Anomaly

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Abstract

- Symmetry and quantization
- $U_A(1)$ anomaly and π_0 decay
- Origin of anomalies
- Chiral and nonabelian anomaly
- Anomally and nonabelian dynamics
- Anomaly cancellation as consistency

Symmetries and Quantum mechanics

- "Quantization, Symmetries, Phase Factor"

  - three melodies of theoretical physics of the 20th century

  (C.N. Yang, Paris TH 2002)

- **Anomalies are consequences of Quantum mechanics and Symmetries;**

- **Anomaly as a certain symmetry, modified by quantum effects, in a subtle and precise way;**

- Consequences

  ⋆ Calculable physical effects;
  ⋆ Nontrivial consistency conditions

Symmetries in Nature

- Classic examples (crystalographic symmetries, left-right symmetry of certain biological systems, etc)

- Symmetries and conservation laws:

  Homogeneity in spacetime (E and p_i conservation); isotropy of space (angular momentum);

- Internal symmetries

  Isospin (p, n as the two quantum states of the same particle); Unitary symmetries and the quark model (u, d, s quarks);

- Important discrete symmetries (parity; time reversal, CP, CPT, *etc.*)

- Symmetries in Nambu-Goldstone mode (spontaneously broken symmetries), *i.e.*, massless pions

$$SU_L(2) \times SU_R(2) \times U_V(1) \times U_A(1) \to SU_{isospin}(2) \times U_V(1) \qquad (1)$$

Symmetries and interactions (gauge principle)

- Charge conservation from the invariance for the electron field

$$\psi(x) \to e^{i\alpha}\,\psi(x);$$

Require invariance under $\alpha = \alpha(x^\mu)$: $\Rightarrow$

$$L = -\frac{1}{4}F_{\mu\nu} + i\,\psi(x)\,\gamma^\mu\,(\partial_\mu - ieA_\mu(x))\,\psi(x), \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

(quantum electrodynamics).

- More general groups ($G = SU(2)$, $SU(3)$, etc.):

Yang-Mills theories

- Standard model of strong and electroweak interactions

(Fritsch, Gell-Mann, Glashow, Weinberg, Salam)

$$G = SU_C(3) \times SU_L(2) \times U_Y(1) \tag{2}$$

Elementary particles in Nature

Quarks and their Charges

Quarks	$SU(3)$	$SU_L(2)$	$U_Y(1)$	$U_{EM}(1)$
$\begin{pmatrix} u_L \\ d'_L \end{pmatrix}, \begin{pmatrix} c_L \\ s'_L \end{pmatrix}, \begin{pmatrix} t_L \\ b'_L \end{pmatrix}$	$\underline{3}$	$\underline{2}$	$\frac{1}{3}$	$\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$
$u_R, \quad c_R, \quad t_R$	$\underline{3}$	$\underline{1}$	$\frac{4}{3}$	$\frac{2}{3}$
$d_R, \quad s_R, \quad b_R$	$\underline{3}$	$\underline{1}$	$-\frac{2}{3}$	$-\frac{1}{3}$

Table 1: The primes indicate that the mass eigenstates are different from the states transforming as multiplets of $SU_L(2) \times U_Y(1)$. They are linearly related by Cabibbo-Kobayashi-Maskawa mixing matrix.

Leptons and Their Charges

Leptons	$SU_L(2)$	$U_Y(1)$	$U_{EM}(1)$
$\begin{pmatrix} \nu'_{e\,L} \\ e_L \end{pmatrix}, \begin{pmatrix} \nu'_{\mu\,L} \\ \mu_L \end{pmatrix}, \begin{pmatrix} \nu'_{\tau\,L} \\ \tau_L \end{pmatrix}$	$\underline{2}$	-1	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$
$e_R, \quad \mu_R, \quad \tau_R$	$\underline{1}$	-2	-1

Table 2: The primes indicate again that the mass eigenstates are different from the states transforming as multiplets of $SU_L(2) \times U_Y(1)$, as required by the observed neutrino oscillations.

