Particules Élémentaires, Gravitation et Cosmologie

Année 2004-2005

Interactions fortes et chromodynamique quantique I: Aspects perturbatifs

Cours II: 8 mars 2005

1. Summary of previous course
2. Dirac fermions
3. Classical QED and QCD
4. Quantum Corrections and renormalization of the gauge couplings
5. IR triviality vs. asymptotic freedom

1. Short summary of course no. 1

1.1 *P* (Poincaré) and its representations in terms of:

a) Single-particle states: $(m \neq 0, J)$, $(m=0, h)$

b) Fields: $(r) = (j_- , j_+)$: $\phi(x)$, $A_\mu(x)$, $\psi_\alpha(x)$, $\chi_\alpha(x)$

1.2 EM, Weak & Strong int. <=> massless J=1 quanta

$\Rightarrow$ Gauge theories

1.3 Construction of the generic gauge theory:

a) Choose *G* => gauge bosons

b) Assign all l.h. ferm's to some (r) of *G* (r.h. in (r*))

c) Assign all bosons to reps. of *G* (always (r +r*))

m_B always possible, m_F iff (r) is real (QED, QCD, ~~EW~~)

2. Dirac fermions

- The (two-component) fermions we have discussed last time, ψ_α and χ_β are called Weyl fermions. For vector-like theories it is quite convenient to work with (four-component) Dirac spinors, made out of a ψ_α and a χ_β

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \chi_\beta \end{pmatrix}$$

if $\chi_\beta = \varepsilon_{\alpha\beta}\, \psi^*_\alpha$ one talks about Majorana fermions

The Dirac spinor is useful if all its 4 components belong to the same rep. of *G*

For Majorana fermions this means that (r)=(r*)

For chiral theories better stay with Weyl…

For vector-like theory (r)=(s)+(s*)

$$\Psi = \begin{pmatrix} \psi^{(s)} \\ \chi^{(s)} = (\tilde{\psi}^{(s*)})^* \end{pmatrix}$$

Let us see, for instance, how the u-quark Dirac spinor is constructed in QCD

$$\Psi_u^{(3)} = \begin{pmatrix} u^{(3)} \\ (u^{c(3^*)})^* \end{pmatrix} = \begin{pmatrix} u_L^{(3)} \\ u_R^{(3)} \end{pmatrix}$$

Then the h.c. of Ψ(x γ^0) is also a Dirac spinor in the (3*):

$$\bar{\Psi}_u^{(3^*)} = \left(u^{c(3^*)}, (u^{(3)})^*\right) = \left(\bar{u}_R^{(3^*)}, \bar{u}_L^{(3^*)}\right)$$

from which we can form either scalar and psedoscalar:

$$\bar{\Psi}_u(1,\gamma_5)\Psi_u = u^{c(3^*)}u^{(3)} \pm (u^{(3)})^*(u^{c(3^*)})^*$$

or vector and axial-vector gauge-invariants:

$$\bar{\Psi}_u(\gamma^\mu,\gamma^\mu\gamma_5)\Psi_u = (u^{(3)})^*u^{(3)} \pm u^{c(3^*)}(u^{c(3^*)})^*$$

In the latter each fermion couples to its antiparticle

3. Classical QED & QCD

3.1 Action principle, Lagrangian

All physical systems we know can be specified, both at the classical and at the quantum level, in terms of an « Action » S, a function of the dof of the system at all times. In FT it's a « functional » of the $\phi_i(x)$ i.e.

$$S\ (\phi_i(x)) \text{ is a (real) number for a given set of } \phi_i(x)$$

The classical (Euler-Lagrange) field equations are obtained by extremizing S (subject to some contraints): $\delta S(\phi_i(x))/\delta\phi_i(x) = 0$ for all i, x.

If we want our theory to respect some symmetries, S has to be invariant under the transformations that define that symmetry. In our case we want our symmetries to include $P \times G$

P forces S to be of the form: $S = \int d^4x\, L(\phi_i(x))$ where L is a scalar density, i.e. transforms as a scalar field under P (d^4x is itself invariant), but typically is a product of elementary fields and their derivatives.

