# Green-Schwarz anomaly cancellation

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# Plan of the talk

# Introduction

- The theory of general relativity for gravity was formulated by Einstein in 1915.
- It is a four-dimensional theory that extends the theory of special relativity.
- While special relativity is invariant under the transformations of the Lorentz group, general relativity is invariant under an arbitrary change of coordinates.
- In the twenties it was proposed by Theodor Kaluza and Oskar Klein to unify electromagnetism with gravity by starting from general relativity in a five-dimensional space-time and compactify the extra-dimension on a small circle.
- In this way one obtains general relativity in four dimensions, a vector gauge field satisfying the Maxwell equations and a scalar.
- This idea of extra dimensions was not pursued in the years after.
- In the sixties and seventies, when I started to work in the physics of the elementary particles, everybody was strictly working in four dimensions.

- Also the dual resonance model, being a model for hadrons, was obviously formulated in four dimensions.
- and nobody questioned this point until the paper by Lovelace in which he proposed $d = 26$ for avoiding problems with unitarity.
- But even then almost nobody took his paper seriously !!
- Little by little string theorists started to accept $d = 26$ and $d = 10$.
- But at the beginning, this was for them a source of embarrassment.
- Only later with the change of perspective from hadron physics to the theory unifying gauge theories with gravity [Scherk and Schwarz, 1975] extra dimensions started to be accepted.
- In the same year Cremmer and Scherk compactified the six extra dimensions of the NSR model.
- They write: we have proved that this proposal is compatible with everyday experience.
- At this point it became clear that the existence of extra dimensions was an important prediction of string theory, rather than a source of embarrassment.

- The formulation of supergravity in 1976 and especially of those supergravities with higher amount of supersymmetry, as the $N = 8$ supergravity, opened the possibility of unifying, in a consistent quantum theory, gauge theories with gravity in the framework of ordinary field theory.
- Due to the cancellation of bosonic and fermionic contributions, those theories have in fact a much higher degree of convergence.
- In particular, it was conjectured that $N = 8$ supergravity was a finite theory of gravity.
- It included a $SU(8)$ gauge group that was big enough for particle phenomenology.
- This discussion on the finiteness of $N = 8$ supergravity is still going on today, but this is not the subject of this lecture.
- For this reason string theory was abandoned.
- In 1978 the Lagrangian of a unique and most symmetric supergravity, 11-dimensional supergravity, was written down by Cremmer, Julia and Scherk here in Paris.

- By compactifing the seven extra dimensions on various manifolds one obtained a number of four-dimensional theories.
- The problem that one soon met was the difficulty of having four-dimensional theories with chiral fermions.
- But quarks and leptons are chiral fermions and therefore any phenomenological viable theory must contain chiral fermions.
- The difficulty of generating chiral fermions in the Kaluza-Klein reduction of 11-dimensional supergravity led many to go back to string theory.
- It was already known that type IIB and type I superstrings had chiral fermions in $d = 10$.
- On the other hand, chiral fermions have the tendency to generate gauge and gravitational anomalies.
- The question was: are those string theories free from gauge and gravitational anomalies?

- The presence of an anomaly means that a current, conserved in the classical theory, is not conserved anymore when we quantize the theory.
- Here we want to stress that there are two kinds of anomalies.
- Those associated with a current corresponding to a global symmetry as for instance the axial $U(1)$ current in QCD.
- This kind of anomaly is not a problem and actually welcome, as in the case of the $U(1)$ current in QCD, in order to reproduce the correct spectrum of pseudoscalar mesons.
- Also an anomaly in the axial vector current is essential for getting the decay rate $\pi^0 \rightarrow 2\gamma$ in agreement with the experiments.
- These anomalies have an observable and welcome effect.

- Other anomalies are those associated to currents coupled to a gauge field and to gravitons.
- The lack of conservation of these currents implies that the gauge symmetry or the invariance under arbitrary coordinate transformation is broken at the quantum level.
- This may imply that the unphysical non-transverse components of the gauge field and of the graviton are not decoupled from the physical spectrum giving rise to zero or negative norm states in the physical spectrum.
- This is in contradiction with the positivity of the probability.