Higgs doublet

Higgs doublet	$SU_L(2)$	$U_Y(1)$	$U_{EM}(1)$
$\begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$	$\underline{2}$	1	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Table 3: Higgs doublet scalars and their charges

Gauge Boson Masses

photon	gluons	$W^\pm$ (GeV)	Z (GeV)
0	0	80.425 ± 0.038	91.1876 ± 0.0021

Table 4:

Chirality

- Fundamental particles come as chiral fermion (e_L, e_R, ν_L) (Feynman, Gell-Mann)
- Dirac fermion $\sim \psi_L + \psi_R$
- Parity violation in (electro-) weak interactions (Fig.)
- Chiral symmetry: quarks u_L, d_L, u_R, $d_R \Rightarrow$

$$G_F = SU_L(2) \times SU_R(2) \times U_L(1) \times U_R(1)$$

- G_F Spontaneously Broken to (Pions: $\pi^{\pm}$, π^0)

$$G_{manifest} = SU_{isospin}(2) \times U_B(1)$$

Chiral fermion

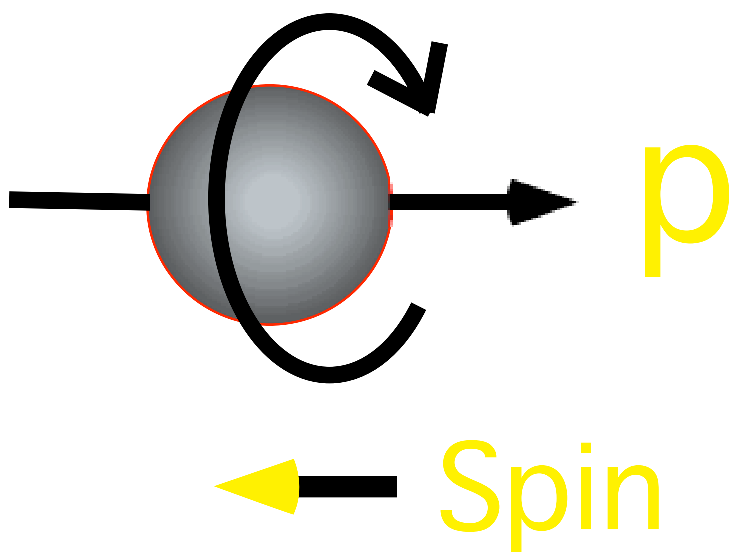

Figure 1:

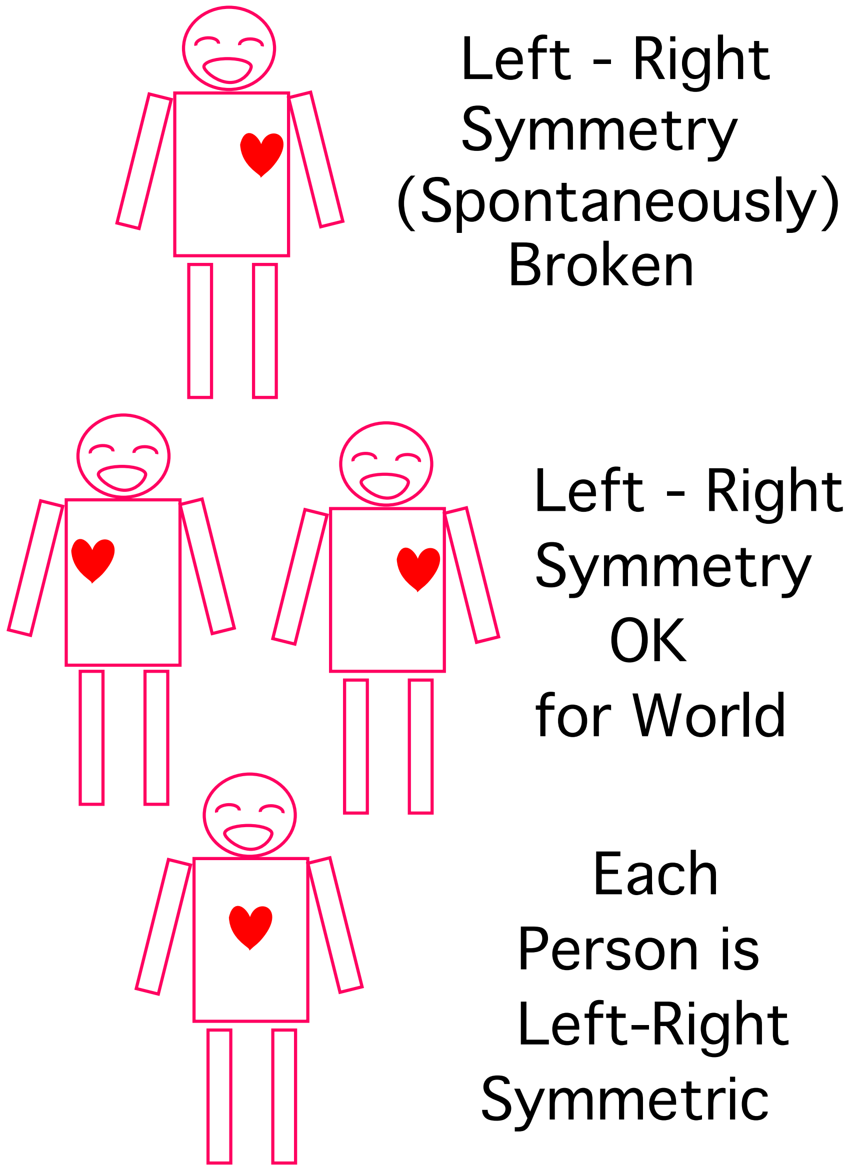
Left - Right
Symmetry
(Spontaneously)
Broken
Left - Right
Symmetry
OK
for World
Each
Person is
Left-Right
Symmetric

