Particules Élémentaires, Gravitation et Cosmologie

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II: aspects non-perturbatifs

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Théories de jauge sur le réseau: une introduction

- Discretization: the general idea
- Discretization and symmetries
- Hamiltonian vs. Lagrangian formulation
- Links, plaquettes, Wilson action
- Gauge invariant observables, Wilson loop
- Confinement criterions

Discretization: the general idea

- Since all analytic methods have failed so far to provide an solution of QCD, even at large-N, we are forced to try brute-force numerical methods.
- On a computer we have to replace the infinite number of d.o.f. by a large but finite number of them (common to other areas of physics..)
- In QFTs the number of d.o.f. is $(\text{infinity})^2$:
  1. space-time is infinitely large;
  2. there are infinitely-many points even in a finite region.

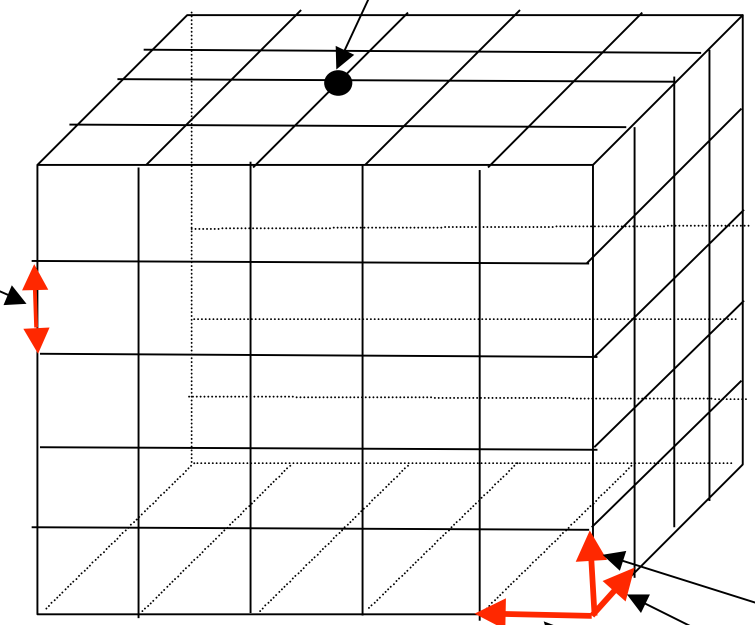
➡ We have to work in a finite volume and discretize space (-time). This is what lattice-QCD does

If we replace the whole of space(-time) by a finite, periodic grid, our generic field $\phi(x)$ of the continuum theory becomes $\phi(n_1,n_2,n_3,n_4\,(t))$ with some identification: ($n_1 = n_1 + N_1$ etc.)

The (hyper)cube is actually a torus!

$(n_1,n_2,n_3) = (3,2,5)$

This size, called the lattice size, is denoted by a



Eucl. time ?

$$\phi(x) \rightarrow \phi(n_1,n_2,n_3,n_4) \Leftrightarrow \phi(n_1\vec{e}_1 + n_2\vec{e}_2 + n_3\vec{e}_3 + n_4\vec{e}_4)$$

Discretization and symmetries

- When we discretize space(-time) certain symmetries of the continuum theory are necessarily lost (continuous traslations, rotations, Lorentz transformations)
- We should demand that they are recovered, to better and better accuracy, in a suitably defined «continuum limit»
- On the other hand, we should make sure that the «sacred» local symmetries are not broken by the discretization, otherwise we introduce spurious d.o.f. that are very hard to eliminate even in the continuum limit (= basic idea of K.Wilson, 1974)

Hamiltonian vs. Lagrangian LGT

- In the Hamiltonian approach only space is discretized: time is kept continuous. One writes down a lattice-Hamiltonian and tries to solve numerically for eigenvalues, eigenvectors, time evolution etc.
- In the Lagrangian approach one first goes over to so-called Euclidean time ($t \to \tau = -it$) so that the metric becomes Euclidean and the action purely imaginary ($S \to S_E = i\,S$). Next, one discretizes this 4D-space (usually with different n's in space and time directions)
- One then carries out Feynman's path integral where each path is weighted by $\exp(iS) \Rightarrow \exp(-S_E)$. For QCD S_E is positive semidefinite, making the integrals converge.
- Finally one reinterprets the results in Minkowskian time

Links, plaquettes & their continuum analogues

In the continuum we have introduced the concept of a covariant derivative ⟶ $(D_\mu)^i_j = \partial_\mu \delta^i_j + ig\, A^a_\mu (T^a)^i_j$

