Particules Élémentaires, Gravitation et Cosmologie

Année 2005-2006

Interactions fortes et chromodynamique quantique

II: aspects non-perturbatifs

Cours V: 7 mars 2006

Solution du problème U(1) et action efficace à grande N

- Short reminder from previous lecture
- General considerations on gauge-invariant correlators
- Large-N solution of the U(1) problem: masses & mixings
- Effective actions:
  1. General case
  2. Large-N QCD (continued in PDV 2nd seminar)

Reminder from last week

- When the coupling constant α_s is large there is no reason to trust perturbation theory (i.e. the loop expansion)
- Can we find, in this regime, another small expansion parameter?
- At first sight the answer looks negative: QCD depends just on one dimensionful constant, Λ_{QCD}
- However, by considering a whole infinite family of QCD-like theories based on the gauge group SU(N) (rather than SU(3)) one can consider a large-N limit
- Unlike powers of α_s, powers of N (and of N_f) do not correspond to the number of loops, but to some more global, topological property of the Feynman diagrams

- We discussed two kinds of 1/N expansions:

1. When fields are N-dim vectors of an O(N) symmetry
2. When fields are NxN matrices of an SU(N) symmetry

A 2-D example of the former (CP^N) was discussed by PDV.

QCD belongs to the second, more difficult case

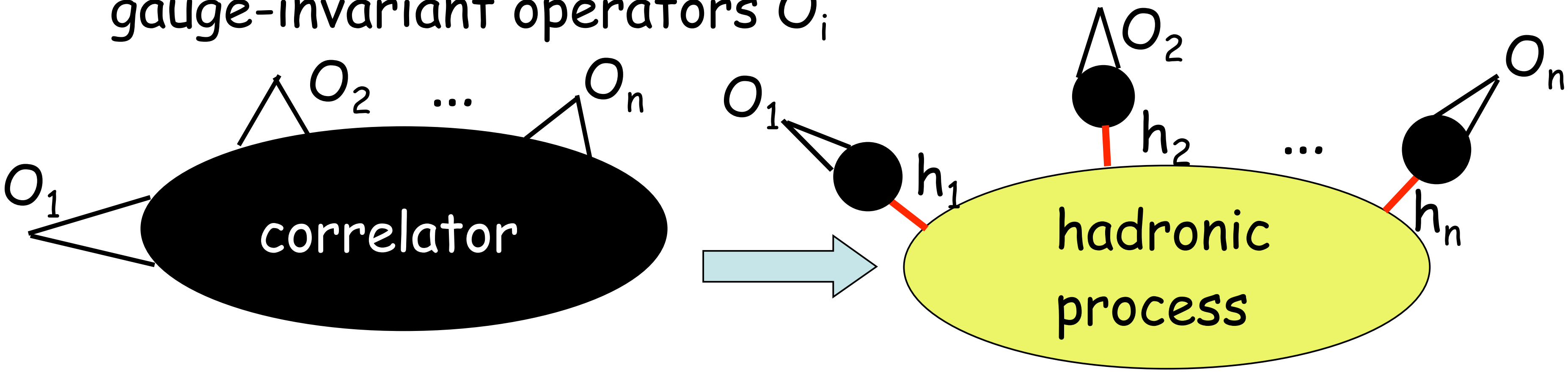
Within QCD we discussed two large N limits:

1. N -> infinity with g^2N and N_f fixed: both quark loops and non planar diagrams are suppressed ('t-Hooft expansion)
2. N -> infinity with g^2N and N_f /N fixed: quark loops included, but non planar diagrams are killed (TE)

Finally, we mentioned that the anomaly-induced mass^2 parameter a is expected to be $O(N_f/N)$, opening the way to a perturbative solution of the U(1) problem (in 't-H. exp.)

