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II: aspects non-perturbatifs
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Les problèmes U(1)/CP et les instantons

- A short reminder from last week
- More on the Goldstone theorem
- A U(1) problem in QCD?
- CP-conservation in QCD?
- Instantons decide..

Short reminder from last week

- We have seen how gauge invariance (plus the restriction to a minimal number of derivatives) automatically enforces certain global symmetries in QCD
- These symmetries are not always «seen» in Nature
- In particular, if we consider QCD with N_f massless (or light) quarks, we expect:

1. A (slightly broken) $U(N_f)_V$ symmetry: this is OK (isospin..)
2. Parity doublets and/or massless fermions as a consequence of the full G_F symmetry: this is not OK

$$G_F = U(N_f)_L \otimes U(N_f)_R = SU(N_f)_L \otimes U(1)_L \otimes SU(N_f)_R \otimes U(1)_R$$

- We then argued that the above problem can be solved if G_F is spontaneously broken to $U(N_f)_V$

We have also seen how to compute the number of NG bosons according to Goldstone's theorem:

If a continuous global symmetry **G** is spontaneously broken down to its subgroup **S** (meaning S|0> =|0>), there must a massless (NG) boson for each generators of G outside S:

$$N_{NG} = \text{(number of NG-bosons)} = \dim G - \dim S$$

How does this apply to QCD?

$$G_F = U(N_f)_L \otimes U(N_f)_R = SU(N_f)_L \otimes U(1)_L \otimes SU(N_f)_R \otimes U(1)_R$$

has 2 N_F^2 generators. If the unbroken subgroup S is the subgroup $U(N_f)_V$ of G_F, the number of NGB should be :

$$N_{NG} = 2N_F^2 - N_F^2 = N_F^2 \quad \text{(hence 4 or 9).}$$

But how do we take into account explicit breaking?

The Ward-Takahashi-identity and NG bosons
(more on Goldstone's theorem)

Let us denote by $J_\mu(x)$ the conserved (Noether) current associated with a generic symmetry, and consider the (2-point) correlation function:

$$T_\mu^A(x) = \langle 0 | T(J_\mu(x) A(0)) | 0 \rangle$$

(A(y) is a generic gauge invariant local operator)

It is easy to show that $T_\mu{}^A(x)$ satisfies the exact (WT) identity:

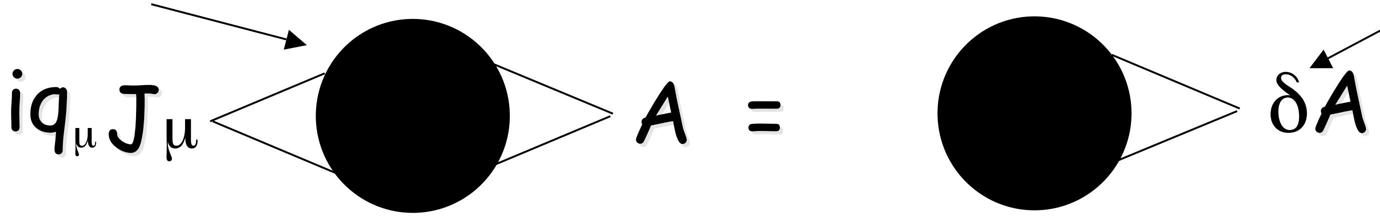
$$\partial_\mu T_\mu^A(x) = \langle 0 | \delta A | 0 \rangle \delta^4(x)$$

After FT:

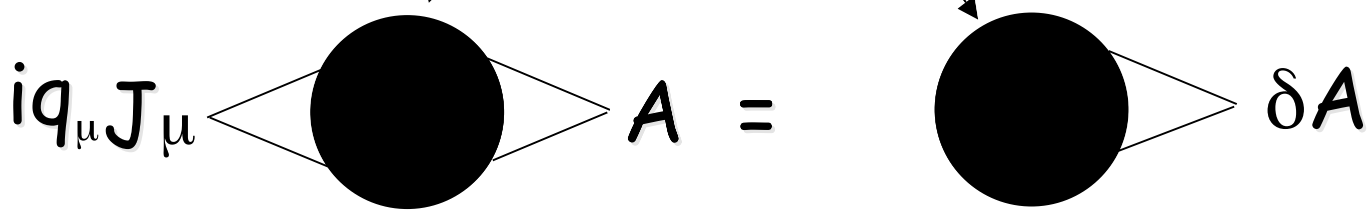
$$i q_\mu T_\mu^A(q) = \langle 0 | \delta A | 0 \rangle$$

