquad \hat{A}^0_\mu(x) = (1 - Z) A^0_\mu(x)$$

Topological susceptibility with GW fermions in the chiral limit [L. G., Rossi, Testa, Veneziano 01]

- The renormalized AWIs read

$$\langle \partial^*_\mu \hat{A}^0_\mu(x) \hat{O} \rangle = 2N_{\rm f} \langle \hat{Q}(x) \hat{O} \rangle - \langle \delta^x_A \hat{O} \rangle$$

- Taking $\hat{O} = \hat{Q}$

$$\begin{aligned} \chi(p) &= \sum_x {\rm e}^{-ipx} \langle \hat{Q}(x) \hat{Q}(0) \rangle + {\rm CT}(p) \\ &= \frac{1}{2N_{\rm f}} \sum_x {\rm e}^{-ipx} \langle \partial^*_\mu \hat{A}^0_\mu(x) \hat{Q}(0) \rangle + {\rm CT}(p) \end{aligned}$$

in the full theory the requirement is

$$\chi(0) = 0$$

- ${\rm CT}(p)$ is a $2^{\rm nd}$ order polynomial in p^2 (counter-terms) to make the int. corr. fnc. finite
- For GW fermions, the AWI $\Longrightarrow$ ${\rm CT}(0) = 0$

The Witten–Veneziano mechanism [Witten 79, Veneziano 79]

- Under the “hypothesis” that the limit $N_\mathrm{f}/N_\mathrm{c} \to 0$ is smooth

$$\frac{F_\pi^2 m_{\eta'}^2}{2N_\mathrm{f}}\Bigg|^{M=0}_{\frac{N_\mathrm{f}}{N_\mathrm{c}}=0} = \chi^{\mathrm{YM}}$$

- Note that in the limit $N_\mathrm{f}/N_\mathrm{c} \to 0$:
  1. $\mathrm{U}(1)_\mathrm{A}$ is restored
  2. η' is a Nambu–Goldstone boson $\Longrightarrow m_{\eta'} = 0$
  3. At first order in $N_\mathrm{f}/N_\mathrm{c}$, $m_{\eta'}^2 = \mathcal{O}(N_\mathrm{f}/N_\mathrm{c})$

- For the topological charge operator suggested by Ginsparg–Wilson fermions

$$\frac{F_\pi^2 m_{\eta'}^2}{2N_\mathrm{f}}\Bigg|^{M=0}_{\frac{N_\mathrm{f}}{N_\mathrm{c}}=0} = \lim_{\substack{V\to\infty \\ a\to 0}} \frac{1}{V}\Big\langle (n_+ - n_-)^2 \Big\rangle^{\mathrm{YM}}$$

Numerical challenge

- A Monte Carlo computation of

$$\chi_{\rm L}^{\rm YM} = \frac{1}{V}\left\langle (n_+ - n_-)^2 \right\rangle^{\rm YM}$$

is challenging for several reasons

- $L \sim 1$ fm and $a \sim 0.08$ fm $\Longrightarrow$ $\dim[D] \sim 2.5\ 10^5$: computing and diagonalizing the full matrix not feasible

- A standard minimization would require high precision to beat contamination from quasi-zero modes

- At large V the probability distribution has a width which increases linearly with V

$$P_Q = \frac{1}{\sqrt{2\pi V \chi_{\rm L}^{\rm YM}}} e^{-\frac{Q^2}{2V\chi_{\rm L}^{\rm YM}}} \{1 + O(V^{-1})\}$$

$\Longrightarrow$ computing $\chi_{\rm L}^{\rm YM}$ requires very high statistics

Algorithm for zero-mode counting [L. G., Hoelbling, Lüscher and Wittig 02]