General considerations on gauge-invariant correlators

- If colour is confined (evidence to be given later) all QCD observables are encoded in the correlation functions of gauge-invariant operators O_i

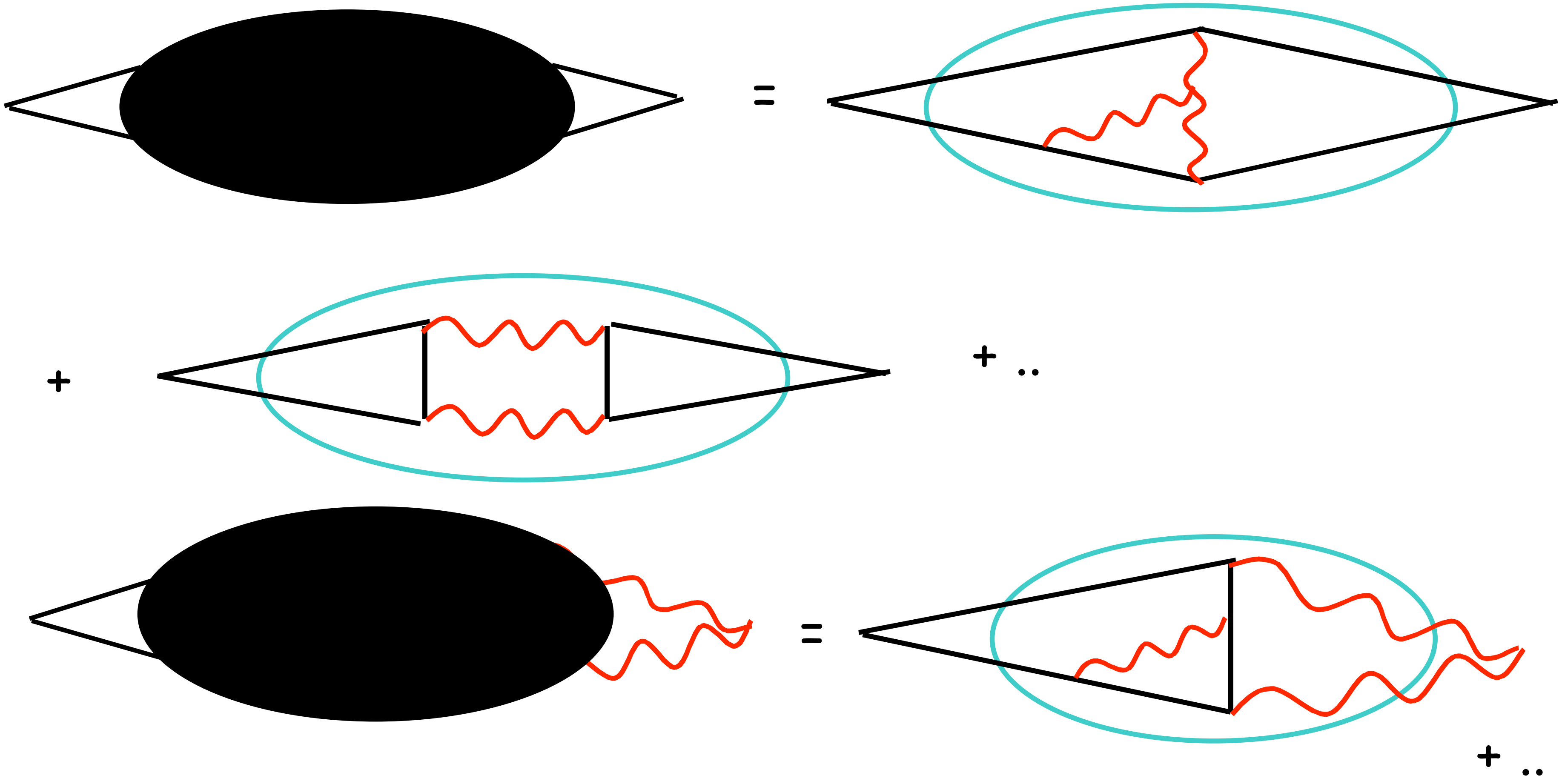


Each O_i creates or annihilates various hadrons and thus the correlators contain all possible information on hadrons (masses, couplings, etc.). Actually our large-N counting was based on such gauge-invariant correlators.