symm. var. of A

In pictures:



$$iq_\mu T_\mu^A(q) = \langle 0|\delta A|0\rangle$$



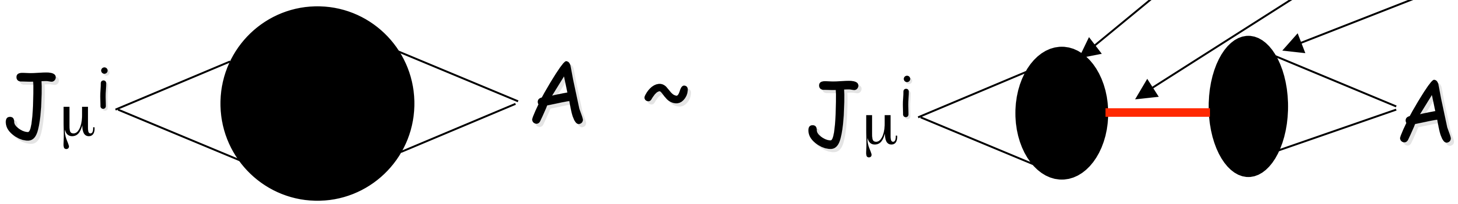
- Assume now that the symmetry is sp. broken i.e. that $\langle 0|\delta A|0\rangle \neq 0$ and consider the limit $q \to 0$ in WT identity. It clearly requires $T_\mu^A(q)$ to be singular in that limit:

$$T_\mu^A(q) = -i\,\frac{q_\mu}{q^2}\langle 0|\delta A|0\rangle$$

In (D=4) QFT such a singular behaviour can only come from an "intermediate" massless boson (π):

$$T_\mu^A(q) = \langle 0|J_\mu|\pi\rangle \frac{-i}{q^2}\langle \pi|A(0)|0\rangle$$

$$T_\mu^A(q) \sim \langle 0|J_\mu|\pi\rangle \frac{1}{q^2}\langle \pi|A(0)|0\rangle = -iq_\mu F_\pi \frac{1}{q^2}\langle \pi|A(0)|0\rangle$$



$$T_\mu^A(q) = -i\,\frac{q_\mu}{q^2}\langle 0|\delta A|0\rangle$$

- Comparing with WTI gives immediately the crucial result:

$$F_\pi\langle \pi|A(0)|0\rangle = \langle 0|\delta A|0\rangle \neq 0$$

In words: a condensate (VEV) that breaks spontaneously a (continuous, global) symmetry implies the existence of a NG boson π that couples both to the conserved current (F_π) and to any local operator A whose variation δA condenses.

We now turn on a small (explicit) symmetry breaking term so that: $\partial_\mu J_\mu = P \sim \delta L$

We will argue that, if the condensate is smooth in the breaking, physics is also smooth. The WTI has an extra term:

$$iq_\mu T^A_\mu(q) = T^{PA}(q) + \langle 0|\delta A|0\rangle$$

where $T^{PA}(q) = \langle 0|T(P(x)A(0))|0\rangle_{FT}$

However, it is still saturated by the PNGB contribution:

$$\langle 0|J_\mu|\pi\rangle = -iq_\mu F_\pi \; implies \; \langle 0|P|\pi\rangle = \mu^2 F_\pi$$

so that

$$\frac{q^2}{q^2-\mu^2} F_\pi \langle \pi|A(0)|0\rangle = \frac{\mu^2}{q^2-\mu^2} F_\pi \langle \pi|A(0)|0\rangle + \langle 0|\delta A|0\rangle$$

i.e. again $F_\pi \langle \pi|A(0)|0\rangle = \langle 0|\delta A|0\rangle \neq 0$