3.2 Imposing the gauge symmetry

Invariance under *G* turns out to be very restrictive. Let us just consider the examples of QED and QCD. In QED the field strength tensor:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = -F_{\nu\mu} \Rightarrow (\vec{E}, \vec{B})$$

is invariant under the (abelian) gauge transformation:

$$A_\mu \longrightarrow A_\mu + \partial_\mu \varepsilon$$

To add J=1/2 Dirac fermions in a *G*-invariant way we have to replace normal derivatives by covariant derivatives:

$$\partial_\mu \longrightarrow D_\mu = \partial_\mu - iqA_\mu \longrightarrow \partial_\mu + ieA_\mu$$

so that both Ψ and its cov.derivative transform under *G* by picking up the phase factor exp(iqε(x)). This leads to the famous QED action:

$$S^{(QED)} = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu} + \int d^4x\, [\bar{\Psi} i\gamma^\mu D_\mu \Psi - m_e \bar{\Psi}\Psi]$$

The QCD lagrangian is qualitatively very similar

$$S^{(QCD)} = -\frac{1}{4}\int d^4x\, F^a_{\mu\nu} F^{a,\mu\nu} + \int d^4x\, [\bar{\Psi}_i\, \gamma^\mu (iD_\mu)^i_j \Psi^j - \bar{\Psi}_i m_q \Psi^i]$$

where: $$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_s f^a_{bc} A^b_\mu A^c_\nu = -F^a_{\nu\mu}$$

$$(D_\mu)^i_j = \partial_\mu \delta^i_j + g_s\, A^a_\mu (T^a)^i_j \qquad [T_a, T_b] = i f^c_{ab} T_c$$

Thus f^c_{ab} are the structure constants of the gauge group (here SU(3)).
(NB: while the T_a depend on the rep. the structure constants do not)
$S^{(QCD)}$ is invariant wrt SU(3) gauge transformations under which Ψ (and its cov.derivative) is rotated by the x-dependent SU(3) matrix
$U^i{}_j = \exp(ig\varepsilon^a(x)T^a)^i{}_j$ and the gauge field undergoes the appropriate generalization of the abelian gauge transformation:

$$T^a A^a_\mu \to U(T^a A^a_\mu)U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger$$

which is also rep. independent…

3.3 Comments

Let us write down the two lagrangians, actions:

$$S^{(QED)} = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu} + \int d^4x\, [\bar{\Psi} i\gamma^\mu D_\mu \Psi - m_e \bar{\Psi}\Psi]$$

$$S^{(QCD)} = -\frac{1}{4}\int d^4x\, F^a_{\mu\nu}F^{a,\mu\nu} + \int d^4x\, [\bar{\Psi}_i\, \gamma^\mu (iD_\mu)^i_j \Psi^j - \bar{\Psi}_i m_q \Psi^i]$$

1. They look so similar and yet the physics they imply is so different!
2. Main difference: photons are not self-coupled. Gluons are, since $S^{(QCD)}$ contains terms with three or four $A^a_\mu(x)$'s (and coupling g, g^2)
3. => A non-linear interacting theory even in the absence of quarks (YM theory): this gluon-self interaction will make all the difference…

4. Quantum corrections and renormalization of the gauge couplings

The full quantum theory follows from the classical action through a Feynman path-integral:

$$A(i \to f) \sim \int_{\phi_{in}}^{\phi_{fin}} d[\phi(x)] exp(iS(\phi)/\hbar)$$

Since the classical field equations are $\delta S(\phi(x))/\delta\phi(x) = 0$, they correspond to saddle points of the functional integral: they provide the semi-classical approximation to the full quantum theory.

4.1 Tree-level calculations

The classical action allows one to compute several quantities of the quantum theory in the so-called semiclassical (or tree-level) approximation. For the process i -> f the calculation proceeds as follows:

