Particules Elémentaires, Gravitation et Cosmologie
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Gravitation et Cosmologie: le Modèle Standard

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Outils mathématiques de base en RG

- The group of GCT, scalars, vectors, tensors
- Invariant integrals and covariant differentiation
- Divergence, curl, covariant Gauss theorem
- Curvature tensors

Reminder from last week

In principle one could derive all equations of GR by first going to L.I.C. and then transforming to arbitrary ones. This is quite cumbersome.

There is an equivalent easier procedure. All equations should be covariant under GCT (i.e. should look the same in all coordinate systems) and should reduce to those of SR if we replace $g_{\mu\nu}$ by $\eta_{\mu\nu}$

In order to use this latter method we have to get acquainted with some mathematical tools for working in four-dimensional spaces with a general metric (and even sometime topological) structure.

The group of General Coordinate Transformations

The basic symmetry group of GR is the group of General Coordinate Transformations (GCT):

$$x^\mu \to \tilde{x}^\mu(x^\lambda) \equiv \tilde{x}^\mu(x)$$

There are essentially no restrictions on the allowed GCT except that they should be invertible and differentiable (also called diffeomorphisms)

Note analogy with gauge transformations (both are "local" transformations)

Scalars

$$\mathrm{GCT}: \mathrm{x}^\mu \rightarrow \tilde{\mathrm{x}}^\mu(\mathrm{x}^\lambda) \equiv \tilde{\mathrm{x}}^\mu(\mathrm{x})$$

The simplest objects are scalars. They transform trivially. For a constant this means that it is the same number in all coordinate systems.

For a scalar field, S(x), it means that it takes the same value at the same physical point

$$S(x) \rightarrow \tilde{S}(x) \quad \text{with} \quad \tilde{S}(\tilde{x}) = S(x)$$

Vectors

$$x^\mu \to \tilde{x}^\mu(x^\lambda) \equiv \tilde{x}^\mu(x)$$

The next simplest objects are vectors. While x^μ itself is not a vector (unlike in SR!) dx^μ is a contravariant vector

$$dx^\mu \to d\tilde{x}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} dx^\nu \quad \text{i.e.} \quad \tilde{W}^\mu(\tilde{x}) = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} W^\nu(x)$$

$$V_\mu(x) \equiv \frac{\partial \phi(x)}{\partial x^\mu} \;\; ; \;\; V_\mu(x) \to \tilde{V}_\mu(x) \;\; \text{with} \;\; \tilde{V}_\mu(\tilde{x}) = \frac{\partial x^\nu}{\partial \tilde{x}^\mu} V_\nu(x)$$

is a covariant vector: an important distinction in GR!

If we "contract" a contravariant vector with a covariant vector we get a scalar, e.g.

$$V_\mu(x) W^\mu(x) = S(x) \;;\; V_\mu(x) dx^\mu = dS(x)$$

Tensors

As usual we get tensors by multiplying vectors: since we have two kinds of vectors we can get three kinds of tensors: covariant, contravariant, or mixed. But, like with vectors, we can define the transformation properties of tensors independently of their origin:

$$\tilde{T}^{\nu_1 \nu_2 \dots}_{\mu_1 \mu_2 \dots}(\tilde{x}) = \frac{\partial \tilde{x}^{\nu_1}}{\partial x^{\rho_1}} \frac{\partial \tilde{x}^{\nu_2}}{\partial x^{\rho_2}} \dots \frac{\partial x^{\sigma_1}}{\partial \tilde{x}^{\mu_1}} \frac{\partial x^{\sigma_2}}{\partial \tilde{x}^{\mu_2}} \dots T^{\rho_1 \rho_2 \dots}_{\sigma_1 \sigma_2 \dots}(x)$$

The "contraction" procedure also applies to tensors, e.g. we can get a scalar from a mixed tensor:

$$\tilde{T}^{\mu}_{\mu}(\tilde{x}) = \frac{\partial \tilde{x}^{\mu}}{\partial x^{\rho}} \frac{\partial x^{\sigma}}{\partial \tilde{x}^{\mu}} T^{\rho}_{\sigma}(x) = \delta^{\sigma}_{\rho} T^{\rho}_{\sigma}(x) = T^{\rho}_{\rho}(x)$$

NB: Symmetries in indices at same height preserved!