We finally apply this general result to the case A= P

Recalling: $F_\pi \langle \pi | A(0) | 0 \rangle = \langle 0 | \delta A | 0 \rangle \neq 0$

$\partial_\mu J_\mu = P \qquad \langle 0 | J_\mu | \pi \rangle = -i q_\mu F_\pi \; ; \; \langle 0 | P | \pi \rangle = \mu^2 F_\pi$

we find:

$$F_\pi \langle 0 | P | \pi \rangle = \mu^2 F_\pi^2 = \langle 0 | \delta P | 0 \rangle \sim \langle 0 | \delta(\delta L)) | 0 \rangle$$

When applied to QCD with, say, three light quarks (u,d,s) this general formula becomes the celebrated (DGMOR) formula for the masses of PNGB in QCD to leading order in m_i

Since there are N_f^2 currents of the (SB) kind we have considered we expect as many PNGB $\pi_{ij} \sim q^*_i q_j$ of masses:

$$\mu_{ij}^2 = -F_\pi^{-2} (m_i + m_j) \langle \bar{\psi} \psi \rangle \; ; \; i, j = 1, 2, \ldots N_f$$

The U(1) problem

If this were the whole story the spectrum of the lowest-mass states would consist of the following PNGBs ($m_u < m_d < m_s$):

uu* :	$\mu^2 \sim 2\, m_u$	??
ud*, du*:	$\mu^2 \sim (m_u + m_d)$	$\pi^{\pm}$
dd* :	$\mu^2 \sim 2\, m_d$	??
us*, su*:	$\mu^2 \sim (m_u + m_s)$	K^+, K^-
ds*, sd	$\mu^2 \sim (m_d + m_s)$	K^0, K^{0*}
ss*:	$\mu^2 \sim 2\, m_s$	??

Experimentally we see: 3 very light pions (135-140 MeV); 4 moderately light kaons and the η (495, 548 MeV); a quite heavy η' (958 MeV). In terms of quark content all works fine for the off-diagonal PNGB; but in the diagonal, neutral sector, the mass eigenstates are heavily mixed in the quark basis:

$$\pi^0 \sim \frac{1}{\sqrt{2}}[uu^* - dd^*]$$

$$\eta \sim \frac{1}{\sqrt{6}}[uu^* + dd^* - 2ss^*]\,,\ \eta' \sim \frac{1}{\sqrt{3}}[uu^* + dd^* + ss^*]$$

(actually the latter two are true up to a mixing angle $\theta_P \sim 11^o$)

Something strange is going on in the flavourless channels (uu*,dd*,ss*) forcing states of different quark content to mix heavily.

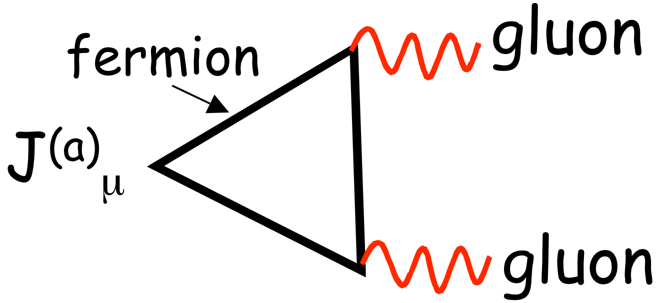
On the other hand the experimental eigenstates look very natural in terms of SU(2) and/or SU(3) symmetries that are slightly broken by quark masses and EM interactions.

The three pions form an SU(2) triplet, and, together with the kaons and the η, they form an SU(3) octet. The η' is a singlet of both SU(2) and SU(3).

Incidentally: this is NOT what happens in sectors with other QN: vector & tensor mesons are "ideally mixed", meaning that they are almost "pure" quark states

Recall from previous lecture and seminar: the flavourless channel is precisely the one where the ABJ anomaly appears (since gluons are flavour-blind)!

Q: Does the additional explicit breaking provided by the anomaly account for the peculiar pattern of masses and mixing in the flavourless pseudo-scalar sector?