Let us look at U(1) problem in terms of their large-N limit!

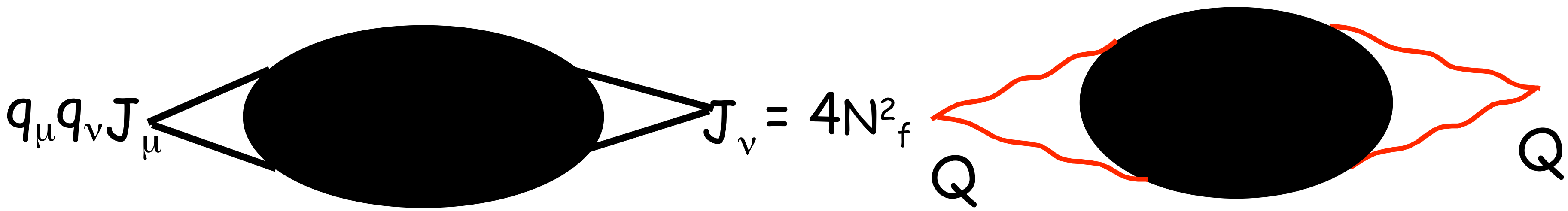
Expanding correlators at large N

Let us consider various correlators and expand them in N_f/N and/or $1/N$. For instance:

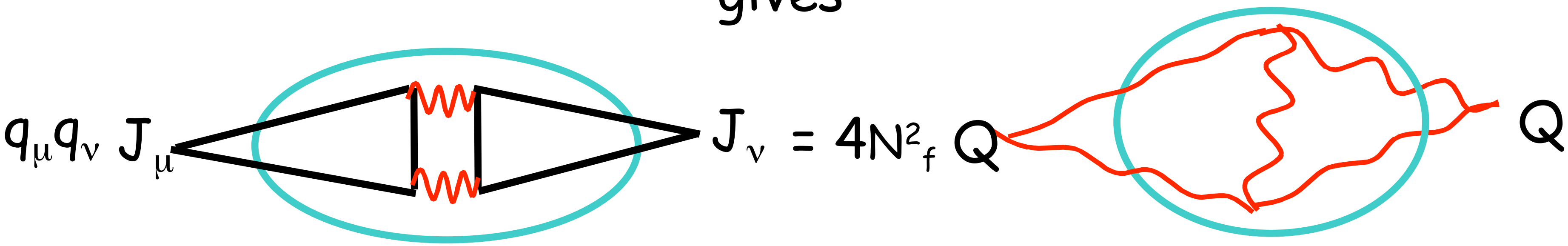


Expanding the WT identities (m_i=0)

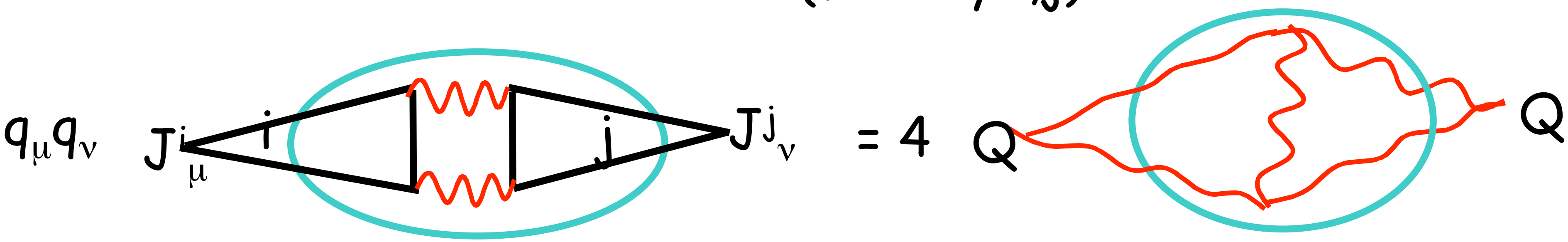
- WTIs can also be expanded in N_f/N and/or $1/N$. In particular:



gives



or even (for any i,j)