- In finite V null prob. for $n_+ \neq 0$ and $n_- \neq 0$

- Simultaneous minimization of Ritz functionals associated to

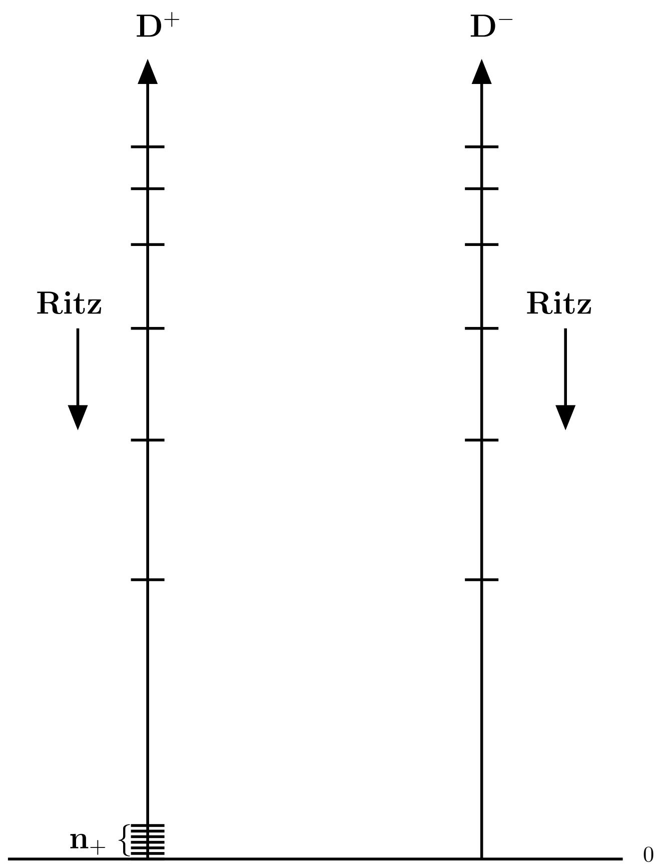
$$D^{\pm} = P_{\pm} D P_{\pm} \qquad P_{\pm} = \frac{1 \pm \gamma_5}{2}$$

  to find the gap in one of the sectors

- Run again the minimization in the sector without gap and count zero modes

- No contaminaton from quasi-zero modes

- Adaptive precision for computing D during the minimization [see also L. G., Hoelbling, Rebbi 01]



Monte Carlo computation of χ_L^{YM} [L.G., Lüscher, Weisz, Wittig 03; Del Debbio, Pica 03; Del Debbio, L. G., Pica 04]

$s = 0.4$

β	L/a	L [fm]	N_{conf}	$\langle Q^2 \rangle$	$r_0^4 \chi_L^{YM}$
6.0	12	1.12	2452	1.63(5)	0.065(2)
6.1791	16	1.12	1138	1.59(8)	0.063(3)
5.8989	10	1.12	1460	1.74(7)	0.070(3)
6.0938	14	1.12	1405	1.54(6)	0.062(3)
5.8458	12	1.49	2918	5.6(2)	0.072(2)
6.0	16	1.49	1001	5.6(3)	0.071(4)
6.1366	20	1.49	963	4.8(2)	0.060(3)
5.9249	14	1.49	1284	5.6(2)	0.071(3)
5.8784	16	1.86	1109	15.0(7)	0.078(4)
6.0	20	1.86	931	13(1)	0.066(5)
6.0	14	1.30	1577	3.0(1)	0.065(3)

$s = 0.0$

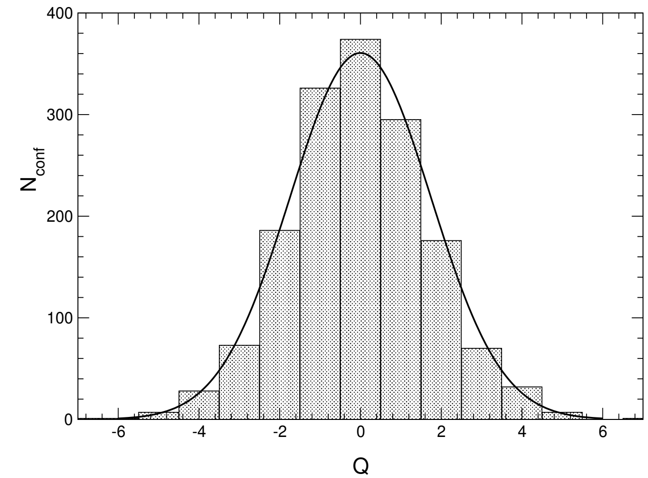
β	L/a	L[fm]	N_{conf}	$\langle Q^2 \rangle$	$r_0^4 \chi_L^{YM}$
5.9	12	1.34	1349	2.8(1)	0.054(2)
5.95	12	1.22	1291	1.96(8)	0.055(2)
6.0	12	1.12	3586	1.49(4)	0.060(2)
6.1	16	1.26	962	2.5(1)	0.060(3)
6.2	18	1.22	1721	2.11(8)	0.059(2)