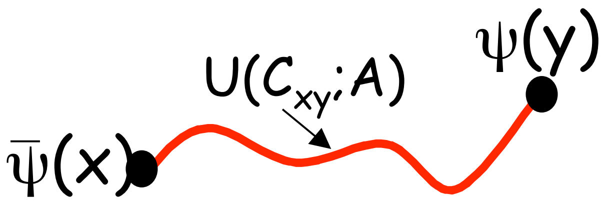
so that ψ and $D_\mu\psi$ have the same gauge transformation (T^a are the generators in rep. of ψ). On the lattice derivatives become finite differences: we need to make objects like $\bar{\psi}(x)\psi(y)$ gauge invariant by using the gauge field. Same question can be asked in the continuum where the solution is well known:

$$\bar{\psi}(x)\psi(y) \to \bar{\psi}(x) U(C_{xy}; A_\mu) \psi(y)$$

where

$$U(C_{xy}; A_\mu) = Pexp\left(ig \int_{C_{xy}} A^a_\mu T^a dx^\mu \right)$$

C_{xy} is a path from x to y, P denotes path ordering. Pictorially, it's like a string connecting a quark at y to an antiquark at x

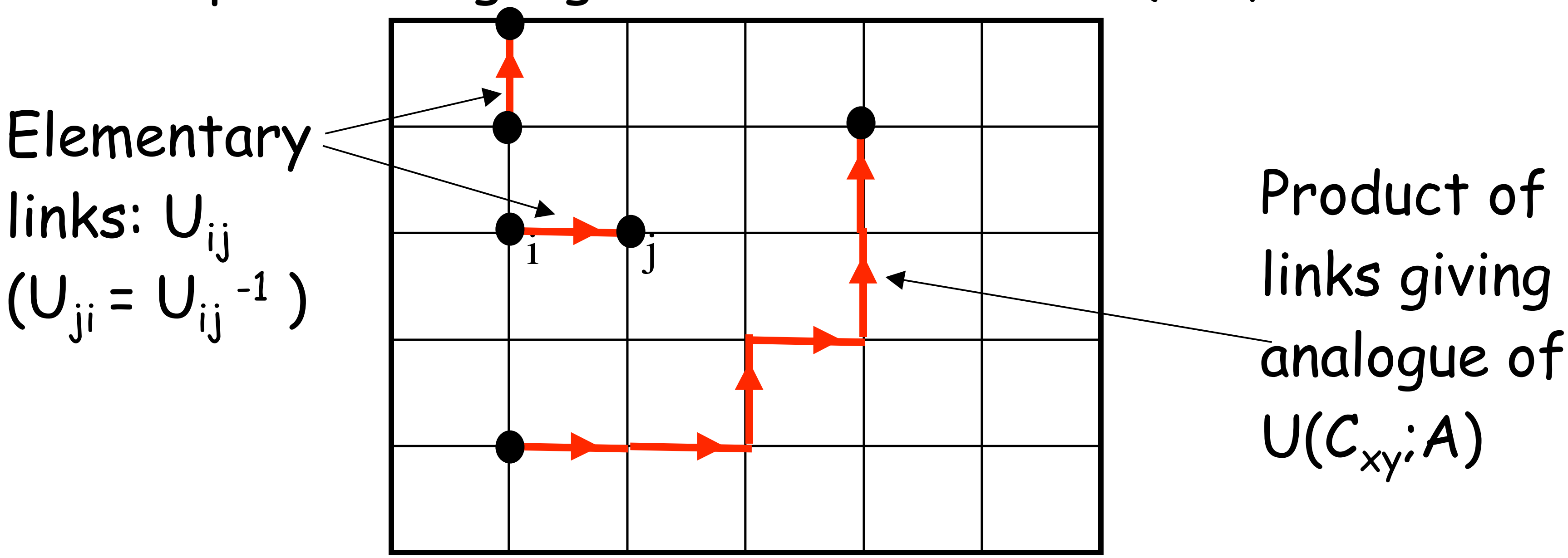


Under a gauge transformation G:

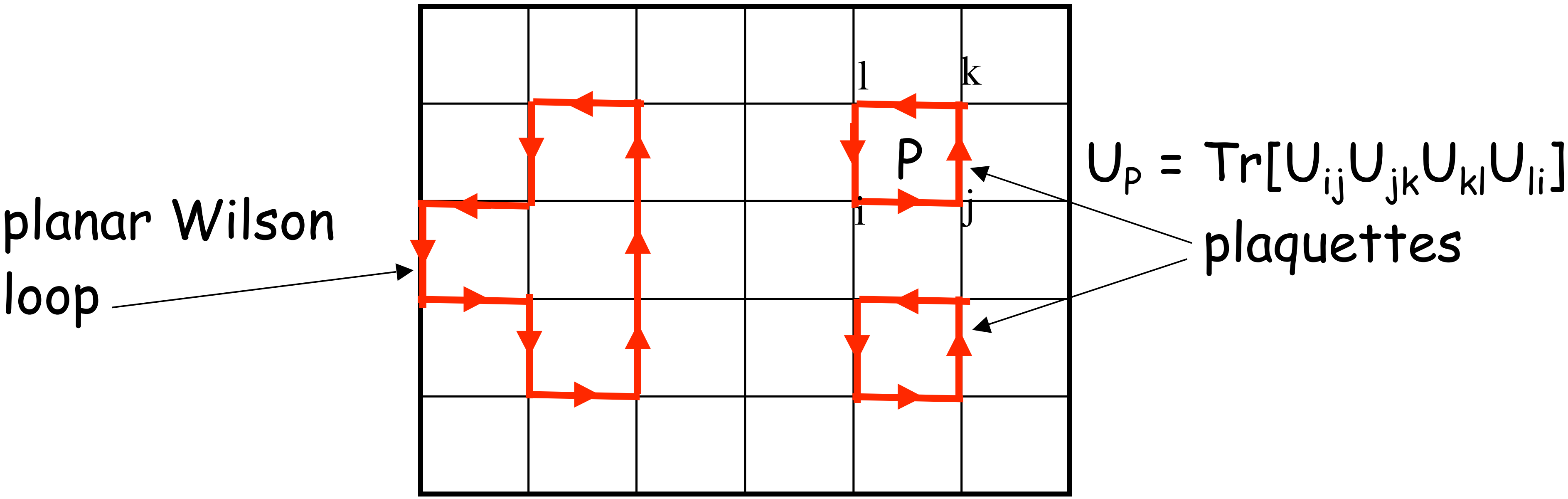
$\psi(y) \dashrightarrow G_y\psi(y)$, $\bar{\psi}(x) \dashrightarrow \bar{\psi}(x)G_x^{-1}$

$U \dashrightarrow G_x U\, G_y^{-1}$, $\bar{\psi}(x)U\psi(y)$ invariant

A discretized version of these $U(C_{xy};A_\mu)$ can be defined on the lattice. When x and y are nearest neighbours and one chooses the shortest C_{xy}, they are called link variables and can be used to replace the gauge fields themselves (w/ μ=direction ij)



It is straightforward to construct gauge-invariant quantities out of links. A product of links along a continuous path transforms as a gauge transformation at one end and its inverse at the other end. If the two ends coincide and we take a trace, we get a gauge-invariant quantity called a Wilson loop. The simplest example is the so-called plaquette, made of four links on a plane: it can be used to construct a lattice analogue of the gauge kinetic term (gauge action) of the continuum



The Wilson action

Since the plaquette is the simplest gauge invariant operator it is not surprising that, in the limit in which we send the lattice spacing to zero, it reduces to something proportional to the YM Lagrangian. More precisely, one finds (in SU(N)):

$$Tr\left[U_P + U_P^\dagger\right] \to 2N - \frac{g^2}{4} a^4 F^a_{\mu\nu} F^a_{\mu\nu} \quad \text{(no sum over } \mu,\nu\text{)}$$

Thus: $$\frac{2N}{g^2} \sum_P \left(1 - \frac{1}{2N} Tr\left[U_P + U_P^\dagger\right]\right) \to \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}$$

This is the celebrated Wilson action.

$$S_W(U_{ij}) = \frac{2N}{g^2}\sum_P\left(1 - \frac{1}{2N}Tr\left[U_P + U_P^\dagger\right]\right) =$$

$$= -\frac{1}{g^2}\sum_P\left(Tr\left[U_P + U_P^\dagger\right] - 2N\right)$$

It contains a parameter, $1/g^2$ that appears like the inverse temperature β in a stat. mech. system!