The crucial assumption

We now assume that, thanks to topologically non-trivial gauge fields contributing to the path integral, the topological susceptibility of the "pure-glue" theory (i.e. without quarks) is non-vanishing, even at leading order in 1/N. In formulae:

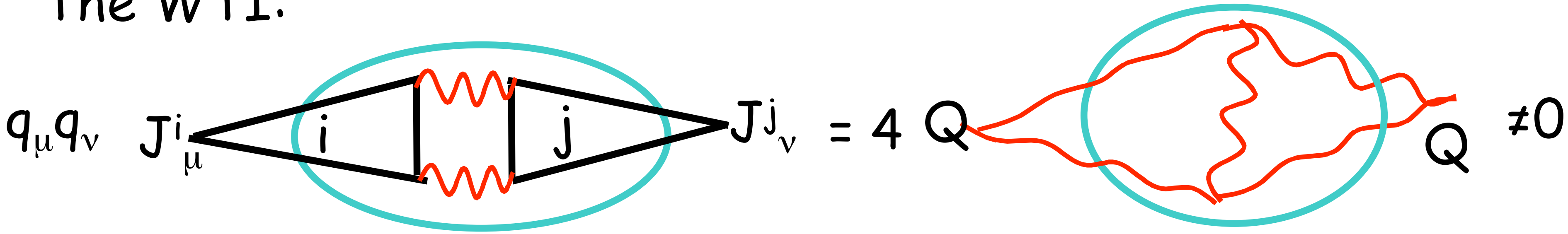
$$\chi_t^{YM} \equiv \langle \int d^4x \, Q(x) Q(0) \rangle_{no\ quarks} = \frac{\langle \nu^2 \rangle}{V_4} = c \Lambda_{QCD}^4$$

with c≠0, O(1)

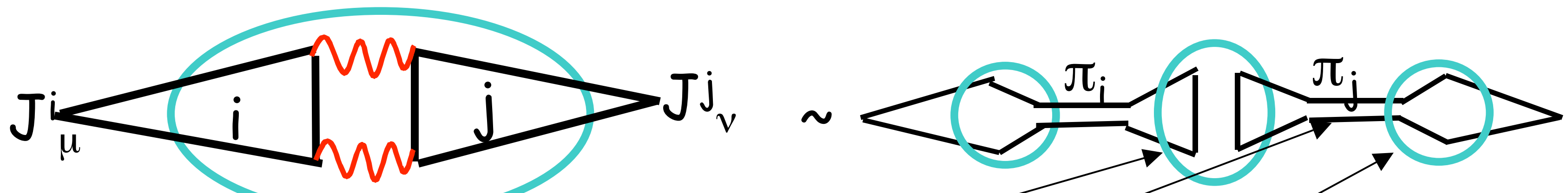
It turns out that such a simple and innocent-looking assumption has far-reaching consequences. We will now show (in two different ways) that it leads to a formula for the mass of the 9th NG boson (essentially the η')

1^{st} derivation of mass formula

The WTI:



implies that the $J_\mu J_\nu$ correlator behaves like $q_\mu q_\nu/q4$ at small q. This double pole can only come from two intermediate massless NG bosons, coupled to the i^{th} and j^{th} currents, and with a non-trivial mass-matrix mixing. In pictures and formulae:



$$q_\mu q_\nu \sqrt{2}F_\pi q_\mu \frac{1}{q^2}\langle \pi_i | M^2 | \pi_j \rangle \frac{1}{q^2}\sqrt{2}F_\pi q_\nu = 2F_\pi^2 a = 4\chi_t^{YM}$$