# A quick look at the abelian axial anomaly

- Let us consider a fermion in interaction with an external vector and axial vector fields. The Lagrangian is given by

$$L = \bar{\Psi}\gamma^{\mu} D_{\mu}\Psi \qquad\qquad D_{\mu} = \partial_{\mu} + iV_{\mu} + iA_{\mu}\gamma_5$$

- We work in a $d$ dimensional euclidean space ($d$ is even) with

$$\gamma_{d+1} = \gamma_1\gamma_2\gamma_3\cdots\gamma_d \qquad \gamma_{d+1}^2 = (-1)^{d/2}$$

- $L$ is invariant under $U(1)$ local vector transformations

$$\Psi(x) \to e^{-i\Lambda(x)}\Psi(x) \;\; ; \;\; \bar{\Psi}(x) \to \bar{\Psi}(x)e^{+i\Lambda(x)} \;\; ; \;\; V_{\mu} \to V_{\mu} + \partial_{\mu}\Lambda(x)$$

- and local axial vector transformations

$$\Psi(x) \to e^{-i\alpha(x)\gamma_5}\Psi(x) \; ; \; \bar{\Psi}(x) \to \bar{\Psi}(x)e^{-i\alpha(x)\gamma_5} \; ; \; A_{\mu} \to A_{\mu}+\partial_{\mu}\alpha(x)$$

- In the quantum theory the vector symmetry is maintained, while the axial one has an anomaly.

- This can be seen by computing the functional integral over the fermions:

$$e^{W(V_\mu, A_\mu)} \equiv \int d\psi d\bar{\psi}\ e^{-\int d^d x\ \bar{\psi}(\gamma^\mu D_\mu + im)\psi}$$

- The functional integral can be computed and one gets:

$$W(V_\mu, A_\mu) = \mathrm{Tr} \log \left(\gamma^\mu D_\mu + im\right)$$

- In the quantum theory the vector symmetry can be maintained, but then the axial vector symmetry has an anomaly:

$$\delta_V W(V_\mu, 0) = 0 \quad ; \quad \delta_A W(V_\mu, 0) \neq 0$$

- In particular, one gets:

$$\delta_A W(V_\mu, 0) = i\,\mathrm{Tr}\left[\left(\gamma^\mu(\partial_\mu + iV_\mu) + im\right)^{-1} \gamma^\nu \partial_\nu \alpha \gamma_{d+1}\right]$$

- The previous expression can be expanded in powers of the field $V_\mu$ to get ($\not{C} \equiv \gamma^\mu C_\mu$)

$$\delta W(V_\mu) = iTr\left\{(\not\partial + im)^{-1}\left[1 - i\not{V}(\not\partial + im)^{-1}\right.\right.$$
$$\left.\left. - \not{V}(\not\partial + im)^{-1}\not{V}(\not\partial + im)^{-1} + \ldots\right]\not\partial\alpha\gamma_5\right\}$$

- If $d = 2$ only the second term is not vanishing and one gets:

$$\delta_A W(V_\mu) = \mathrm{Tr}\left((\not\partial + im)^{-1}\not{V}(\not\partial + im)^{-1}\ \not\partial\alpha\gamma_5\right)$$
$$= i\int d^2x \int d^2y\alpha(x)V_\mu(y)\int \frac{d^2q}{(2\pi)^2}\mathrm{e}^{iq\cdot(x-y)}q_\nu\Gamma^{\mu\nu}(q)$$

where

$$q_\nu\Gamma^{\mu\nu}(q) = \int \frac{d^2r}{(2\pi)^2}\frac{Tr[((\not{r} + \not{q}) - m)\gamma^\mu(\not{r} - m)\ \not{q}\gamma_5]}{(r^2 + m^2)((r + q)^2 + m^2)}$$

- By suitably regularizing the previous integral we get

$$q_\nu \Gamma^{\mu\nu} = \frac{1}{\pi} \epsilon^{\mu\nu} q_\nu$$

- This implies that:

$$\delta_A W(V_\mu) = -\frac{1}{2\pi} \int d^2x \alpha(x) \epsilon^{\mu\nu} F_{\mu\nu}(x) \;\; ; \;\; F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$$

  that gives the axial anomaly in two dimensions.
- In conclusion, classically $\delta_A S = 0$, while quantum mechanically $\delta_A W \neq 0$.
- The next contribution in the expansion is a term with three propagators that gives the axial anomaly in $d = 4$ and so on.