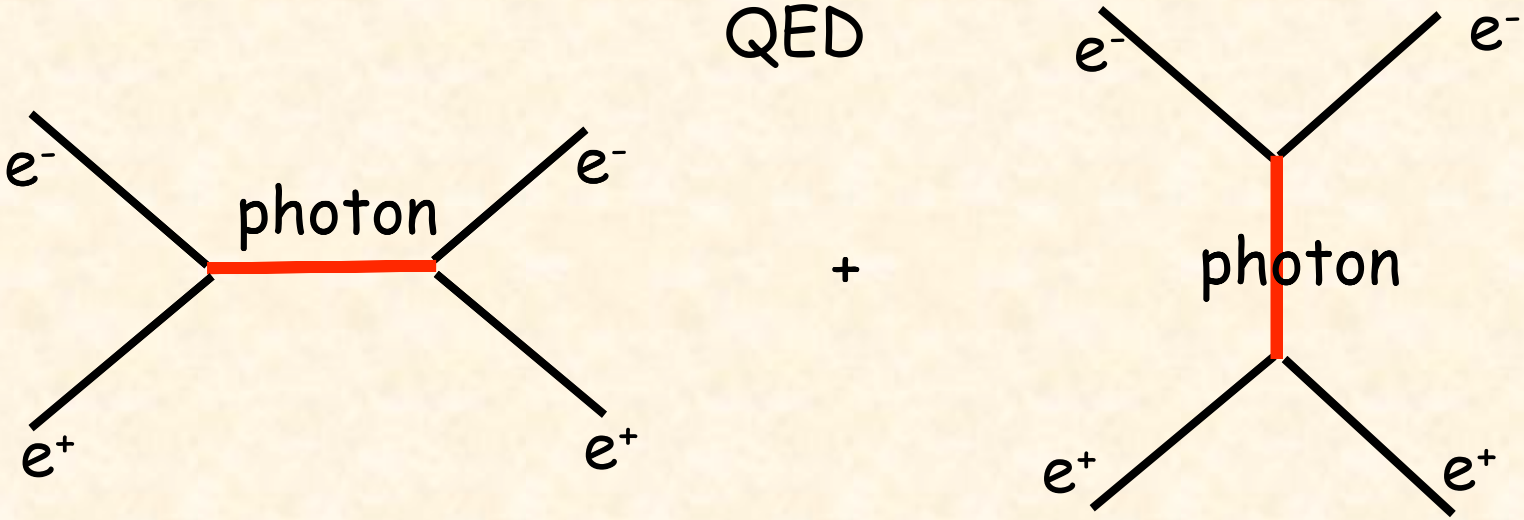
a) Draw all tree diagrams with i and f as external legs

b) Associate to each diagram an amplitude (=complex number) using some simple (Feynman) rules derived from the classical Lagrangian (after choosing a gauge, but physical amplitudes will be gauge-indep.)

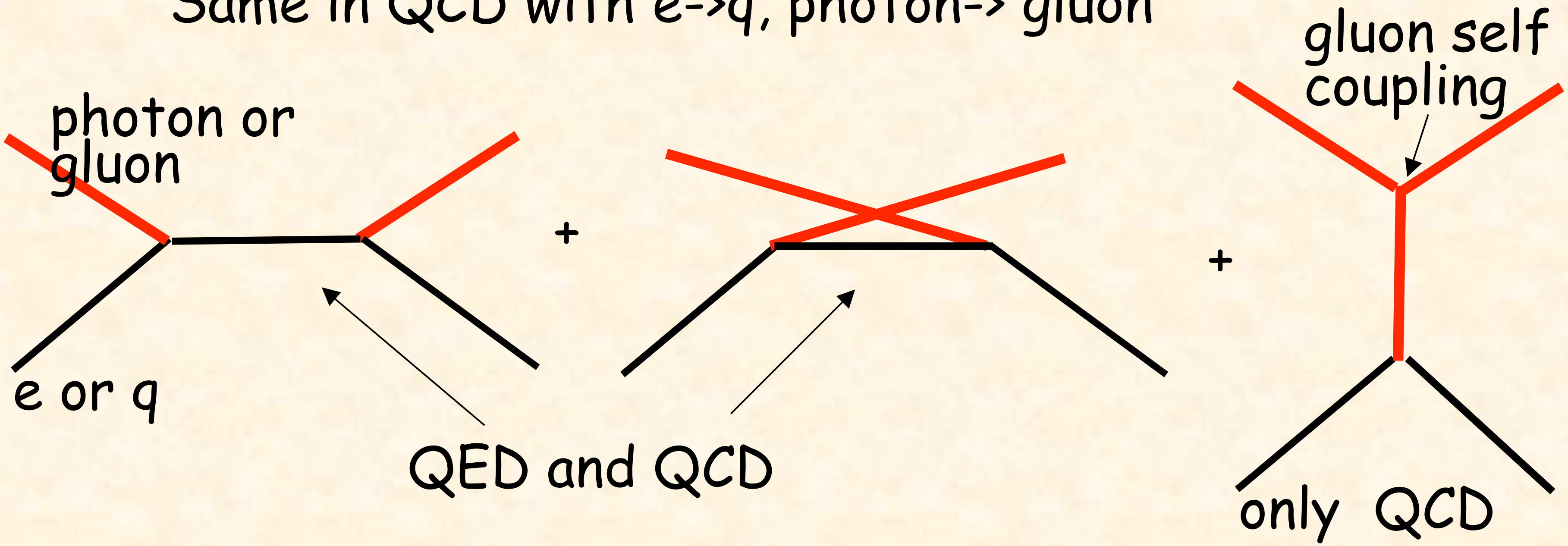
c) Sum the contribution of each diagram and **then** take the absolute square to compute the probability for the process (this is how quantum interference is included)