The choice of the action is not unique. Many other lattice expressions go to the usual continuum action in the small-a limit. The choice is only a matter of simplicity and/or convenience (see below)

Gauge-invariant observables, Wilson loops

For any given gauge invariant operator $O_n(U)$ the VEV will be given by the Feynman path integral (now a real average!)

$$\langle O_n(U) \rangle = Z^{-1} \int \prod_{ij} dU_{ij}\, O_n(U_{ij}) e^{-S_W(U_{ij})}$$

NB: division by Z removes disconnected «vacuum» diagrams

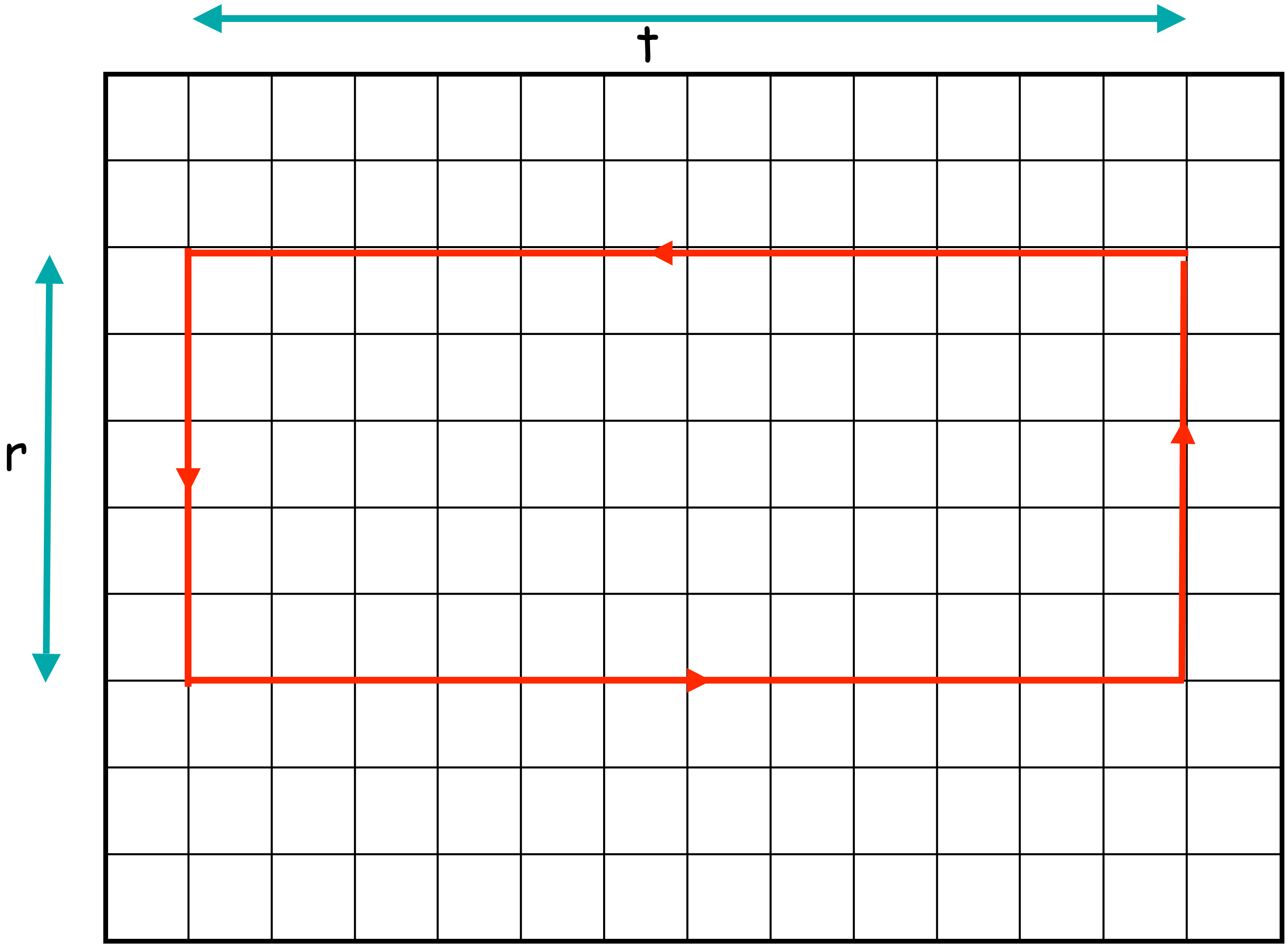
$$Z = \int \prod_{ij} dU_{ij} e^{-S_W(U_{ij})}$$

A complete set of such operators (in the absence of quarks) is provided by the Wilson loops and products thereof:

$<W(C)> = <\mathrm{Tr}\,(U_1 U_2 .. U_n)_C>$; $<W(C_1)\, W(C_2) .. W(C_m)> =$

$< W(C_1) > < W(C_2) > .. + <W(C_1)\, W(C_2)>< .. > + .. + < ... >_{\mathrm{conn.}}$

Physical interpretation of the Wilson loop (when large)