# Few words on forms

- For treating the gauge and gravitational anomalies it is very convenient to use a formalism based on forms.
- Instead of working with the Yang-Mills potential $A_\mu$ and field strength $F_{\mu\nu}$, it is more convenient to introduce the following one- and two-forms:

$$A \equiv A_\mu dx^\mu \quad ; \quad F \equiv F_{\mu\nu}\frac{dx^\mu \wedge dx^\nu}{2} \quad ; \quad F = dA + A \wedge A$$

  and work with them.
- $A$ and $F$ are matrices corresponding to a representation of the gauge group.
- $A$ and $F$ transform as follows under a gauge transformation:

$$\delta A = d\Lambda + [A, \Lambda] \quad ; \quad \delta F = [F, \Lambda]$$

- In the following we will consider only the fundamental and the adjoint representation of the group and we denote with $\mathrm{tr}$ the trace in the fundamental and with $\mathrm{Tr}$ the trace in the adjoint.

- Analogously in the case of gravity, instead of working with the spin connection $\omega_{\mu ab}$, one introduces the one-form $\omega$ and the two-curvature form $R$:

$$\omega = \omega_\mu dx^\mu \quad ; \quad R = R_{\mu\nu} \frac{dx^\mu \wedge dx^\nu}{2} = d\omega + \omega \wedge \omega$$

- They are also matrices transforming according to the fundamental representation of the Lorentz group $SO(9,1)$.
- The wedge products $\omega \wedge \omega$ and $A \wedge A$ represent also the product of matrices.
- In the following, for the sake of simplicity, we omit to explicitly write the $\wedge$ symbol.
- $\omega$ and $R$ transform as follows under a local Lorentz rotation:

$$\delta\omega = d\Theta + [\omega, \Theta] \quad ; \quad \delta R = [R, \Theta]$$

- This description of gravity is different from the usual one where one introduces the metric, the Christoffel symbols and the curvature tensor $R_{\mu\rho,\nu\sigma}$ and from it one computes the Ricci tensor $R_{\mu\nu}$. $R_{\mu\nu}$ above is not the Ricci tensor!

## Simple calculations with forms

$$
\begin{aligned}
F \equiv F_{\mu\nu}\frac{dx^{\mu} \wedge dx^{\nu}}{2} &= dA + A^2 \\
&= \partial_{\mu}A_{\nu}dx^{\mu} \wedge dx^{\nu} + A_{\mu}A_{\nu}dx^{\mu} \wedge dx^{\nu} \\
&= \frac{dx^{\mu} \wedge dx^{\nu}}{2}\left[\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + [A_{\mu}, A_{\nu}]\right]
\end{aligned}
$$

Remember that

$$
dx^{\mu} \wedge dx^{\nu} = -dx^{\nu} \wedge dx^{\mu}
$$

# Anomaly cancellation in type IIB superstring theory

- Type IIB superstring is a chiral theory because both the two gravitinos and dilatinos are chiral fermions in ten dimensions.
- It has also a chiral boson because it contains in the spectrum a four-index R-R potential $C_4$ with self-dual field strength:

$$\tilde{F}_5 = {}^*\tilde{F}_5 \quad ; \quad F_5 = dC_4 - \frac{1}{2}C_2 \wedge F_3 + \frac{1}{2}B_2 \wedge F_3$$

  Remember that it contains also a R-R $C_0$ and $C_2$ with field strength $F_3 = dC_2$ and a NS-NS $B_2$ with field strength $H_3 = dB_2$.
- The contribution of these chiral fields to the gauge and gravitational anomalies were computed by Alvarez-Gaumé and Witten in 1984.
- In type IIB there is no gauge field and therefore no gauge anomaly.