a

$$q_\mu q_\nu \sqrt{2} F_\pi q_\mu \frac{1}{q^2} \langle \pi_i | M^2 | \pi_j \rangle \frac{1}{q^2} \sqrt{2} F_\pi q_\nu = 2F_\pi^2 a = 4\chi_t^{YM}$$

$$F_\pi^2 a = 2\chi_t^{YM} \qquad\qquad a = 2\,\frac{\chi_t^{YM}}{F_\pi^2}$$

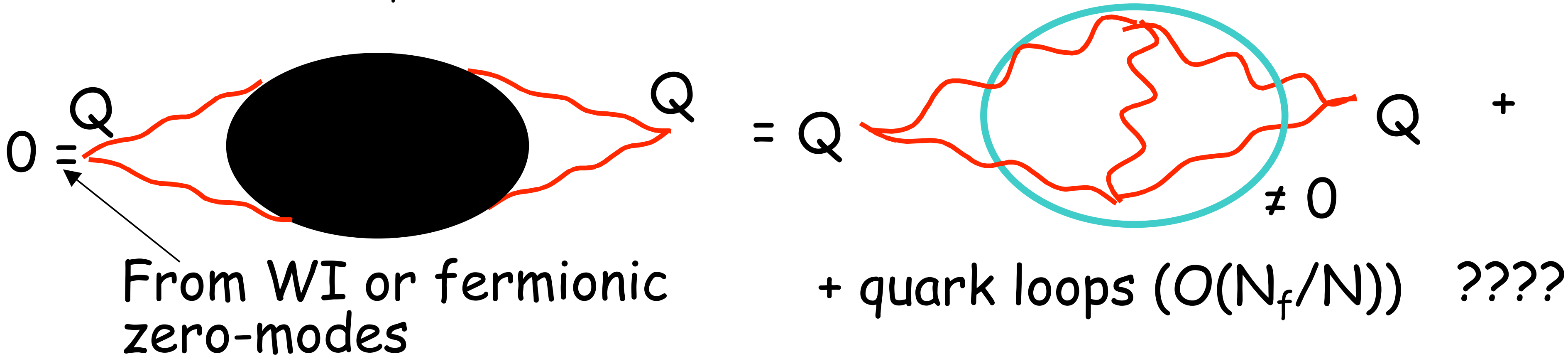
A very strong result: the non vanishing of χ_t in YM theory implies the existence of N_f massless neutral bosons.
Out of these, (N_f -1) remain massless while one gets a mass:

$$m_{\eta'}^2 = N_f a = \frac{2N_f \chi_t^{YM}}{F_\pi^2}$$

In other words: $\chi_t^{(YM)} \neq 0$ => SSB and a solution to U(1) problem!

2^{nd} derivation of mass formula (Witten)

Same assumption about $\chi_t^{(YM)}$ combined with the statement that, instead, $\chi_t = 0$ in full massless QCD .



This looks paradoxical, even inconsistent: how can something subleading cancel something leading? Is the “crucial assumption” just inconsistent? No, there is a way out: the cancellation is supposed to occur just at q=0. If the formally subleading diagrams contain a light particle, this can enhance their contribution and a cancellation becomes possible.

Let us try

$$\chi_t^{QCD} \sim \chi_t^{YM} + \frac{F_\pi^2}{2N_f}\frac{m_{\eta'}^4}{q^2 - m_{\eta'}^2} \rightarrow 0 \; if \; \chi_t^{YM} = \frac{F_\pi^2 m_{\eta'}^2}{2N_f}$$

which is the same formula as before: →

$$m_{\eta'}^2 = \frac{2N_f \chi_t^{YM}}{F_\pi^2}$$

NB: since F_π^2 =O(N), χ_t^{YM} =O(1), $m_{\eta'}^2$=O(Nf/N)

Using this relation we have, at small q:

$$\chi_t^{QCD}(q) \sim \frac{\chi_t^{YM} q^2}{q^2 - m_{\eta'}^2}$$

The limits q -->0 and N --> infinity do not commute:

χ_t^{QCD} = O(1) (=0) if we take N = infinity (q -->0) first

Masses and mixings: the real world (?)