Strong-CP problem in QCD

There is an apparently unrelated problem in QCD: the so-called strong-CP problem

It's the question of whether QCD naturally conserves or violates CP (changing each l.h. particle in its r.h. antiparticle). The mass term in the QCD Lagrangian appears to break CP unless $m_{ff'}$ is a real matrix. The problem is that even a very small CP violation would induce an unacceptably large electric dipole moment of the neutron.

Is there a "natural" solution to the strong-CP problem (other that putting by hand real masses)? Masses come from the EW sector of the SM and there is no reason for them to be real.

Weinberg's argument for automatic CP cons.

Since physics is invariant under field redefinitions, we can ask: Can we bring the quark mass matrix to a real form by redefining the quark fields?
Let us start by performing a non anomalous $SU(N_f)_L \times SU(N_f)_R \times U(1)_V$ transformation.

It is easy to show that such a transformation can bring $m_{ff'}$ to a diagonal form, $m_{ff'} = m_f \delta_{ff'}$ but, in general, with complex m_f. Further non anomalous transformations can eliminate all but one common phase β: $m_f = |m_f| e^{i\beta}$. We cannot do better than that. We need $U(1)_A$ in order to eliminate also β. This is precisely what Weinberg did!

•Weinberg knew of course that such a $U(1)_A$ transformation has an anomaly. If we insist on doing it, the classical action does become CP invariant, but the effective action picks us a term (see previous lecture):

$$\delta S_{eff} = 2N_f\beta \int d^4x\, Q(x)\,;\, Q(x) \equiv \frac{\alpha}{16\pi}\varepsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}$$

•This terms beaks CP (C=+1, P=-1): what we threw out of the door is coming back from the window and we seem to have gained nothing. In fact, why did we not add a term $\delta L = \theta Q$ to the Lagrangian from the start?

• The point is that Q can be written as the divergence of a current K_μ : $Q = \partial_\mu K^\mu$ (see also seminar).

•We normally neglect total derivatives since they do not contribute to the field equations. If this is the case for Q, the extra piece is irrelevant and CP is automatically conserved!

Back to U(1) problem

(peut-on avoir le beurre et l'argent du beurre?)

If, the axial U(1) anomaly reduces the true global symmetry of massless QCD to:

$$SU(N_f)_L \times SU(N_f)_R \times U(1)_V$$

the number of NG bosons is reduced to $N_F^2 - 1$ (by excluding the η') and this may solve the U(1) problem. However, if

$$\partial_\mu J_\mu = 2N_f Q \; and \; Q = \partial_\mu K_\mu$$

we can define a new conserved U(1) current:

$$J^c_\mu \equiv J_\mu - 2N_f K_\mu \quad \text{with} \quad \partial_\mu J^c_\mu = 0$$

.. and prove again the necessity of a ninth NG boson. Is this really so?

Let us go back to the WTI before we turned on masses but now including the anomaly i.e. let us set P = 2 N_f Q:

$$\partial_\mu J_{\mu 5} = 2N_f Q \ , \ Q = \partial_\mu K_\mu \qquad J^c_\mu \equiv J_\mu - 2N_f K_\mu$$

The WTI in the presence of explicit breaking was

$$iq_\mu T^A_\mu(q) = T^{PA}(q) + \langle 0|\delta A(0)|0\rangle$$

$$T^{PA}(q) = \langle 0|T(P(x)A(0))|0\rangle_{FT}$$

Inserting P = 2 N_f Q gives the «anomalous WTI»

$$iq_\mu \langle 0|T(J^c_\mu(x)A(0)|0\rangle_{FT} = \langle 0|\delta A|0\rangle \neq 0$$

The necessity of a massless boson coupled to $J_\mu{}^c$ looks again inevitable…and it is! Are we back to having a U(1) problem?