- r_0 is a lattice reference scale of ≈ 0.5 fm, $\beta = 6/g^2$
- To keep stat. err. under control $N_{conf} \gtrsim 1000 \Longrightarrow \Delta\chi_L^{YM}/\chi_L^{YM} \lesssim 5\%$ for every lattice
- To keep systematic errors under control:
  1. Finite volume corrections: $L > 1$ fm
  2. Finite lattice spacing effects: $a = 0.068 \div 0.124$ fm, two values of s

Volume dependence [L.G., Lüscher, Weisz, Wittig 03]

- Probability distribution at large volume

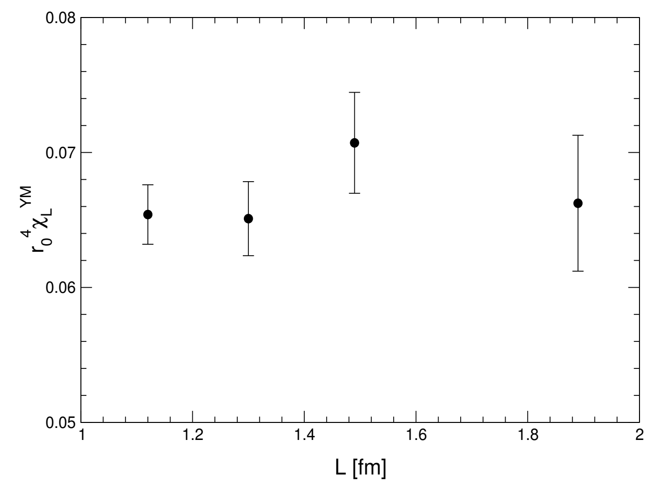
$$P_Q = \frac{1}{\sqrt{2\pi\langle Q^2\rangle}} e^{-\frac{Q^2}{2\langle Q^2\rangle}} \{1 + O(V^{-1})\}$$



- Mass gap in the pure gauge theory $m_g \sim 1.5$ GeV

- χ_L^{YM} goes to the infinite-volume limit as $e^{-m_g L}$

- For $L \gtrsim 1$ fm, χ_L^{YM} is indep. of L within stat. errors



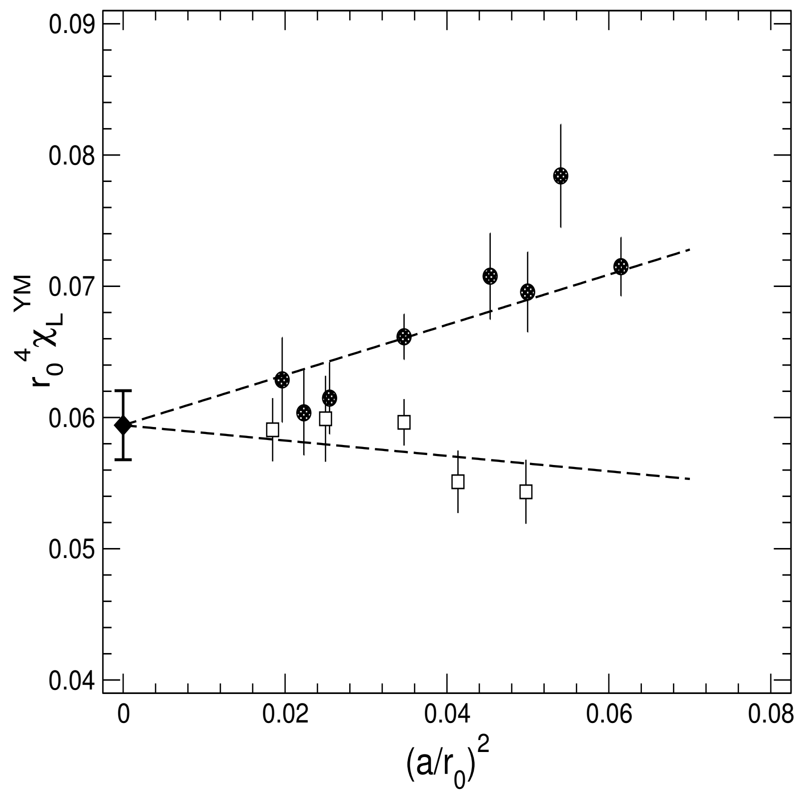
Continuum extrapolation [Del Debbio, L. G., Pica 04]

