Particules Elémentaires, Gravitation et Cosmologie
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Gravitation et Cosmologie: le Modèle Standard

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Les équations cosmologiques d'Einstein

- The cosmological "principle"
- Cosmological metric and energy-momentum tensor
- The cosmological Einstein equations
- Equations of state and energy redshifts
- Einstein's real blunder

The cosmological principle

Soon after the proposal of GR (1915) physicists thought of applying it to the Universe as a whole. This is even more justified after having checked GR for isolated systems.
At first sight, however, the task looks like an impossible one, if compared, say, to the static, spherically symmetric case we have already discussed.
Some drastic approximations are clearly necessary.

The Universe is indeed a very complicated system, dominated by (almost) empty space, but also containing dense objects: clusters, galaxies, stars, planets.

Also, we do not know if and where the Universe ends...

Physicists were bold enough to make the assumption that, on average over very large regions, the Universe is the same at every point and in each direction. This assumption of homogeneity and isotropy is known as the

"Cosmological Principle"

The expansion of the Universe was not known at the time. Yet, even if the prevailing view was that the Universe was also the same at all times (i.e. static), the option of a time evolution was left open.

For consistency, the cosmological principle must apply separately to both sides of EE i.e. to:

1. The geometry of space-time;
2. The matter content of the Universe.

Let us see how this can be translated in mathematical terms.

The cosmological principle for the metric

One can show that, in suitable coordinates, the most general metric describing an isotropic, homogeneous Universe can be reduced to the following (Lemaître, Friedmann, Robertson, Walker, or FRW) form:

$$
\begin{aligned}
ds^2 &= g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \right] \\
d\Omega^2 &= d\theta^2 + sin^2\theta d\phi^2 \quad ; \quad K = 0, \pm 1
\end{aligned}
$$

Here r is a dimensionless radial coordinate while the "scale factor" a(t) has dimensions of length

K distinguishes spatially flat (K=0) from open (K=-1) and closed (K=1) Universes. Only the latter one has a finite volume: $V = 2\pi^2 a^3$ (for K=0, r can be given dimensions of length with a(t) dimensionless)

A few comments

The time t, usually called cosmic or proper time, is the one measured by a clock at rest in these coordinates;

Such a clock falls freely, since $\Gamma^i_{00} = 0$

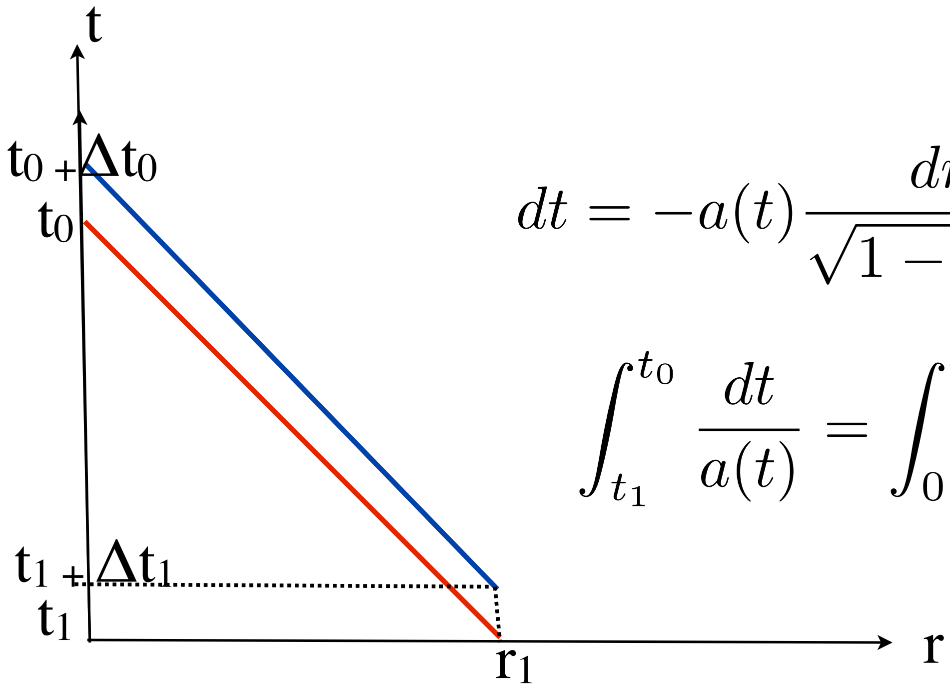
$$ds^2 = -dt^2 + g_{ij}dx^i dx^j = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2 d\Omega^2\right]$$