The metric tensor

As anticipated, a most important tensor in GR is the metric tensor since it contains all the information about the geometry of space-time and gravity. The fact that it is a covariant two-index symmetric tensor follows from its definition:

$$g_{\mu\nu}(x) \equiv \eta_{ab} \frac{\partial \xi_X^a}{\partial x^\mu} \frac{\partial \xi_X^b}{\partial x^\nu} = g_{\nu\mu}(x)$$

Its inverse, denoted by $g^{\mu\nu}$, is a contravariant two-index symmetric tensor. They are used to raise and lower tensor indices. For instance:

$$g^{\mu\nu} V_\rho = T_\rho^{\mu\nu} \; ; \; g^{\mu\rho} V_\rho = T_\rho^{\mu\rho} \equiv V^\mu$$

Integration (easier than differentiation)

Suppose we want to integrate some integrand over a region of space-time. The naive integration measure is not invariant under GCT:

$$d^4x \to d^4\tilde{x} = |det\frac{\partial \tilde{x}^\mu}{\partial x^\nu}| d^4x$$

We can compensate, however, this lack of invariance by introducing a weight in the integration measure:

$$|det g_{\mu\nu}| \equiv (-g) \to (-\tilde{g}) = |det\frac{\partial x^\nu}{\partial \tilde{x}^\mu}|^2(-g) = |det\frac{\partial \tilde{x}^\mu}{\partial x^\nu}|^{-2}(-g)$$

Consequently: $$d^4x\sqrt{-g(x)} = d^4\tilde{x}\sqrt{-\tilde{g}(\tilde{x})}$$

is a good (invariant) integration measure (not unique!)

Invariant integrals

The above results tell us how to construct invariant integrals as functionals of various fields, i.e. precisely what we shall soon need in order to formulate an action principle for GR. Indeed:

$$\int_{R(x)} d^4x \sqrt{-g(x)} S(x) = \int_{\tilde{R}(\tilde{x})} d^4\tilde{x} \sqrt{-\tilde{g}(\tilde{x})} \tilde{S}(\tilde{x})$$

if the two regions correspond to the same physical domain in the two coordinate systems. Setting S=1 we get the simplest (but uninteresting) action for GR (just a cosmological constant term). In order to do better we have to consider differentiation

(Covariant) Differentiation

We have already seen that the gradient of a scalar is a covariant vector. How about the derivative of a vector?

Unfortunately it is not a tensor! Indeed:

$$V_\mu(x) = \frac{\partial \tilde{x}^\rho}{\partial x^\mu} \tilde{V}_\rho(\tilde{x}) \Rightarrow$$

$$V_{\mu,\nu}(x) \equiv \frac{\partial}{\partial x^\nu} V_\mu(x) = \frac{\partial \tilde{x}^\rho}{\partial x^\mu} \frac{\partial \tilde{x}^\sigma}{\partial x^\nu} \frac{\partial}{\partial \tilde{x}^\sigma} \tilde{V}_\rho(\tilde{x}) + \frac{\partial^2 \tilde{x}^\rho}{\partial x^\nu \partial x^\mu} \tilde{V}_\rho(\tilde{x})$$

and the last term is unwanted. It turns out that one can fix this problem using the affine connection Γ:

$$V_{\mu;\nu}(x) \equiv V_{\mu,\nu}(x) - \Gamma^\rho_{\mu\nu} V_\rho = \frac{\partial \tilde{x}^\rho}{\partial x^\mu} \frac{\partial \tilde{x}^\sigma}{\partial x^\nu} \tilde{V}_{\rho;\sigma}(\tilde{x})$$

(Covariant) Differentiation

The previous formula defines the covariant derivative of a covariant vector (giving a covariant tensor)

$$V_{\mu;\nu}(x) \equiv V_{\mu,\nu}(x) - \Gamma^{\rho}_{\mu\nu} V_{\rho}(x)$$

For a contravariant vector (note change of sign!):

$$V^{\mu}_{\ ;\nu}(x) \equiv V^{\mu}_{\ ,\nu}(x) + \Gamma^{\mu}_{\rho\nu} V^{\rho}(x) = g^{\mu\rho}(x) V_{\rho;\nu}(x)$$

is a mixed tensor. In general the covariant derivative adds a lower index.