- Combined fit of the form [$\chi^2_{\rm dof} = 0.73$]

$$r_0^4 \chi_{\rm L}^{\rm YM}(s) = r_0^4 \chi^{\rm YM} + c_1(s) \frac{a^2}{r_0^2}$$

gives

$$r_0^4 \chi^{\rm YM} = 0.059 \pm 0.003$$



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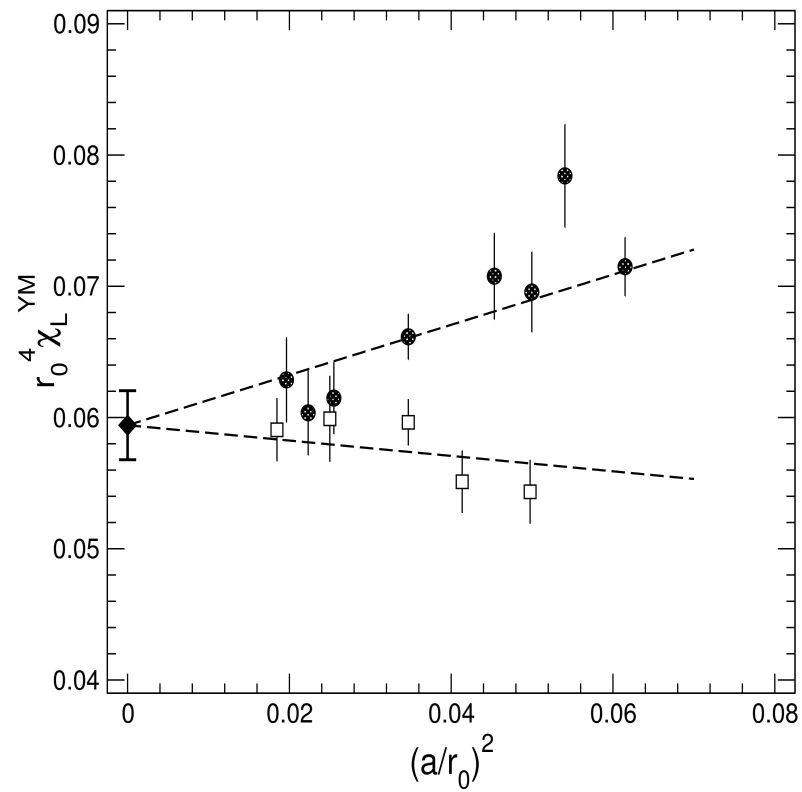
- By setting the scale $F_{\rm K} = 113(1)$ MeV

$$\boxed{\chi^{\rm YM} = (191 \pm 5 \text{ MeV})^4}$$



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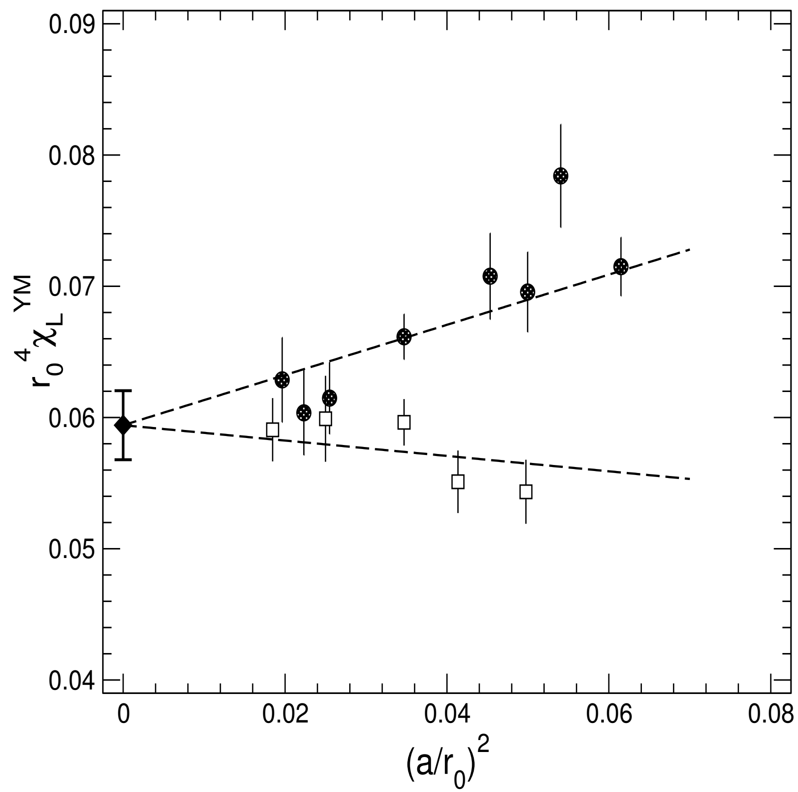