Figure 2:

$U(1)$ (axial) Anomaly

Invariance under

$$\psi(x) \to e^{i\gamma^5\alpha}\psi_(x) \qquad \gamma^5 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad \psi(x) = \begin{pmatrix} \psi_R(x) \\ \psi_L(x) \end{pmatrix}$$

$$\Rightarrow$$

$$\partial_\mu J_5^\mu = 0 + \frac{e^2}{16\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{e^2}{8\pi^2}\mathbf{E}\cdot\mathbf{H}, \qquad J_5^\mu = \bar{\psi} i\gamma_5\gamma^\mu\psi$$

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}; \qquad \epsilon^{0123} = 1.$$

The three point function (Fig.)

$$G^{\mu;\lambda,\nu} \equiv \langle T J_5^\mu(x) J^\lambda(y) J^\nu(z)\rangle \tag{3}$$

with a linearly divergent one-loop contribution, satisfies:

$$\partial_\mu^x G^{\mu;\lambda,\nu} \neq 0, \qquad \text{if} \qquad \partial_\lambda^y G^{\mu;\lambda,\nu} = \partial_\nu^z G^{\mu;\lambda,\nu} = 0. \tag{4}$$

(Fukuda, Miyamoto, Steinberger ('49), Schwinger ('51), Adler, Bell, Jackiw ('68))

Triangle anomaly and $\pi^0 \to \gamma\gamma$

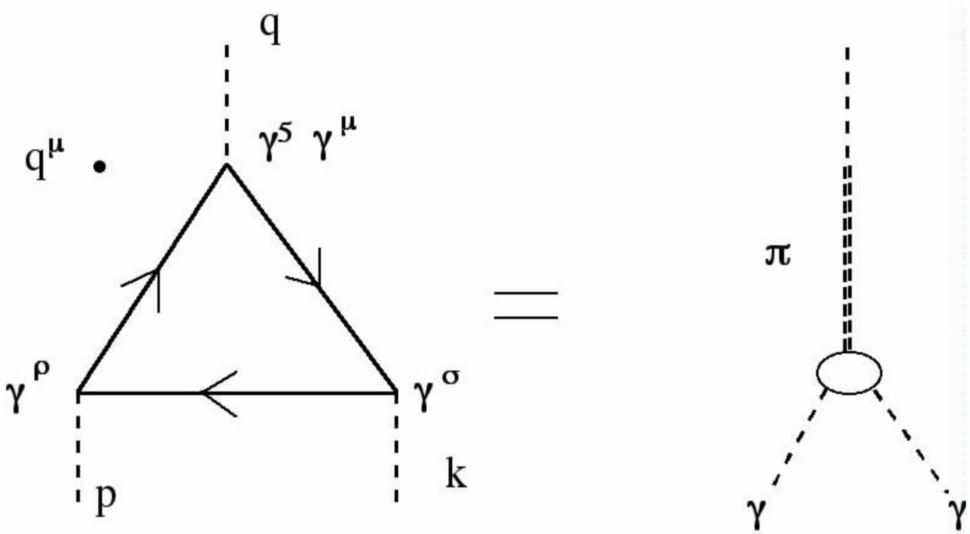

Figure 3:

$$\Gamma(\pi^0 \to 2\gamma) = \frac{\alpha^2}{64\pi^3}\frac{m_\pi^3}{F_\pi^2} N_c^2 \left[\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 \right]^2 \simeq 1 \cdot 10^{-5}\,\mathrm{MeV},$$

vs

$$\Gamma^{exp}(\pi^0 \to 2\gamma) = \frac{\hbar}{\tau} \simeq \frac{1}{1.2} \cdot 10^{-5}\,\mathrm{MeV}.$$

Sutherland-Veltman's theorem

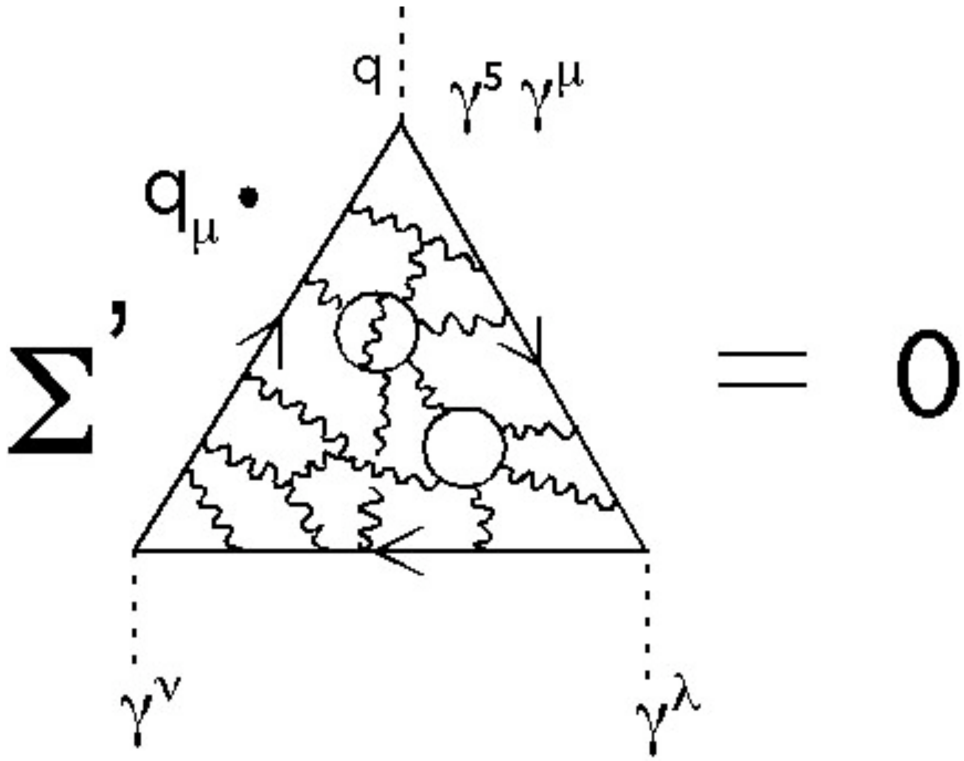

Figure 4:

Origin of the Anomaly

- Locality (*and Causality*) of fundamental interactions: fact of Nature (String theory?)