As we shall see the scale factor a(t) is directly related to the famous redshift (Hubble, 1929). A very important quantity in cosmology is the Hubble parameter: $H(t) \equiv \frac{\dot{a}}{a}$ (with an overdot = d/dt) Today ($t = t_0$) we have:

$$H(t_0) \equiv H_0 = 100\ h\ \text{km}\ s^{-1}\text{Mpc}^{-1} \quad ; \quad h \sim 0.72 \pm 0.05$$

$$H_0^{-1} = 9.778 \times 10^9\ h^{-1}\text{years}$$

The cosmological redshift



$$dt = -a(t)\frac{dr}{\sqrt{1-Kr^2}}$$

$$\frac{\Delta t_0}{a(t_0)} = \frac{\Delta t_1}{a(t_1)}$$

$$\int_{t_1}^{t_0}\frac{dt}{a(t)} = \int_0^{r_1}\frac{dr}{\sqrt{1-Kr^2}}$$

$$\nu_{em.} = \frac{a(t_0)}{a(t_1)}\nu_{rec.} \equiv (1+z)\nu_{rec.}$$

z (> 0 for exp. U.) is called the redshift

For nearby sources:

$$a(t_1) \sim a(t_0) - \dot{a}(t_0)(t_0 - t_1) = a(t_0)\left(1 - H_0(t_0 - t_1)\right)$$

giving Hubble's law: $z = H_0(t_0 - t_1) \sim H_0 d$

The l.h.s. of EE

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu}$$

$$ds^2 = -dt^2 + g_{ij} dx^i dx^j = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \right]$$

It is "easy" to compute the different components of the Ricci (or Einstein) tensor corresponding to the above metric:

$$R_{00} = 3\frac{\ddot{a}}{a} \ ; \ R_{0i} = 0 \ ; \ R_{ij} = -\left(2H^2 + \frac{\ddot{a}}{a} + 2\frac{K}{a^2}\right) g_{ij}$$

$$R = -6\left(H^2 + \frac{\ddot{a}}{a} + \frac{K}{a^2}\right)$$

$$G_{00} \equiv R_{00} - \frac{1}{2} g_{00} R = R_{00} + \frac{1}{2} R = -3\left(H^2 + \frac{K}{a^2}\right)$$

$$G_{0i} = 0 \ ; \ G_{ij} = \left(H^2 + 2\frac{\ddot{a}}{a} + \frac{K}{a^2}\right) g_{ij}$$

The cosmological principle for matter

For consistency, matter has to be distributed homogeneously and isotropically.

Assuming, for simplicity, matter to behave like a perfect fluid (this is also the case for field-theoretic models of matter), its energy-momentum tensor must take the following simple form (in our coordinate system):

$$T_{00} = \rho(t) \qquad ; \qquad T_{0i} = 0 \; ; \; T_{ij} = g_{ij}(t,r)\; p(t)$$

$$g_{ij}(t,r) dx^i dx^j \quad = \quad a^2(t) \left[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \right]$$

At this point it is straightforward to write down the cosmological version of Einstein's equations.