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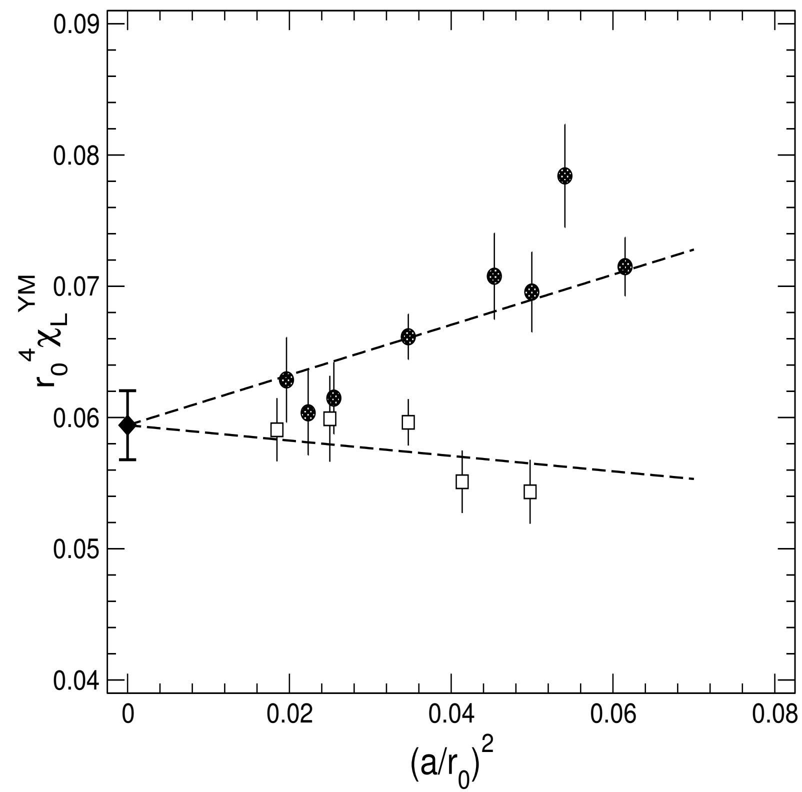

- The (leading) QCD anomalous contribution to $m_{\eta'}^2$, explains the bulk of its large experimental value as conjectured by Witten and Veneziano

Summary for the $U(1)_A$ problem

- A precise and unambiguous implementation of the Witten–Veneziano formula can be derived at the non-perturbative level in QCD
- Ultraviolet power-divergent subtractions fixed (avoided) without ambiguities
- Under the "smooth-quenching hypothesis", formula can be derived from dispersion relation (no reference to large N_c)
- With Ginsparg–Wilson fermions

$$\left.\frac{F_\pi^2 m_{\eta'}^2}{2N_f}\right|_{\frac{N_f}{N_c}=0}^{M=0} = \lim_{\substack{V\to\infty \\ a\to 0}} \frac{1}{V}\left\langle (n_+ - n_-)^2 \right\rangle^{\rm YM}$$

- A Monte Carlo non-perturbative computation with the Neuberger operator gives

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- The (leading) QCD anomalous contribution to $m_{\eta'}^2$ explains the bulk of its large experimental value as conjectured by Witten and Veneziano

- $K \to \pi\pi$ amplitudes can be parameterized [CP conservation implies $A_I = A_I^*$]

$$
\begin{aligned}
-i\,T[K^+ \to \pi^+\pi^0] &= \tfrac{1}{2}\sqrt{3}A_2 e^{i\delta_2} \\
-i\,T[K^0 \to \pi^+\pi^-] &= \sqrt{\tfrac{1}{3}}A_0 e^{i\delta_0} + \sqrt{\tfrac{1}{6}}A_2 e^{i\delta_2} \\
-i\,T[K^0 \to \pi^0\pi^0] &= -\sqrt{\tfrac{1}{3}}A_0 e^{i\delta_0} + \sqrt{\tfrac{2}{3}}A_2 e^{i\delta_2}
\end{aligned}
$$

$$
-iT[K^0 \to (\pi\pi)_I] = A_I e^{i\delta_I} \qquad T[(\pi\pi)_I \to (\pi\pi)_I]_{l=0} = 2\,e^{i\delta_I}\sin\delta_I
$$

