Particules Elémentaires, Gravitation et Cosmologie
Année 2008-'09

Gravitation et Cosmologie: le Modèle Standard

Cours 6: 30 janvier 2009

Dynamique de l'expansion

- Explicit solutions with K=0, w = constant
- Solutions with K≠0 and different sources
- Luminosity distance and deceleration parameter
- Generalized Hubble law and evidence for cosmic acceleration

Solutions with K=0 and w=p/ρ = const.

Equations become $$H^2 = \frac{8\pi G}{3}\rho \ ; \ \dot{\rho} = -3H\rho(1+w)$$

which also imply: $$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho(1+3w)$$

Energy cons. eqn. gives $$\rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)}$$

Defining $$x = \frac{a}{a_0} = \frac{1}{1+z}$$ the H constraint gives:

$$\left(\frac{1}{x}\frac{dx}{dt}\right)^2 = H_0^2 \ x^{-3(1+w)}$$ hence the explicit (+) solution:

$$H_0(t_0 - t) = \int_x^1 dy \ y^{-1+\frac{3(1+w)}{2}} = \frac{2}{3(1+w)}\left(1 - x^{\frac{3(1+w)}{2}}\right)$$

$$H_0(t_0 - t) = \int_x^1 dy\ y^{-1+\frac{3(1+w)}{2}} = \frac{2}{3(1+w)}\left(1 - x^{\frac{3(1+w)}{2}}\right)$$

Note that a(t*)=x(t*)=0, with $(t_0 - t*) = \frac{2}{3(1+w)} H_0^{-1}$

Redefining the origin of time by t* =0 we can rewrite sln as:

$$x = \frac{a}{a_0} = \left(\frac{3(1+w)}{2} H_0 t\right)^{\frac{2}{3(1+w)}} = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}$$

for w ≠-1,
while for w=-1

$$\rho = \rho_0\ ;\ \frac{a}{a_0} = e^{H_0(t-t_0)}\ ;\ H_0 = \sqrt{\frac{8\pi G \rho_0}{3}}$$

Except in this last case the geometry becomes singular (e.g. R diverges) at a finite proper time in our past: the Big Bang!
Easy to check that some invariants diverge for t-->0:

$$R = -6\left(H^2 + \frac{\ddot{a}}{a} + \frac{K}{a^2}\right)\ ;\ R_{\mu\nu}R^{\mu\nu} \sim 12\left(H^4 + (\frac{\ddot{a}}{a})^2 + \frac{\ddot{a}}{a}H^2\right)$$

Solutions with K≠0 and several sources

Recall $$H^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\sum_i \rho_i \qquad \dot{\rho}_i = -3H\rho_i(1+w_i)$$

$$\rho_i = \rho_{i,0}\left(\frac{a}{a_0}\right)^{-3(1+w_i)}$$

Using again

$$\rho^{(cr)} \equiv \frac{3H^2}{8\pi G} = \sum_i \rho_i + \rho_K \ ; \ \rho_K = -\frac{3K}{8\pi G a^2}$$

$$\Omega_i \equiv \frac{\rho_i}{\rho^{(cr)}} \qquad \sum_i \Omega_i = 1 \qquad x = \frac{a}{a_0} = \frac{1}{1+z}$$

we get

$$\left(\frac{1}{x}\frac{dx}{dt}\right)^2 = H_0^2\ \left(\Omega_{\Lambda,0} + \Omega_{K,0}x^{-2} + \Omega_{m,0}x^{-3} + \Omega_{r,0}x^{-4} + \ldots\right)$$

Identifying t=0 with a =0 (z= infinity) this gives:

$$H_0 t(z) = \int_0^{\frac{1}{1+z}} \frac{dx}{x}\ \left(\Omega_{\Lambda,0} + \Omega_{K,0}x^{-2} + \Omega_{m,0}x^{-3} + \Omega_{r,0}x^{-4} + \ldots\right)^{-1/2}$$

$$H_0 t(z) = \int_0^{\frac{1}{1+z}} \frac{dx}{x} \frac{1}{\sqrt{\Omega_{\Lambda,0} + \Omega_{K,0} x^{-2} + \Omega_{m,0} x^{-3} + \Omega_{r,0} x^{-4}}}$$

Note that this integral converges at x=0 (unless there is nothing but a cosmological constant)

It gives the time of emission of light arriving to us today with redshift z.