- Combining the contribution to the PNGB masses from quark masses to the one from the anomaly we got:

M^2 =

	π_{uu}	π_{dd}	π_{ss}
π_{uu}	$\mu_{uu}^2 + a$	a	a
π_{dd}	a	$\mu_{dd}^2 + a$	a
π_{ss}	a	a	$\mu_{ss}^2 + a$

If $\mu_{uu}^2, \mu_{dd}^2 \ll \mu_{ss}^2, a$, we have one approximate eigenstate:

$$\pi^0 = \frac{1}{\sqrt{2}}[\bar{u}u - \bar{d}d]$$

with mass $(\mu_{uu}^2 + \mu_{dd}^2)/2$ i.e. the same as $\pi^\pm$ (isospin is saved!)

The remaining 2x2 matrix is easily diagonalized

We easily find:

$$M^2(\eta',\eta) = \frac{1}{2}[2m_K^2 + 3a \pm \sqrt{(2m_K^2 - a)^2 + 8a^2}]$$

$$3a = m_{\eta'}^2 + m_\eta^2 - 2m_K^2 \sim 0.72 GeV^2$$

Inserting this value of a we get (w/ exp. numb. in parenthesis):

$$m_\eta^2 = 0.27(0.30) GeV^2 \ , \ m_{\eta'}^2 = 0.95(0.92) GeV^2 \ , \ \phi_P = 14^o(11^o)$$

$$\eta = cos\phi \ \eta_8 + sin\phi \ \eta_1 \ , \ \eta' = cos\phi \ \eta_1 - sin\phi \ \eta_8$$

$$\eta_8 = \frac{1}{\sqrt{6}}[\bar{u}u + \bar{d}d - 2\bar{s}s] \ , \eta_1 = \frac{1}{\sqrt{3}}[\bar{u}u + \bar{d}d + \bar{s}s]$$

Finally, from $2\chi_t^{(YM)} = aF_\pi^2$, we get: $\chi_t^{(YM)} \sim (180 \text{ MeV})^4$

This "reasonable" number has been confirmed in lattice QCD!