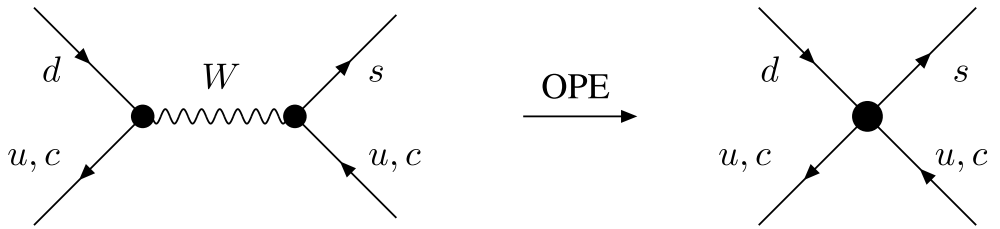
- Experimental results indicate the so-called the $\Delta \mathrm{I} = 1/2$ rule

$$
\left|\frac{A_0}{A_2}\right| \simeq 22.1
$$

- Why interesting? Because in this ratio the interplay between strong and electro-weak interactions of the Standard Model gives an opportunity for a refined test of NP QCD
- More than 30 years without a reliable computation from first principles (!)

The $H_{\rm eff}^{\Delta S=1}$ with an active charm

- By using the Operator Product Expansion



$$iA_I e^{i\delta_I} = \langle (\pi\pi)_I \mid H_{\rm eff}^{\Delta S=1} \mid K^0 \rangle$$

- The CP-conserving $\Delta S = 1$ eff. Hamiltonian is [Gaillard, Lee 74; Altarelli, Maiani 74]

$$H_{\rm eff}^{\Delta S=1} = \sqrt{2} G_F V_{ud} V_{us}^* \Big\{ \sum_{\sigma=\pm} k_1^\sigma Q_1^\sigma + k_2 Q_2 \Big\}$$

$$Q_1^\pm = \Big[(\bar{s}\gamma_\mu P_- u)(\bar{u}\gamma_\mu P_- d) \pm (\bar{s}\gamma_\mu P_- d)(\bar{u}\gamma_\mu P_- u) \Big] - \Big[u \to c \Big]$$

$$Q_2 = (m_u^2 - m_c^2) \Big[m_d (\bar{s} P_+ d) + m_s (\bar{s} P_- d) \Big]$$

- For $m_s \pm m_d \neq 0$

$$\bar{s}P_{\pm}d = \partial_\mu \left[\frac{1}{m_s - m_d} \bar{s}\gamma_\mu d \pm \frac{1}{m_s + m_d} \bar{s}\gamma_\mu \gamma_5 d \right]$$

  and it does not contribute in MEs which preserve four-momentum

- In physical matrix elements

$$H_{\rm eff}^{\Delta S=1} = \sqrt{2} G_F V_{ud} V_{us}^* \left\{ k_1^+ Q_1^+ + k_1^- Q_1^- \right\}$$

- The Wilson coefficients are known at NLO in α_s [Buras et al. 92; Ciuchini et al. 94]

- A non-perturbative determination of the matrix elements $\langle (\pi\pi)_I | \widehat{Q}_1^{\pm} | K^0 \rangle$ of the properly renormalized operators is needed

$K \to \pi\pi$ at leading order in ChPT in the $SU(4)$ limit

- If we define the matrix elements

$$k_1^\pm \langle \pi^+ | \hat{Q}_1^\pm | K^+ \rangle = -\frac{F^2}{2} M_K^2 g_1^\pm$$

at leading order in ChPT (and for simplicity in the unphysical $SU(4)$ limit)

$$\begin{aligned} \langle \pi^+ \pi^- | \hat{Q}_1^\pm | K^0 \rangle &= \frac{iF}{2\sqrt{2}} \left\{ M_K^2 - M_\pi^2 \right\} g_1^\pm \\ \langle \pi^0 \pi^0 | \hat{Q}_1^\pm | K^0 \rangle &= \pm \frac{iF}{2\sqrt{2}} \left\{ M_K^2 - M_\pi^2 \right\} g_1^\pm \end{aligned}$$

- At this order in ChPT

$$\left| \frac{A_0}{A_2} \right| = \frac{1}{2\sqrt{2}} \left\{ 1 + 3 \frac{g_1^-}{g_1^+} \right\}$$

and if compared with experimental results $g_1^- / g_1^+ \sim 20.5$

- An order of magnitude estimate would suggest $g_1^- \sim g_1^+$ (!)