<W(C)> gives exp(-V(r) t) = order parameter for confinement

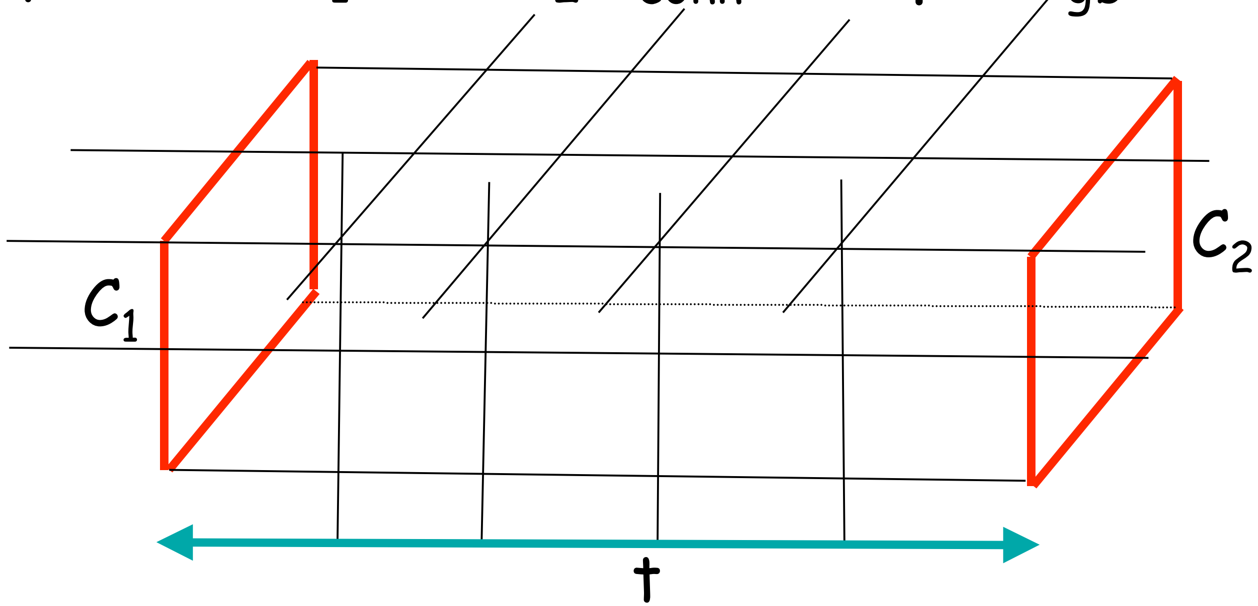
Here V(r) corresponds to the static potential of two heavy sources (e.g. two heavy quarks) in some given rep. of the gauge group (the one appearing in the def. of W(C)). Two cases:

1. If V(r) --> 0 at large r, W(C) ~ exp(-c P), where P is the length/perimeter of C. This is the case of no confinement (believed to be realized in QED)

2. If V(r) ~ T r at large r, W(C) ~ exp(-TA) where A is the area encircled by C and T is the so-called string tension. Such a linearly rising potential obviously confines (we would like to prove this to be the case for QCD)

Similarly, the connected correlators of Wilson loops give crucial information about «glueball» masses and interactions. If colour is confined, there should be a non-perturbative mass gap of order Λ_{QCD}, while in perturbation theory the massless gluons give a continuum of states extending all the way down to E=0.

Example: $\langle W(C_1)\, W(C_2)\rangle_{conn} \sim \exp(-m_{gb} t)$ @ large t



Weak and strong coupling expansions

Starting from Wilson's action we can consider two limits where we can perform analytic expansions/calculations :

1. Weak-coupling expansion

Very similar to the loop expansion (perturbation theory) of the continuum. Actually more complicated, since it does not respect many of the continuum symmetries. Its main use is that it allows to connect the lattice parameters to those in the continuum

2. Strong coupling expansion

This has no continuum analogue. The surprise is that it automatically leads to colour confinement (even for a U(1) gauge symmetry!). It may contain lattice artefacts..

The MonteCarlo method

- The real thing is neither the strong-coupling nor the weak coupling expansion. One would like to perform the actual integrals. This can only be done numerically.
- However, since the integrals are over an enormous number of variables, they cannot be done by standard numerical integration techniques either
- The MonteCarlo (MC) method replaces more conventional techniques by a sampling of the integrand (including the exp(-S) factor) at some «randomly» chosen «points». If the sample is well chosen one gets a good estimate of the integral. There are techniques to accelerate convergence (see L. Giusti's seminars for more details). This is how calculations are actually done.