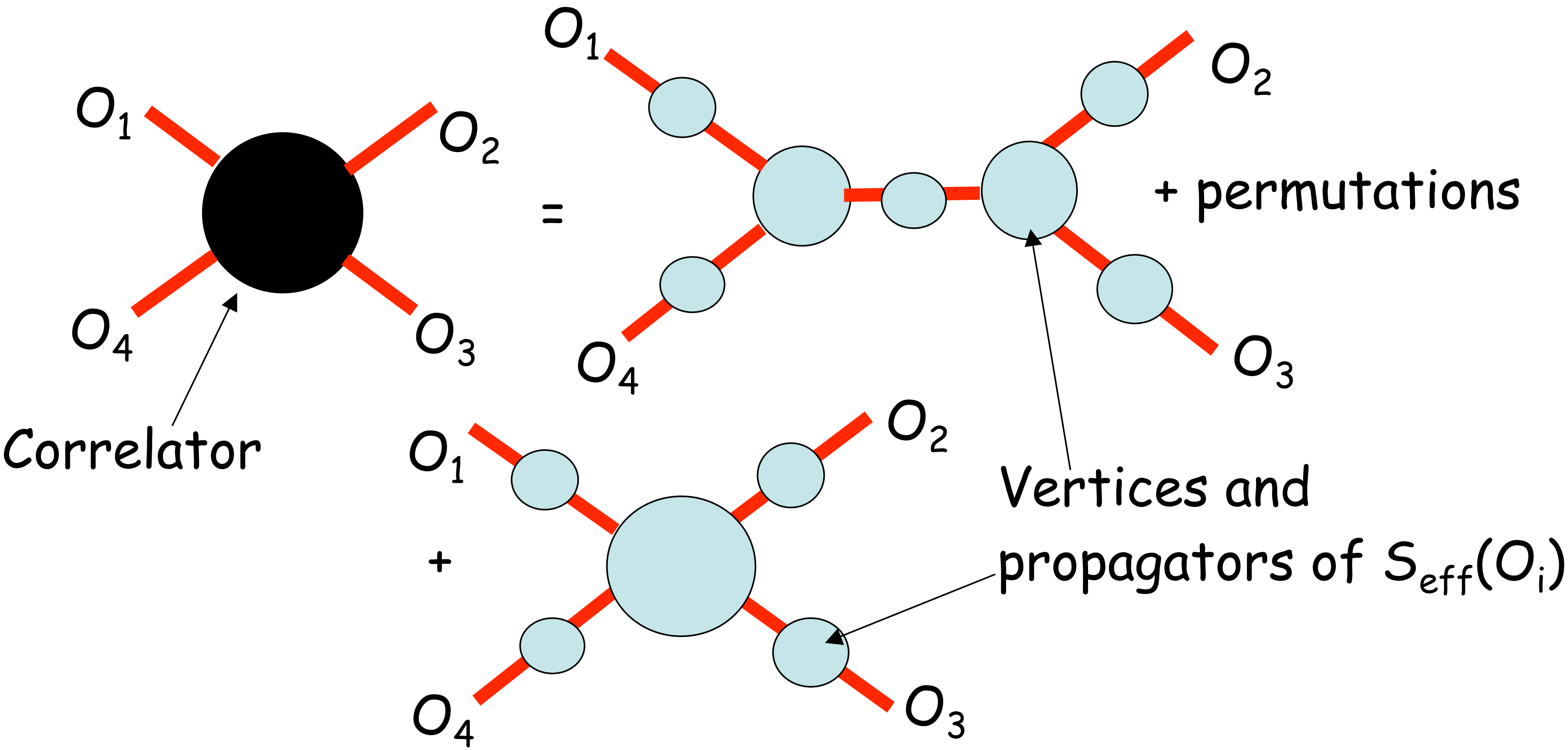
Unreasonable success of the WV formula?

- Does the above (WV) formula work too well?
- After all, it is supposed to represent a leading order result in N_f/N, a quantity which is effectively 1.
- Q: Can we justify WV at finite N in a different way?
- A: Yes, by simply identifying, in the WTIs, powers of N_f
- One finds that the WV formula should also hold, at leading order in N_f, even if N is finite. Leading order in N_f means the so-called quenched approximation, much used in lattice QCD, in which quark-loops are neglected.
- The validity of WV would be related to the fact that, for computing the effect of the anomaly on PNG boson masses, the quenched approximation is, as in many other cases, accurate.

Effective actions for composite operators: general considerations

- An effective action (or Lagrangian) for composite operators, $S_{eff}(O_i)$, can be defined rigorously from the gauge-invariant correlators defined earlier, through a Legendre transform.
- $S_{eff}(O_i)$ does depends on the set of operators O_i chosen as its arguments, but the correlators, obtained by summing tree-diagrams with vertices and propagators derived from $S_{eff}(O_i)$, do not depend on that choice
- $S_{eff}(O_i)$ shares the symmetries of S but takes also into account loop-induced anomalies; finally, its dependence on various parameters appearing in S is often very simple.

We will now describe how all this works for QCD.