The cosmological Einstein Equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu}$$

$$T_{00} = \rho(t) \quad ; \quad T_{0i} = 0 \; ; \; T_{ij} = g_{ij}(t, r) \; p(t)$$

$$G_{00} \equiv R_{00} - \frac{1}{2} g_{00} R = R_{00} + \frac{1}{2} R = -3\left(H^2 + \frac{K}{a^2}\right)$$

$$G_{0i} = 0 \; ; \; G_{ij} = \left(H^2 + 2\frac{\ddot{a}}{a} + \frac{K}{a^2}\right) g_{ij}$$

The (0i) equation is 0=0; the (00) and (ij) equations give:

$$H^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho \; ; \; H^2 + \frac{K}{a^2} + 2\frac{\ddot{a}}{a} = -8\pi G p$$

implying: $$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

General comments on the CEEs

$$H^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho \; ; \; \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

imply: $\dot{\rho} = -3H(\rho + p)$ $\qquad H(t) \equiv \frac{\dot{a}}{a}$

1. Hamiltonian constraint: expansion as negative kinetic energy
2. Matter resists expansion, helps collapse (if $\rho+3p > 0$)
3. Energy conservation (dilution + work) is automatic (consequence of Bianchi identities)
4. Two equations for three unknowns?

The equation of state: examples

The missing equation is provided by the detailed model of matter through the so-called equation of state, typically a relation between p and ρ. If the equation of state is simply

$$p = w\ \rho \qquad \text{(with w a constant)}$$

one finds immediately: $\rho \sim a^{-3(1+w)}$

Important examples

- Cold (non relativistic) matter ("dust") $w = 0\ ;\ \rho \sim a^{-3}$
- Relativistic matter ("radiation") $w = 1/3\ ;\ \rho \sim a^{-4}\ ;\ (T^{\mu}_{\mu} = 0)$
- Cosm. const. ("vacuum energy") $w = -1\ ;\ \rho = \text{const.}\ ;\ (T_{\mu\nu} \sim g_{\mu\nu})$

A (minimally coupled) scalar field

An interesting and relevant example: Starting from:

$$S = -\int d^4x\sqrt{-g}\left(\frac{1}{2}\partial_\mu\phi\partial_\nu\phi g^{\mu\nu} + V(\phi)\right)$$

we find:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\partial_\rho\phi\;\partial^\rho\phi + V(\phi)\right)$$

If ϕ depends only on time:

$$\rho(t) \quad = \quad T_{00} = \frac{1}{2}\dot{\phi}^2 + V$$

$$g_{ij}\;p(t) \quad = \quad T_{ij} = g_{ij}(\frac{1}{2}\dot{\phi}^2 - V)$$

$$w = \frac{\dot{\phi}^2 - 2V}{\dot{\phi}^2 + 2V}$$

A dynamical eos: goes from "stiff matter" (w=1) when kinetic terms dominates to cosm. const. (w=-1) when the potential energy dominates. Inflation is much based on this example.

Critical density and fractions

Let us rewrite the Friedmann equation

$$H^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\sum_i \rho_i$$ in the form

$$\rho^{(cr)} \equiv \frac{3H^2}{8\pi G} = \sum_i \rho_i + \rho_K \ ; \ \rho_K = -\frac{3K}{8\pi G a^2}$$