- They actually computed an auxiliary quantity $A_{12}$ (it is a 12-form) for the different chiral fields.
- The anomaly can then be computed from $A_{12}$ following the following procedure.
- From $A_{12}$ one determines first $A_{11}$ such that:

$$A_{12} = dA_{11}$$

  $A_{11}$ is a 11-form.
- From $A_{11}$ one determines $A^1_{10}$ such that:

$$\delta A_{11} = dA^1_{10}$$

- $A^1_{10}$ gives the anomaly in ten dimensions that is linear in the parameters of the gauge transformation (this is why the index 1).
- $\delta$ means gauge and gravitational variations:

$$\delta A = d\Lambda + [A, \Lambda] \quad ; \quad \delta F = [F, \Lambda]$$
$$\delta \omega = d\Theta + [\omega, \Theta] \quad ; \quad \delta R = [R, \Theta]$$

- In the following we write the contributions of the various chiral fields.
- The contribution to the anomaly of the gravitino is given by:

$$A_{12}^{\frac{3}{2}}(R) = -\frac{11 \cdot 9 \cdot 5}{2^6} \left[ \frac{\mathrm{tr}(R^6)}{5670} + \frac{\mathrm{tr}(R^2)\mathrm{tr}(R^4)}{4320} + \frac{\left(\mathrm{tr}(R^2)\right)^3}{10368} \right]$$
$$+\frac{\mathrm{tr}(R^2)}{3 \cdot 2^7} \left[ \mathrm{tr}(R^4) + \frac{1}{4} \left( \mathrm{tr}(R^2) \right)^2 \right]$$

- That of a spin $\frac{1}{2}$ is equal to

$$A_{12}^{\frac{1}{2}}(R, F) = -\frac{1}{720}\mathrm{Tr}(F^6) + \frac{1}{24 \cdot 48}\mathrm{Tr}(F^4)\mathrm{tr}(R^2)$$
$$-\frac{1}{2^8}\mathrm{Tr}(F^2) \left[ \frac{1}{45}\mathrm{tr}(R^4) + \frac{1}{36} \left( \mathrm{tr}(R^2) \right)^2 \right]$$
$$+\frac{n}{2^6} \left[ \frac{1}{5670}\mathrm{tr}(R^6) + \frac{1}{4320}\mathrm{tr}(R^2)\mathrm{tr}(R^4) + \frac{1}{10368} \left( \mathrm{tr}(R^2) \right)^3 + \right]$$

$n$ is the number of generators of the gauge group.

- Finally that of the antisymmetric tensor is equal to

$$A^{A}_{12}(R) = \frac{1}{2835}\left[\frac{496}{128}\mathrm{tr}(R^6) - \frac{588}{4\cdot 64}\mathrm{tr}(R^2)\mathrm{tr}(R^4) + \frac{140}{8\cdot 64}\left(\mathrm{tr}(R^2)\right)^3\right]$$

- The anomaly in type IIB is proportional to the following combination:

$$A^{TOT}_{12} \sim A^{\frac{3}{2}}_{12}(R) - A^{\frac{1}{2}}_{12}(R,0) + A^{A}_{12}(R)$$

  Remember that type IIB superstring is a closed string theory without gauge fields.
- The relative minus sign between gravitino and dilatino is due to their opposite chirality.
- Using the previous formulas it is easy to check that $A^{TOT}_{12} = 0$
- No gravitational anomaly in type IIB superstring !!