At this level, the difference between QED and QCD looks more quantitative than qualitative (more diagrams more channels). However, the semiclassical approximation is not enough: it violates, for instance, unitarity (Σ prob. =1)

Examples:



Same in QCD with e->q, photon-> gluon



4.2 Adding loops and the effective action

Unitarity is nicely restored (order by order in h) by adding « loop-corrections » that correspond to paths that are not quite the classical ones. A very useful way to discuss such loop corrections is the one that goes under the name of the effective action. In other words the effect of loops is encoded in the replacement:

$$S_{cl} \Rightarrow S_{eff}$$

After which one computes again the physical quantities by the same tree-diagram rules but using the «effective Feynman rules » that follow from S_{eff} . But what do we know about S_{eff}?

First: the loop-expansion is an expansion in h:

$$S_{eff} = S_{cl} + O(h) + O(h^2) + \ldots + O(e^{-1/h}) =$$

$$\text{tree} + \text{1-loop} + \text{2 loops} + \ldots + \text{non-perturbative}$$

Second: loop effects contain integrals over unrestricted « internal momenta » (for the path-integral viewpoint over arbitrarily fast oscillations in $\phi(x)-\phi_{cl}(x)$). Such integrals can (and usually do) diverge, a divergence related to the infinitely many degrees of freedom of a QFT

One can take three possible attitudes towards these infinities:

1. Pessimistic: Stop doing QFT (Cf. Lev Landau (~1955): the Lagrangian is dead, it should be buried, with all due honours of course)
2. Old attitude: handle the infinities by cancelling them against the original parameters in the classical action (which must therefore be also infinite) and by fitting their finite sum to experiments. Insist on removing the cutoff.
3. New attitude: admit our ignorance about physics above a certain large but finite energy scale M and simply assume that the «true» physics above M is able to cut-off the divergent integrals (without spoiling some sacred principles, such as gauge invariance). Check then what happens for processes at E« M and how much our predictivity is lost because of our UV ignorance.

The last two attitudes lead to the following distinction among QFT's:

Renormalizable theories are those in which the infinities (or if you prefer a strong dependence upon M) only occur in a finite number of terms in S_{eff} ,indeed only in those already present in S_{cl}. Other terms have a finite limit as M goes to infinity (with corrections that vanish as powers of 1/M and that are negligible at E<<M)

Non-renormalizable theories are those in which more and more terms in S_{eff} blow up as we increase the order in h (in the loop expansion)

For today we will just discuss in a simplified way one of these infinities, the one related to the coupling constants of QED and QCD (next week I will give a more complete and precise treatment)

4.3 Renormalization of gauge couplings in QED & QCD

For this purpose it is more convenient to rescale the gauge fields, $A_\mu(x)$, $A^a_\mu(x)$ by:

$$\hat{A}_\mu = e_0 A_\mu \, , \hat{A}^a_\mu = g_0 A^a_\mu$$

and rewrite S_{cl} as:

$$S_{Cl} = -\frac{1}{16\pi\alpha_0}\int d^4x\, \hat{F}^a_{\mu\nu}\hat{F}^{a,\mu\nu} + \int d^4x\, \bar{\Psi}(i\gamma^\mu \hat{D}_\mu - m)\Psi$$

and the coupling constant only appears as an overall factor

$$\alpha_0 \equiv \frac{e_0^2}{4\pi}, \frac{g_0^2}{4\pi}$$

At one-loop level one finds

$$S_{eff}^{1-loop} = -\frac{1}{16\pi}\int d^4x \left(\frac{1}{\alpha_0} + L_3(-\frac{\Box}{M^2}, \frac{m}{M})\right)\hat{F}^a_{\mu\nu}\hat{F}^{a,\mu\nu} + \dots$$

$$S_{eff}^{1-loop} = -\frac{1}{16\pi}\int d^4x \left(\frac{1}{\alpha_0} + L_3(-\frac{\Box}{M^2}, \frac{m}{M})\right)\hat{F}^a_{\mu\nu}\hat{F}^{a,\mu\nu} + \ldots$$

where the correction is both non-local and divergent. By F.T.