O_1
O_2
O_4
O_3
Correlator
=
O_1
O_2
+ permutations
O_4
O_3
O_1
O_2
Vertices and
propagators of $S_{eff}(O_i)$
+
O_4
O_3

Effective action for large-N QCD

Recipe uses the following ingredients:

1. Convenient choice of operators
2. Lowest number of derivatives (low-energy approximation)
3. Only leading-N structures
4. Right transformation wrt symmetries (including anomalies) and dependence on various parameters

Choice of operators (better too many than too few)

We certainly need to keep fields that couple to our PNG bosons. A convenient (redundant) set is (notation as in PDV)

$$U_{ij} = \bar{\psi}_{Ri}\psi_{Lj} \; , \; U^{\dagger}_{ji} = \bar{\psi}_{Lj}\psi_{Ri}$$

We then have the option to include gluonic operators such as

$$G(x) = \frac{\alpha_s}{4\pi} Tr[F_{\mu\nu}F_{\mu\nu}] \; , \; Q(x) = \frac{\alpha_s}{4\pi} Tr[F_{\mu\nu}\tilde{F}_{\mu\nu}]$$

Given the importance of Q (q in PDV's seminar) we will keep it, but will NOT include G (in SUSY extensions G is also kept)

=> There are $2N_f^2 + 1$ (real) fields in our S_{eff}.

Lowest number of derivatives

(low-energy approximation)

We will limit to 2 the number of derivatives appearing in S_{eff}

Leading large-N structures

This basically restricts U and U⁺ to appear under a single trace. Examples are the kinetic term for U,

$$K\ Tr[\partial_\mu U^\dagger \partial_\mu U]$$

and the potential that forces U to acquire a VEV:

$$V(U,U^\dagger) = \lambda Tr[(UU^\dagger - v^2)^2] \Rightarrow \langle U_{ij} \rangle = v\delta_{ij}$$

Transformation wrt symmetries (including anomalies)

Under a $U(N_f)_L \times U(N_f)_R$ transformation:

$$U \to AUB^\dagger \ , \ U^\dagger \to BU^\dagger A^\dagger \ , \ A^\dagger A = B^\dagger B = 1$$

one must have (very simple because of inclusion of Q!):

$$S_{eff} \to S_{eff} + iQ \, log \, det(AB^\dagger)$$

i.e. S_{eff} is invariant under $SU(N_f)_L \times SU(N_f)_{RL} \times U(1)_V$ but, under $U(1)_A$ ($A = B^+ = e^{i\gamma} \times 1$), it changes by $-2N_f \gamma$ Q. This gives:

$$S_{eff} = S_{eff}^{inv.} + \frac{i}{2} Q \, log \, det \left(\frac{U}{U^\dagger} \right)$$

Dependence on parameters also becomes simple:

$$\frac{\partial S_{eff}}{\partial m_{ij}} = U_{ij}\ ,\ \frac{\partial S_{eff}}{\partial m^*_{ij}} = U^\dagger_{ij}\ ,\ \frac{\partial S_{eff}}{\partial \theta} = -Q$$

At this point we collect all the information and find:

$$S_{eff} = KTr[\partial_\mu U^\dagger \partial_\mu U] - \lambda Tr[(UU^\dagger - v^2)^2] + \frac{Q^2}{2\chi_t^{YM}} +$$
$$+Tr[mU + m^\dagger U^\dagger] + Q\left(\frac{i}{2}\, log\ det\left(\frac{U}{U^\dagger}\right) - \theta\right)$$

By suitable redef. of U we can always bring m_{ij} to a real-diag. form at the price of changing θ (physics depends only on θ - arg det m). Hereafter θ is that physically significant angle!

Final remarks

1. We have added the leading (in 1/N) term proportional to Q^2 Neglecting the U-field (i.e. going to the pure YM theory) it would give a topological suceptibility χ_t^{YM}. This term corresponds to our "crucial assumption" in S_{eff} language.

2. If one of the quark masses were zero, we could redefine the corresponding U, U^+ and rotate away θ without inducing CP violation elsewhere, and without having a massless NG boson. This easy solution, unfortunately, does not seem to work phenomenologically, but the axion-based solution is essentially of the same type (see PDV seminar no.2)

3. It is convenient to: i) rescale the fields by constant dimensionful factors to make them canonical; ii) take the limit λ -> infinity (eliminate scalars, non-linear model, see PDV no.2)