Renormalization pattern for $Q_1^{\pm}$ with Ginsparg–Wilson fermions

- We want to renormalize the operators

$$Q_1^{\pm} = \Big[(\bar{s}\gamma_\mu P_- u)(\bar{u}\gamma_\mu P_- d) \pm (\bar{s}\gamma_\mu P_- d)(\bar{u}\gamma_\mu P_- u)\Big] - \Big[u \to c\Big]$$

- To select operators with $d \leq 6$:

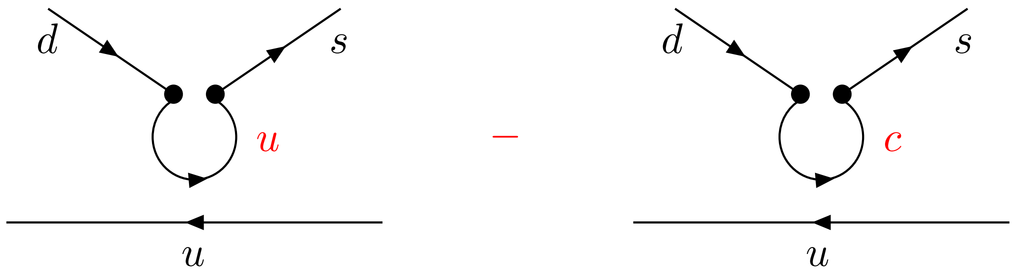
  - Flavour symmetry
  - P, C symmetries
  - Chiral symmetry

- On shell one operator is left (quadratic GIM mechanism)

$$Q_2 = (m_u^2 - m_c^2)\Big[m_d(\bar{s}P_+\tilde{d}) + m_s(\bar{s}P_-\tilde{d})\Big]$$

- No power divergent subtractions are needed with GW fermions

$$\widehat{Q}_1^{\pm} = Z_1^{\pm}\Big\{Q_1^{\pm} + z^{\pm} Q_2\Big\}$$



- Parity-odd and parity-even components renormalize differently
- For parity conserving sector, using flavour and CPS [Maiani et al. 87; Dawson et al. 97]

$$
\begin{aligned}
[\widehat{Q}_1^{\pm}]^{\mathrm{PC}} &= \mathcal{Z}_1^{\pm}[\widetilde{\mathcal{Q}}_1^{\pm}]^{\mathrm{PC}} \\
[\widetilde{Q}_1^{\pm}]^{\mathrm{PC}} &= [Q_1^{\pm}]^{\mathrm{PC}} + \sum_j b_j^{\pm} O_j^{\pm} + z_\tau^{\pm} Q_\tau + \frac{z_s^{\pm}}{a^2} Q_s
\end{aligned}
$$

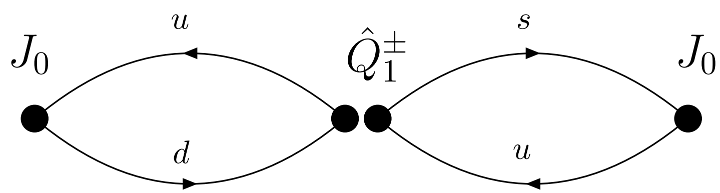
where

$$
Q_\tau = (m_u - m_c)\bar{s}\sigma_{\mu\nu}F_{\mu\nu}d \qquad\qquad Q_s = (m_u - m_c)\bar{s}d
$$