- Divergences $\Rightarrow$ Neccessity of *renormalization*. (renormalizable field theories: paradigm of modern particle physics) (Tomonaga, Dyson, Feynman, Schwinger)

1. Regularize each operator (precise procedure for removing divergences);
2. Fit the value of certain (conveniently chosen) reference quantities to experimental data (*e.g.*, e);
3. Remove the regularization;
4. If the procedure works to all orders, the theory is *renormalizable*
5. Well defined predictions order by order of perturbation

- Axial current operator ("point-splitting" method)

$$i\,\bar{\psi}(x)\,\gamma^5\,\gamma^\mu\,\psi(x) \equiv \lim_{\epsilon\to 0} i\,\bar{\psi}(x+\frac{\epsilon}{2})\,\gamma^5\,\gamma^\mu \exp[\frac{i\,e}{c}\int_{x-\epsilon/2}^{x+\epsilon/2} dx^\mu\,A_\mu\,]\,\psi(x-\frac{\epsilon}{2})$$

1. $\frac{1}{\epsilon}$ divergences when inserted in a graph;
2. $O(\epsilon)$ contriibution from the string bit;
3. Compute the finite, $\epsilon \cdot \frac{1}{\epsilon}$ piece;
4. $\epsilon \longrightarrow 0$
5. Axial anomaly: $\partial_\mu J_5^\mu = \frac{e^2}{16\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}$.

- Other regularizations (with the same result)

1. Momentum space subtractions (BPHZ);
2. Dimensional regularization;
3. Pauli-Villars (regulator fields with big mass);
4. Functional integral method (Fujikawa)

$$e^{-\Gamma(A)} = \int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, e^{-\int\, dx\, \bar{\psi} i\gamma\mathcal{D}^A\psi}, \qquad A_\mu^5 \to A_\mu^5 - \partial_\mu\alpha(x)$$

$$\psi(x) = e^{i\,\gamma^5\,\alpha(x)}\,\psi'(x) \Rightarrow \mathbf{J} = e^{-2\,i\,\int dx\,\alpha(x)\,\sum_n\,\psi_n^*(x)\,\gamma^5\,\psi_n(x)}$$

$$\sum_n \psi_n^*(x)\,\gamma^5\,\psi_n(x) = 0\cdot\infty \xrightarrow{\text{Regular.}} \frac{e^2}{16\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}$$

Chiral anomaly / nonabelian anomaly

$$e^{-\Gamma(A)} = \int \mathcal{D}\psi\,\mathcal{D}\bar{\psi}\,e^{-\int\,dx\,\bar{\psi}i\gamma\mathcal{D}^A\psi}.$$

Gauge transformation

$$A_\mu \to A_\mu - \mathcal{D}_\mu v, \qquad \mathcal{D}_\mu v = \partial_\mu v + ig[A_\mu, v],$$

$$\Gamma(A) \to \Gamma(A) + \int dx\, v^a\, \mathcal{D}_\mu \frac{\delta\Gamma(A)}{\delta\, A^a_\mu}.$$

If the theory is invariant,

$$\frac{\delta\Gamma(A)}{\delta\, v^a} = 0, \quad \therefore \quad \mathcal{D}_\mu\langle J^a_\mu\rangle = 0, \quad J^a_\mu = \frac{\delta\Gamma(A)}{\delta\, A^a_\mu}$$

Actually, the variation is nonvanishing (anomaly!)

$$\begin{aligned}
\delta_{v_L}\Gamma(A) &= \frac{1}{24\pi^2}\int d^4x\,\mathrm{Tr}\,v(x)\,\epsilon^{\lambda\mu\alpha\beta}\,\partial_\lambda\left(A_\mu\partial_\alpha A_\beta + \frac{1}{2}A_\mu A_\alpha A_\beta\right) \\
&= \frac{1}{24\pi^2}\int d^4x \int \mathrm{Tr}\,v(x)\,d(AdA + \frac{1}{2}A^3) \propto \mathrm{Tr}T^a\{T^aT^c\} = d^{abc}(\underline{r})
\end{aligned}$$

Cancellation of anomalies in the standard model

- Standard model (Eq.2) is *chiral*. Anomalies potentially break *gauge invariance*.

- Algebraic cancellations! (Fig.)