N.B. $\Gamma^{\rho}_{\mu\nu}$ itself is NOT a tensor

N.B.' The covariant derivative of $g_{\mu\nu}$ is zero

Curl, Divergence, Gauss

The covariant curl happens to coincide with the usual curl ($\Gamma^\rho_{\mu\nu}$ drops out because of its symmetry in $\mu\nu$)

$$V_{\mu;\nu}(x) - V_{\nu;\mu}(x) = V_{\mu,\nu}(x) - V_{\nu,\mu}(x)$$

The covariant divergence (using the expression of $\Gamma^\rho_{\mu\nu}$) takes a simple form:

$$V^\mu_{\ ;\mu}(x) = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\ V^\mu\right)_{,\mu}$$

This implies a covariant version of Gauss theorem:

$$\int_{R(x)} d^4x\sqrt{-g(x)}V^\mu_{\ ;\mu}(x) = \int_{R(x)} d^4x\left(\sqrt{-g(x)}V^\mu\right)_{,\mu} = \int_{\partial R} d^3\Sigma^\mu \dots$$

The Curvature Tensor

Is there an intrinsic way to decide whether a space-time described by a certain $g_{\mu\nu}(x)$ is curved or just flat (Minkowski) space-time written in complicated coordinates? The answer is yes! The necessary and sufficient condition for a space-time to be equivalent to Minkowski is that a certain four-rank tensor, the (Riemann) curvature tensor, is identically zero.

There are several ways to introduce the curvature tensor. One which is closest to the way the field-strength tensor $F_{\mu\nu}$ arises in gauge theories, is through the commutator of two covariant derivatives

Non abelian field-strength tensor:

$$[D_\mu, D_\nu] = F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

GR's curvature tensor:

$$\begin{aligned} V_{\rho;\mu;\nu} - V_{\rho;\nu;\mu} &= -\left(\Gamma^\lambda{}_{\rho\mu,\nu} - \Gamma^\lambda{}_{\rho\nu,\mu} + \Gamma^\eta{}_{\rho\mu}\Gamma^\lambda{}_{\eta\nu} - \Gamma^\eta{}_{\rho\nu}\Gamma^\lambda{}_{\eta\mu}\right) V_\lambda \\ &\equiv -R^\lambda{}_{\rho\mu\nu} V_\lambda \end{aligned}$$

Also: $$V^\rho{}_{;\mu;\nu} - V^\rho{}_{;\nu;\mu} = R^\rho{}_{\lambda\mu\nu} V^\lambda$$

Introducing the fully covariant tensor

$$R_{\mu\nu\rho\sigma} = g_{\mu\lambda} R^\lambda{}_{\nu\rho\sigma}$$

we can discuss its form and interesting symmetries

After quite some algebra one finds:

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= \frac{1}{2}\left[g_{\mu\rho,\nu\sigma} - g_{\nu\rho,\mu\sigma} - g_{\mu\sigma,\nu\rho} + g_{\nu\sigma,\mu\rho}\right] \\ &+ g_{\alpha\beta}\left[\Gamma^{\alpha}{}_{\mu\rho}\Gamma^{\beta}{}_{\nu\sigma} - \Gamma^{\alpha}{}_{\mu\sigma}\Gamma^{\beta}{}_{\nu\rho}\right] \end{aligned}$$

From which we deduce the following properties

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho} = R_{\rho\sigma\mu\nu}$$

$$R_{\mu\nu\rho\sigma} + R_{\mu\sigma\nu\rho} + R_{\mu\rho\sigma\nu} = 0$$

The Riemann tensor has 20 components

Related lower-rank tensors

The Ricci tensor:

$$R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu} = R_{\nu\mu}$$

Has 10 components (like $g_{\mu\nu}$):

The curvature scalar: $R = g^{\mu\nu} R_{\mu\nu}$

The Riemann tensor can be decomposed into 10 components determined by the Ricci tensor and 10 others (contained in the so-called Weyl tensor). As we shall see, matter determines only the Ricci tensor => space-time can be curved even in the absence of matter sources (gravitational waves!)