$$L_3(-\frac{\Box}{M^2}, \frac{m}{M}) \Rightarrow L_3(\frac{q^2}{M^2}, \frac{m}{M})$$

For $m^2 << q^2 << M^2$ one finds: $L_3 = \beta_0 \ln(M^2/q^2)$ where

$$\beta_0^{(QED)} = +1/3\pi\, n_l > 0$$
$$\beta_0^{(QCD)} = +1/6\pi\, (n_f - 33/2) < 0 \text{ for } n_f < 16,$$

where n_l is the number of e-like leptons and n_f the number of quark flavours. Above formulae suggest defining

$$\alpha_{eff}^{-1}(q^2) = \alpha_0^{-1} + \beta_0 ln\left(\frac{M^2}{q^2}\right) = \beta_0 ln\left(\frac{M^2 e^{\frac{1}{\beta_0\alpha_0}}}{q^2}\right)$$

(In QED we can also go to $q^2 << m^2 << M^2$: $L_3 = \beta_0 \ln(M^2/m_e^2)$)

The minus sign appearing in:

$$\beta_0^{(QCD)} = 1/6\pi\,(n_f - 33/2)$$

is of course the big news (and was worth this year's Nobel prize!). Before it was found by explicit calculations, people had some arguments that the sign of β_0 had to be positive. But actually the arguments break down for the gluonic contribution to β_0 (it is rock-solid, instead, for the fermionic one).

Can we understand the origin of this minus sign physically?

Y. Dokshitzer will give some hints in this direction in his seminar later this morning…

5. Infrared triviality vs. Asymptotic Freedom

Let us consider the consequences of this last formula:

$$\alpha_{eff}^{-1}(q^2) = \alpha_0^{-1} + \beta_0 ln\left(\frac{M^2}{q^2}\right) = \beta_0 ln\left(\frac{M^2 e^{\frac{1}{\beta_0\alpha_0}}}{q^2}\right)$$

for the two possible signs of β_0 assuming $\alpha_0 < 1/|\beta_0|$

$\beta_0 > 0$ (QED or n_f>16):

1. α_0 is the physical coupling at $q^2 \sim M^2$
2. $\alpha_{eff} < \alpha_0$ at $q^2 < M^2$
3. α_{eff} bows up at $q^2 = M^2 \exp(1/\alpha_0 \beta_0) = M_{LP}^2 >> M^2$ where we can trust neither one loop nor QFT. Since we know that $\alpha_{eff}(q^2=0) \sim 1/137$ we find (replacing q^2 by m_e^2 , see prev. page)

$$M_{LP} \sim m_e \exp(137/2\beta_0) = m_e \exp(411\pi/2n_l) >> M_{Pl}$$

Q: Why was L. Landau so worried?

$$\alpha_{eff}^{-1}(q^2) = \alpha_0^{-1} + \beta_0 ln\left(\frac{M^2}{q^2}\right) = \beta_0 ln\left(\frac{M^2 e^{\frac{1}{\beta_0\alpha_0}}}{q^2}\right)$$

$\beta_0 < 0$ (QCD with $n_f < 16$):