- $U_Y(1)^3$ vertex:

$$[\left(\frac{1}{3}\right)^3 \cdot 2 \cdot 3 - \left(\frac{4}{3}\right)^3 \cdot 3 - \left(-\frac{2}{3}\right)^3 \cdot 3] + [(-1)^3 \cdot 2 - (-2)^3] = -6 + 6$$

$$q_L \qquad u_R \qquad d_R \qquad (\nu, e)_L \qquad e_R$$

- $U_Y(1) - SU_L(2) - SU_L(2)$ vertex:

$$\left(\frac{1}{3}\right) \cdot 3 \cdot 1 - 1 = 1 - 1$$

- Quark-lepton universality (Cabibbo angle, KM mixing, CP violation)

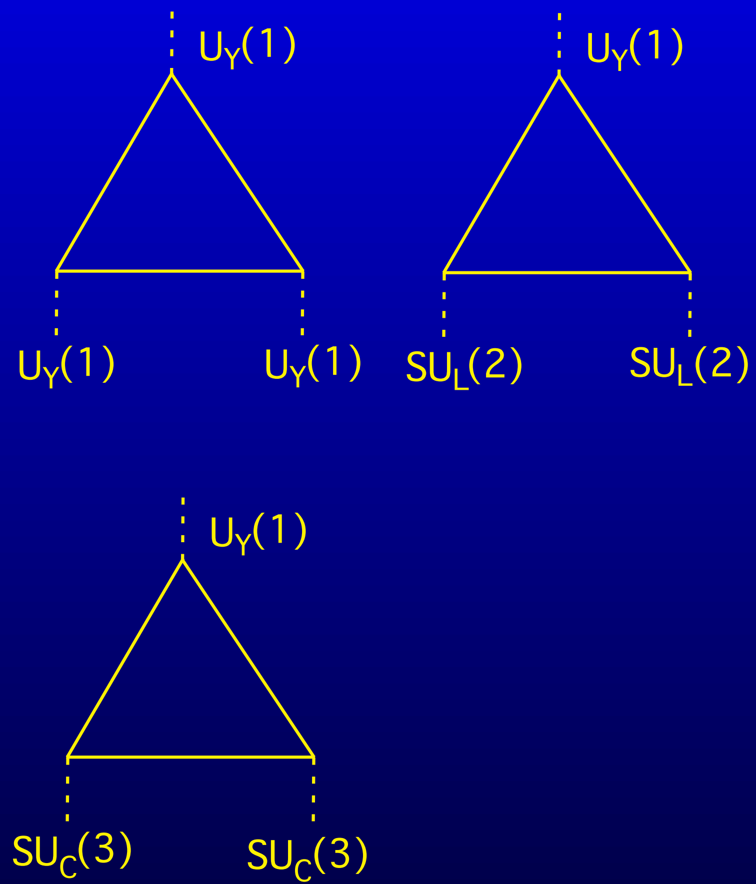

Figure 5:

Wess-Zumino consistency condition, Witten's quantization

- The anomaly as a variation of the effective action (Wess-Zumino)

$$(\delta_{v_1} \cdot \delta_{v_2} - \delta_{v_2} \cdot \delta_{v_1})\,\Gamma(A) = \delta_{[v_1,v_2]}\,\Gamma(A).$$

- $U \sim$ Nambu-Goldstone bosons of $\frac{SU(3)\times SU(3)}{SU(3)} \sim SU(3)$,

$$\pi_5(SU(3)) \sim \mathbb{Z}, \qquad \therefore \quad n \in \mathbb{Z} \qquad (n = N_c = 3 \text{ actually})$$

- Local action on the boundary of $S^4 \sim R^4$

$$\Gamma_{eff} = n \int d\Sigma_{i_1 i_2 \ldots i_5}\, \omega_{i_1 i_2 \ldots i_5} \equiv n\,\Gamma(U),$$

$$\omega_{i_1 i_2 \ldots i_5} = -\frac{i}{240\,\pi^2} \mathrm{Tr}\,(U^{-1}\partial_{i_1}U)(U^{-1}\partial_{i_2}U)\ldots(U^{-1}\partial_{i_5}U),$$

- $\Gamma_{eff} \to \Gamma_{eff}(U, A)$ *e.g.* $\pi_0 \to \gamma\gamma$. $n = N_c = 3$.

't Hooft's "anomaly matching" condition

- Proton $\sim uud$; neutron $\sim udd$;

  Quarks and leptons: are they truly elementary?