and $O_j^{\pm}$ are 4-fermion operators with wrong chirality

- With a broken chirality the GIM mechanism is only linear

Numerical computation of $\bar{g}_1^{\pm}$ in quenched QCD



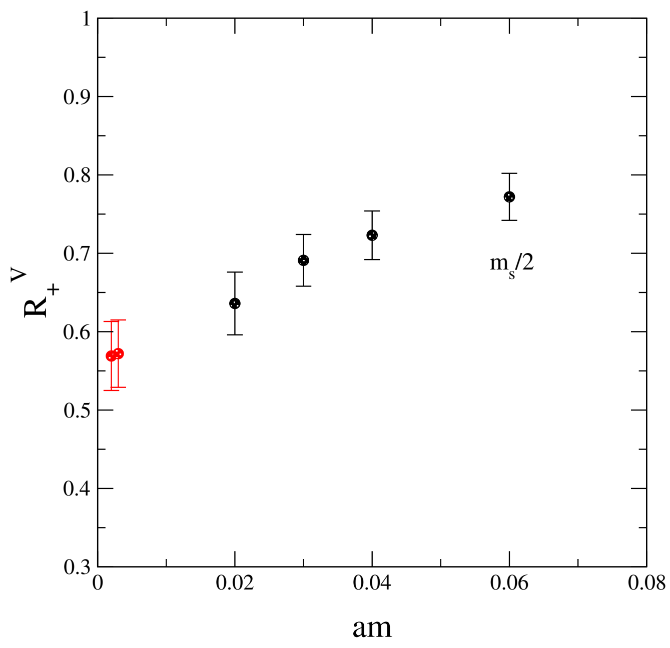
- For $y_0 \ll 0$ and $x_0 \gg 0$

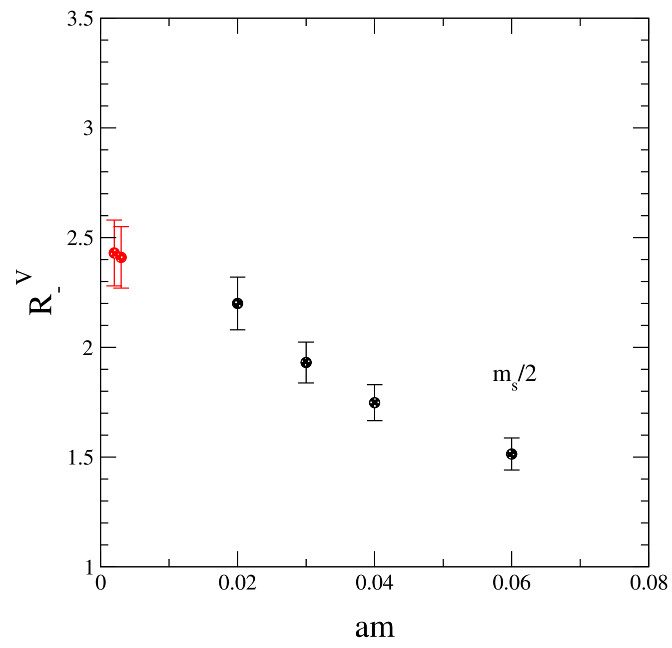
$$R_1^{\pm}(x_0, y_0) = k_1^{\pm} \frac{\sum_{\vec{x},\vec{y}} \langle J_0(x) \hat{Q}_1^{\pm}(0) J_0(y)\rangle}{\sum_{\vec{x}} \langle J_0(x) J_0(0)\rangle \cdot \sum_{\vec{y}} \langle J_0(0) J_0(y)\rangle} \propto g_1^{\pm} + \dots$$

where (flavor index omitted)

$$J_0(x) = \bar{\psi}(x)\gamma_0 P_- \tilde{\psi}(x)$$

First quenched results in the SU(4) limit[L.G. et al. in preparation]





- Chiral corrections tend to be large. By inserting all the relevant factors

$$g_1^+ \sim 0.5\ , \qquad g_1^- \sim 3.0\ , \qquad \frac{g_1^-}{g_+^1} \sim 6$$

- Data suggests a significant enhancement in $|A_0/A_2|$ already in SU(4)
- Departing from the GIM limit important to match the experimental result

Summary for the $\Delta I = 1/2$ rule

- It is crucial to use fermions with exact chiral symmetry to solve the ultraviolet problem and attack the problem
- Technically the problem is difficult because chiral corrections are large and toward the chiral limit the statistical signal tends to disappear
- In the $SU(4)$ limit (GIM) the problem can be solved
- Data suggests a significant enhancement in $|A_0/A_2|$ already in SU(4)
- Necessary to depart from the GIM limit to match the experimental result