If we set z=0 we get the age of the Universe, i.e. the proper time elapsed since a=0.

$$t_0 = \frac{1}{H_0} \int_0^{1} \frac{dx}{x} \frac{1}{\sqrt{\Omega_{\Lambda,0} + \Omega_{K,0} x^{-2} + \Omega_{m,0} x^{-3} + \Omega_{r,0} x^{-4}}}$$

remember: $$\sum_i \Omega_{i,0} = 1$$

$$t_0 = \frac{1}{H_0}\int_0^1 \frac{dx}{x}\;\frac{1}{\sqrt{\Omega_{\Lambda,0}+\Omega_{K,0}x^{-2}+\Omega_{m,0}x^{-3}+\Omega_{r,0}x^{-4}}}$$

$\Omega_{r,0}$ is very small ($O(10^{-4})$) and can be neglected. There are also upper bounds on $\Omega_{K,0}$ (making it small wrt $\Omega_{m,0}$)

In a Universe dominated by matter this gives:

$$t_0 = 2/3\ H_0^{-1} = 6.52 \times 10^9\ h^{-1}\ \text{yr},$$

which appears to be too short compared to the age of certain globular clusters in our galaxy (> 10^{10} yr).

A positive cosmological constant increases t_0. In the (presently favoured) CDMΛ model ($\Omega_{m,0} \sim 0.28$, $\Omega_{\Lambda,0} \sim 0.72$) we get a more confortable:

$$t_0 = 13.4 \pm 1 \times 10^9\ (0.7/h)\ \text{yr}.$$

Luminosity distance and deceleration parameter

Two interesting quantities in cosmology are the so-called luminosity distance d_L and deceleration parameter q_0. They are defined by the equations:

$$l = \frac{L}{4\pi d_L^2} \qquad\qquad q_0 \equiv -\frac{a\ddot{a}}{\dot{a}^2}(t = t_0)$$

where L(l) is the absolute (apparent) luminosity, say of a star or galaxy. The Hubble law, when expanded to second order in z, relates these parameters:

$$d_L(z) = H_0^{-1}\left[z + \frac{1}{2}(1 - q_0)z^2 + O(z^3)\right]$$

On the other hand: $q_0 = \frac{4\pi G(\rho_0 + 3p_o)}{3H_0^2} = \frac{1}{2}(\Omega_{m,0} - 2\Omega_{\Lambda,0})$

Hubble law beyond linear order => information about eq.of state!

General expression for Hubble's law

$$d_L(z) = a_0(1+z)f_K(I(z))\ ;\ f_0(y) = y\ ;\ f_1(y) = \sin y\ ;\ f_{-1}(y) = \sinh y$$

$$I(z) = \frac{1}{a_0 H_0}\int_{\frac{1}{1+z}}^{1} \frac{dx}{\sqrt{\Omega_{\Lambda,0}x^4 + \Omega_{K,0}x^2 + \Omega_{m,0}x + \Omega_{r,0}}}$$

To order z^2 we can approximate siny ~ sinh y ~ y, so that:

$$H_0 d_L(z) = (1+z)\int_{\frac{1}{1+z}}^{1} \frac{dx}{\sqrt{\Omega_{\Lambda,0}x^4 + \Omega_{K,0}x^2 + \Omega_{m,0}x}}$$