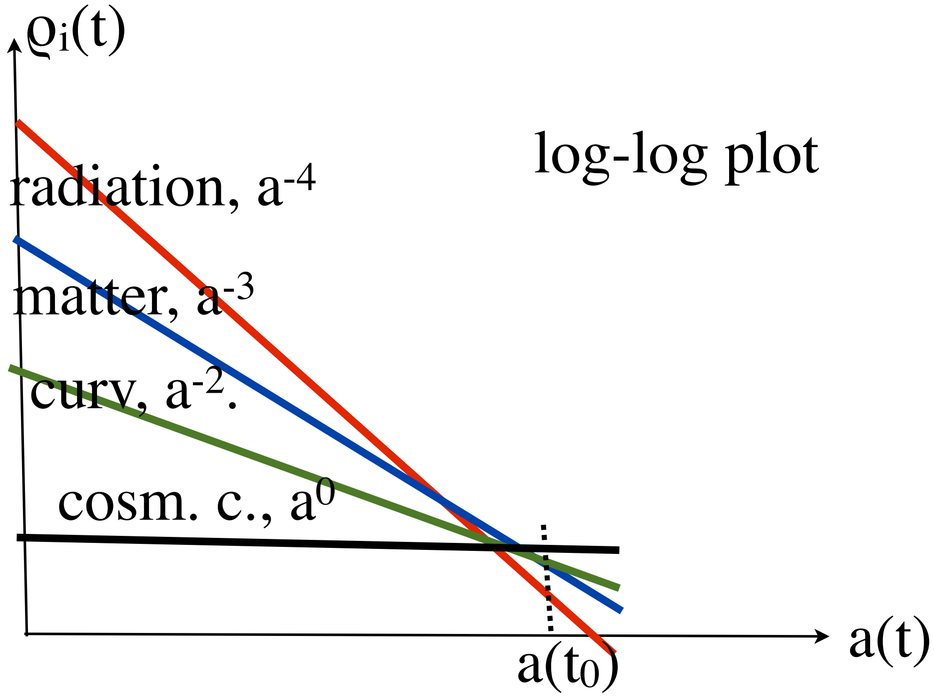
$$\rho_0^{(cr)} \equiv \frac{3H_0^2}{8\pi G} = 1.878 \times 10^{-29} h^2 \mathrm{gcm}^{-3} \qquad \dot{\rho}_i = -3H\rho_i(1+w_i)$$

introducing $\Omega_i \equiv \frac{\rho_i}{\rho^{(cr)}}$

the FE becomes a sum rule $\sum_{i \neq K} \Omega_i = 1 - \Omega_K$

that should hold at all times in spite of the huge variations of the various Ω_i. Also, if we could measure the l.h.s., we would know in which kind of U we are living!

Redshift of different ρ_i



Notes: there is only an upper limit on the curvature contribution. Radiation is small (Ω_r~10^{-4}). Matter and "dark energy" share most of the energy in a 1:2 ratio.

Puzzle: Why are curvature and c.c. not dominating today?

Einstein's real blunder

Friedmann's equations

$$H^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho \; ; \; \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

appear to imply that the Universe cannot be static (H=0). This motivated Einstein to introduce (1917) a "cosmological constant" term so that, for K= +1, the equations:

$$H^2 = \frac{8\pi G}{3}(\rho_m + \rho_\Lambda) - \frac{1}{a^2} \; ; \; \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho_m - 2\rho_\Lambda) \; ; \; \rho_\Lambda = \frac{\Lambda}{8\pi G}$$

admit the static solution: $a = \frac{1}{\sqrt{\Lambda}} \; ; \; \rho_m = 2\rho_\Lambda$

Usual gravitational attraction balanced by repulsion due to Λ

When Hubble discovered the redshift, Einstein took back his proposal calling it a "blunder".

Actually, the blunder was not in the new equations but in the instability of Einstein's static solution against small perturbations away from his condition

$$\rho_m = 2\rho_\Lambda$$

Indeed from:

$$H^2 = \frac{8\pi G}{3}(\rho_m + \rho_\Lambda) - \frac{1}{a^2} \; ; \; \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho_m - 2\rho_\Lambda)$$

we see that, if $\rho_m = (2+\epsilon)\, \rho_\Lambda$ with ϵ positive, the Universe starts contracting and, by the different redshifting of ρ_m and ρ_Λ, ϵ keeps increasing until matter completely dominates the collapse. The opposite happens if ϵ starts slightly negative: in that case the Universe expands faster and faster diluting ρ_m. Evidence that today $\rho_\Lambda \sim 2\rho_m$.