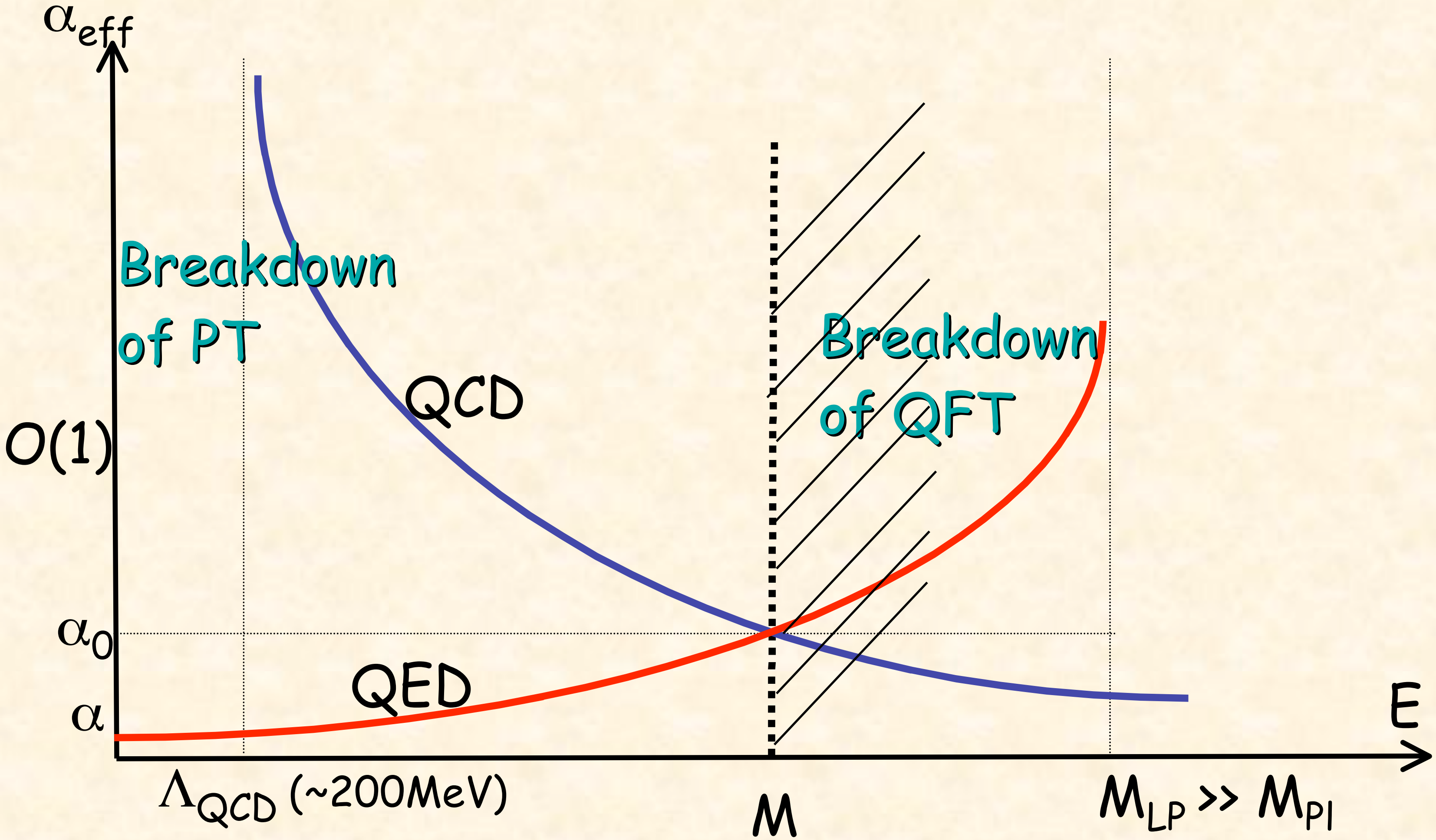
1. α_0 is the physical coupling at $q^2 \sim M^2$
2. $\alpha_{eff} > \alpha_0$ at $q^2 < M^2$
3. α_{eff} bows up at $q^2 = M^2 \exp(1/\alpha_0\, \beta_0) = \Lambda_{QCD}{}^2 \ll M^2$ where we cannot trust one loop but we should trust QCD.

We can observe here the phenomenon of «dimensional transmutation». We started with a dimensionless coupling, α_0 , and ended up with a dimensionful parameter, Λ_{QCD} , with:

$$\alpha_s(q^2) = \frac{1}{|\beta_0| ln\left(\frac{q^2}{\Lambda^2_{QCD}}\right)} = \frac{12\pi}{(33 - 2n_f) ln\left(\frac{q^2}{\Lambda^2_{QCD}}\right)}$$

where I switched to standard notations for the QCD coupling constant

The situation for QED and QCD can be summarized in a graph



If we keep α_0 small we cannot remove M => triviality of QED

If we work at $q^2 >> \Lambda_{QCD}{}^2$ we may hope to use PT for QCD (AF)

Unfortunately, even at high E, life is not that easy: the reason, as we shall see next time (with a preview in the seminar later today), is the possible presence of so-called infrared (and mass) singularities whose existence, physical meaning, and treatment is well known from the early days of QED…

Rappel

Vendredi 11/03, ici à 11h, séminaire de A. Czernecki

« Precision tests of QED and determination of fundamental constants »