  Or composite of even more fundamental constituents (preons) ?

- Suppose some strong gauge forces G_{strong} confine preons into quarks, *e.g.*,

$$q \sim \mathcal{P}\mathcal{P}\mathcal{P}$$

- If $\mathcal{P} \sim \underline{r}$ of the flavor group G_F (representation) then

$$\underline{q} \subset \underline{r} \otimes \underline{r} \otimes \underline{r}.$$

- Fermions are in chiral (not left-right symmetric) representations of G_F

- Introduce hypothetical, weakly coupled gauge bosons of G_F, introducing if necessary hypothetical "leptons" to cancel the anomaly;

- Nontrivial conditions on possible composite modes

$$\sum_{preons} d^{abc}(\underline{r}) = \sum_{quarks} d^{abc}(\underline{r}_q)$$

- Dynamical result (number and types of light composite fermions) must obey the algebraic conditions

- Deep connections between the dynamics and global symmetry

Anomaly Matching

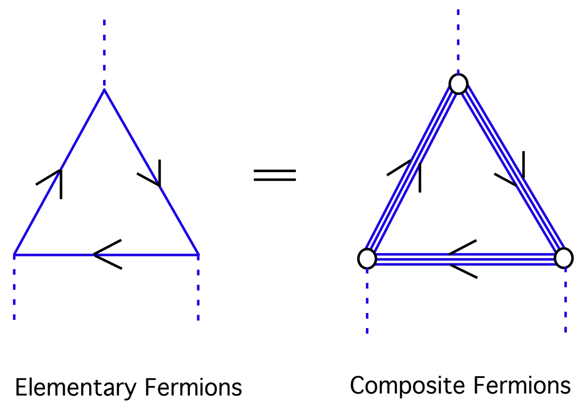

Figure 6:

Anomaly matching condition and $\pi_0 \to \gamma\gamma$

(i) Both consequence of axial (or chiral) anomaly;

(ii) The same "anomaly" content in the high-energy and low-energy theories;

(iii) The anomaly $\to$ singularity at $q^2 = 0$ in the F.T. of the three point function such as (3);

(iv) Singularity either due to

- massless Goldstone boson (G_F spontaneously broken);
- due to pairs of massless fermions (unbroken G_F: 't Hooft);
- or both (supersymmetric theories)

Anomaly and dynamics of QCD

- $U_A(1)$ anomaly in the presence of nonabelian gauge interactions
- $$\frac{\partial}{\partial\mu}J_5^\mu = 0. + \frac{g^2}{32\pi^2}G^a_{\mu\nu}\tilde{G}^{\mu\nu\, a}, \qquad J_5^\mu = \bar{q}i\gamma_5\gamma^\mu q$$
- $$G^a_{\mu\nu} = \frac{\partial}{\partial\mu}A^a_\nu - \frac{\partial}{\partial\nu}A^a_\mu + f^{abc}\, A^b_\mu A^c_\nu$$
- $\pi_2(SU(2)) = \mathbb{Z}$ nonperturbative effects (instantons);
- $G^a_{\mu\nu} \sim$ gluons of QCD $\Rightarrow$ Solution of the "$U(1)$" problem
- Prof. Veneziano's Lecture on the "$U(1)$-Problem"
- $G^a_{\mu\nu} \sim$ electroweak W boson tensor $\Rightarrow$ Particle productions ($\Delta B \neq 0$, $\Delta L \neq 0$).

Summary

- Anomalies are a subtle modification of symmetry by quantum effects;
- Precise predictions (*e.g.*, $\pi_0 \to \gamma\gamma$);
- Locality of fundamental interactions (renormalization theory);
- Chiral structure of the world;
- Standard model of the strong and electroweak interactions based on

$$G = SU_C(3) \times SU_L(2) \times U_Y(1)$$

  quantum-mechanically consistent;
- Quark-lepton universality: origin? (Grand unification? String theory?)
- Dynamical effects on the right hand side of $\partial_\mu J_5^\mu = \frac{e^2}{16\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}$?

Instantons

- Prof. Veneziano's coming lectures
- My next seminar (21st February)