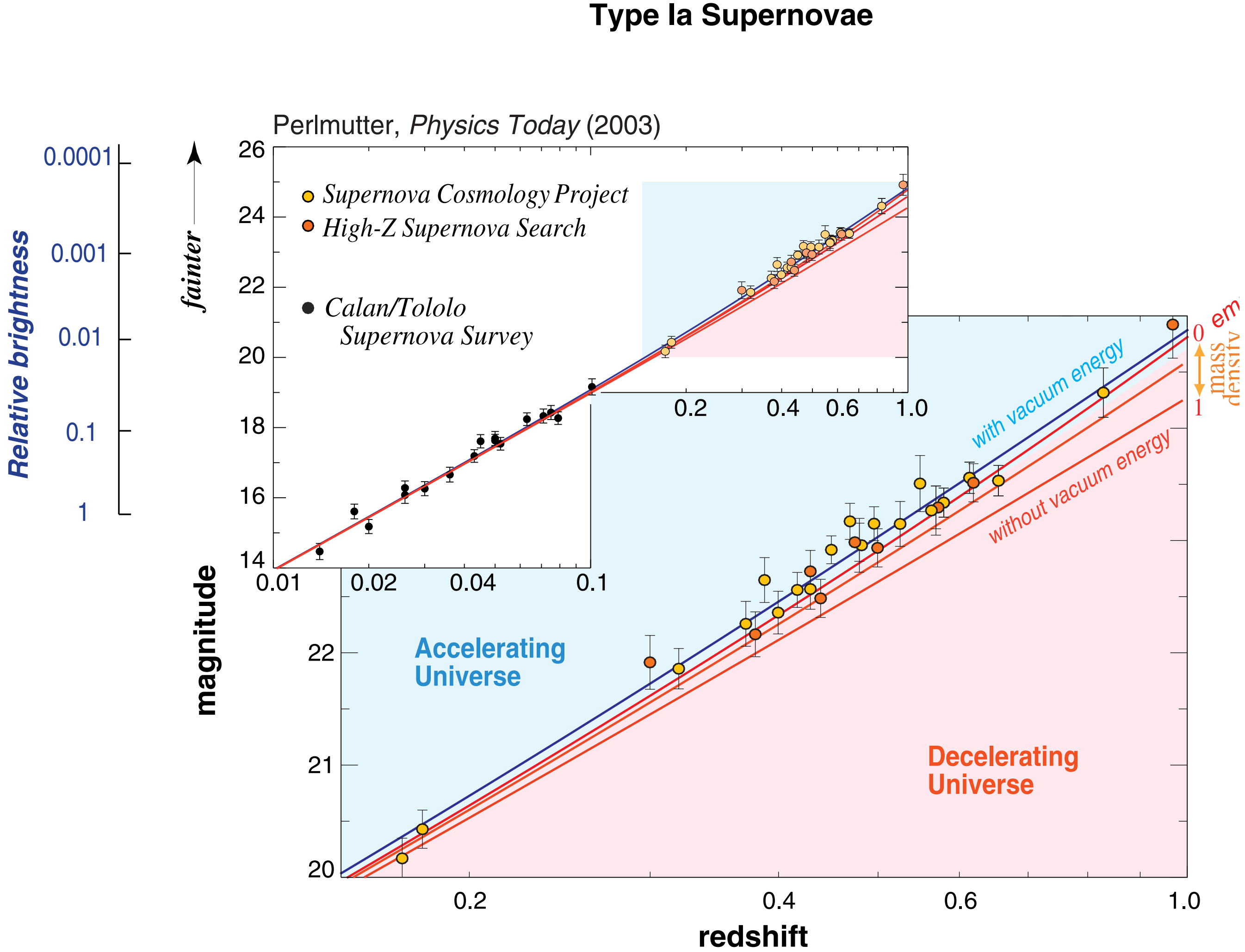
$$\sim (1+z)\int_{\frac{1}{1+z}}^{1} dx \left(1 + \frac{1}{2}(4\Omega_{\Lambda,0} + 2\Omega_{K,0} + \Omega_{m,0})(1-x)\right)$$

$$\sim (1+z)(1 - \frac{1}{1+z}) + \frac{1}{4}(4\Omega_{\Lambda,0} + 2\Omega_{K,0} + \Omega_{m,0})z^2$$

$$\sim z + \frac{1}{2}(1 + 2\Omega_{\Lambda,0} - \Omega_{m,0})z^2 = z + \frac{1}{2}(1-q_0)z^2$$

as previously written

This generalized Hubble law can be checked using Type Ia Supernovae as standard candles: evidence for negative q_0...



Will the Universe expand forever?

We can answer this question (within our model) by looking at:

$$H_0 t(z) = \int_0^{\frac{1}{1+z}} \frac{dx}{x} \frac{1}{\sqrt{\Omega_{\Lambda,0} + \Omega_{K,0} x^{-2} + \Omega_{m,0} x^{-3} + \Omega_{r,0} x^{-4}}}$$

If the quantity inside the √ vanishes at some $x=x^*=a^*/a_0 >1$ then this a^* represents the maximun value reached by the scale factor before the Universe collapses. Neglecting radiation (a very good approximation for $x>1$), this means having a root with $x > 1$ of the cubic equation:

$$\Omega_{\Lambda,0} x^3 + \Omega_{K,0} x + \Omega_{m,0} = 0 \ ; \ \text{for } x > 1 \ ; \ \text{with } \Omega_{\Lambda,0} + \Omega_{K,0} + \Omega_{m,0} = 1$$

Possible only for a sufficiently negative $\Omega_{K,0}$ ($K = +1$). This is excluded by the data (see graph), but what if DE is not just Λ?

Combining Supernovae, CMB and cluster formation models confirms large positive $\Omega_{\Lambda,0}$ (~0.7)