Actually not! The point is that the needed NGB may just couple to K_μ but not to J_μ. It so happens that K_μ is NOT gauge invariant (while its divergence Q is). In a confining world only colour singlet particles coupled to gauge invariant operators have physical meaning; therefore the existence of this massless "ghost" does not mean at all the existence of an experimentally observable massless particle. In detail:

$$iq_\mu \langle 0|T(J^c_\mu(x)A(0)|0\rangle_{FT} = \langle 0|\delta A|0\rangle \neq 0$$

$$iq_\mu \langle 0|K_\mu A|0\rangle \to iq_\mu \langle 0|K_\mu|G_\nu\rangle \frac{g_{\nu\rho}}{q^2}\langle G_\rho|A(0)|0\rangle$$

$$= iq_\mu \lambda_{KG}\, g_{\mu\nu}\, \frac{g_{\nu\rho}}{q^2}\, q_\rho\, \lambda_{GA} = \langle 0|\delta A|0\rangle \neq 0$$

Still we need the operator Q to be non-trivial at zero momentum => a non-trivial "topological" charge: $\nu = \int d^4x\, Q(x)$

Recall now what happened when we tried to rotate away all phases in the quark mass matrix:

$$\delta S_{eff} = 2N_f\beta \int d^4x\, Q(x)\,;\, Q(x) \equiv \frac{\alpha}{16\pi}\varepsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}$$

The non-triviality of $\nu = \int d^4x\, Q(x)$

means that the extra term generated by the anomaly (or a - θ–term) cannot be ignored and can give physical effects

QCD is in a squeeze. Either:

1. ν is trivial and we cannot solve the U(1) problem, or
2. ν is non-trivial and we cannot solve the CP problem (other than by adjusting parameters by hand).

The existence of field configurations of non-vanishing ν, the "instantons", means that QCD chooses the latter alternative

The end of this lecture, as well as today's seminar, deal with those strange objects. Let us first rewrite the gauge part of the QCD action (after going Euclidean: t->it):

$$S_E = \frac{1}{4}\int d^4x\, F^a_{\mu\nu}F^{a,\mu\nu} = \frac{1}{8}\int d^4x\, (F^a_{\mu\nu} \pm \tilde{F}^a_{\mu\nu})^2 \mp \frac{1}{4}\int d^4x\, F^a_{\mu\nu}\tilde{F}^{a,\mu\nu}$$

The first term is positive semi-definite. Its zeros give local minima of the action (and solutions of the field equations) whenever the gauge fields are (anti) self dual:

$$F^a_{\mu\nu} = \pm\tilde{F}^a_{\mu\nu} \Rightarrow S_E = \left|\frac{1}{4}\int d^4x\, F^a_{\mu\nu}\tilde{F}^{a,\mu\nu}\right|$$

There is of course a trivial solution, 0=0, corresponding to a "pure gauge" A^a_μ and to zero action. Non-trivial solutions are called instantons since, in order to have finite action, their field strength must be localized in space and time.

Equivalently, the gauge fields A^a_μ have to go to « pure gauge » at space-time infinity. ST-infinity in 4D is a 3-sphere. For SU(2) group space is also a 3-sphere: having pure gauge at infinity amounts to mapping a 3-sphere to a 3 sphere.
Such a mapping can be trivial (each point in space-time mapped to the trivial element of the group) or it can cover the group's sphere an integer number of times (with an orientation sign): $\Pi_3[SU(2)] = Z$. This topological integer turns out to be our ν, justifying its name (NB: for instantons to exist the gauge group must contain at least an SU(2) subgroup). Since:

$$Q(x) \equiv \frac{\alpha}{16\pi}\varepsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma} = \frac{\alpha}{8\pi}F^a_{\mu\nu}\tilde{F}^{a,\mu\nu}$$ we find:

$$S_E^{inst.} = \left|\frac{1}{4}\int d^4x\, F^a_{\mu\nu}\tilde{F}^{a,\mu\nu}\right| = \frac{2\pi}{\alpha}|\nu|$$

According to Feynman's path integral formulation of QM and QFT, the contribution of instantons to some transition amplitude $A^{inst.}$ must be of the form:

$$A^{inst.} \sim exp(-S_E^{inst.}) \sim exp(-\frac{2\pi}{\alpha_s}|\nu|)$$

We see here the exponential suppression (at small α_s) typical of a non-perturbative tunneling phenomenon.
On the other hand, α_s depends on the typical energy scale of the process. A careful analysis shows that, for an instanton of "size" ρ, the relevant α_s is $\alpha_s(1/\rho)$.
Thus, while small instantons give small transition amplitudes, "large" instantons give large contributions and are certainly capable of accounting for the effects that we are after…