# Anomaly cancellation in type I superstring

- In type I theory the anomaly is given by:

$$A_{12} \sim A^{\frac{3}{2}}_{12}(R) - A^{\frac{1}{2}}_{12}(R) - A^{\frac{1}{2}}_{12}(R, F)$$

- We have no contribution from the antisymmetric tensor (absent in the spectrum of type I), but we have the contribution of the Weyl-Majorana gauginos.
- Using the previous formulas we get:

$$A_{12} \sim \frac{n-496}{2^6}\left[\frac{\mathrm{tr}(R^6)}{5670} + \frac{\mathrm{tr}(R^4)\mathrm{tr}(R^2)}{4320} + \frac{\left(\mathrm{tr}(R^2)\right)^3}{10368}\right]$$
$$+\frac{\mathrm{tr}(R^2)}{3\cdot 2^7}\left[\mathrm{tr}(R^4) + \frac{1}{4}\left(\mathrm{tr}(R^2)\right)^2\right] + \frac{1}{24\cdot 48}\mathrm{Tr}(F^4)\mathrm{tr}(R^2)$$
$$-\frac{1}{2^8}\mathrm{Tr}(F^2)\left[\frac{1}{45}\mathrm{tr}(R^4) + \frac{1}{36}\left(R^2\right)^2\right] - \frac{1}{720}\mathrm{Tr}(F^6)$$

- The first term can be cancelled only if the gauge group has 496 generators.
- This selects groups as $SO(32)$ and $E_8 \times E_8$ or also $E_8 \times (U(1))^{248}$.
- Since type I theory has a $SO(N)$ gauge group, here we consider only $SO(32)$.
- Actually also $SP(N)$ is allowed in type I, but it will be anomalous.
- For the sake of simplicity, we restrict our analysis only to the gauge anomaly that is given by only one term:

$$A_{12} \sim -\frac{1}{720} \mathrm{Tr}(F^6)$$

  The trace is in the adjoint representation of $SO(32)$.
- In $SO(32)$ we have the following identity:

$$\mathrm{Tr}(F^6) = 15 \mathrm{tr}(F^2) \cdot \mathrm{tr}(F^4)$$

  The last two traces are in the fundamental representation of $SO(32)$.

- We get:

$$A_{12} \sim -\frac{1}{48}\mathrm{tr}(F^2) \cdot \mathrm{tr}(F^4)$$

- $A_{11}$ can easily be computed from the observation that

$$d\omega_3 \equiv d\mathrm{tr}\left(AF - \frac{1}{3}A^3\right) = \mathrm{tr}(F^2)$$

$\omega_3$ is called the Chern-Simon three-form.
- Since $d\mathrm{tr}\left(F^4\right) = 0$ one gets:

$$A_{11} \sim -\omega_3 \cdot \mathrm{tr}(F^4)$$

- Using then the following identity:

$$\delta\omega_3 = \mathrm{tr}(d\Lambda dA)$$

one gets:

$$\delta A_{11} = -\delta\omega_3\,\mathrm{tr}(F^4) = -\mathrm{tr}(d\Lambda dA) \cdot \mathrm{tr}(F^4)$$
$$= -d\left(\mathrm{tr}(\Lambda dA) \cdot \mathrm{tr}(F^4)\right) \equiv dA^1_{10}$$

## Few details on the Chern-Simon 3-form

- The Chern-Simon 3-form is given by:

$$\omega_3 \equiv \mathrm{tr}\left(AF - \frac{1}{3}A^3\right) = \mathrm{tr}\left(\frac{1}{2}A_\mu F_{\nu\rho} - \frac{1}{3}A_\mu A_\nu A_\rho\right) dx^\mu \wedge dx^\nu \wedge dx^\rho$$

- From it one gets:

$$d\omega_3 = \mathrm{tr}\left(dAF - AdF - \frac{1}{3}dAA^2 + \frac{1}{3}AdAA - \frac{1}{3}A^2 dA\right)$$
$$= \mathrm{tr}\left(F^2 - A^2F - A(dAA - AdA) - A^2 dA\right) = \mathrm{tr}\left(F^2\right)$$

- It is also easy to show that

$$\delta\omega_3 = \mathrm{tr}\left(d\Lambda dA\right)$$

where

$$\delta A = d\Lambda + [A, \Lambda] \;\; ; \;\; \delta F = [F, \Lambda]$$

- The anomaly in ten dimensions is then given by:

$$A^1_{10} \sim -\mathrm{tr}(\Lambda dA) \cdot \mathrm{tr}(F^4)$$

- The previous calculation shows that there is an anomaly.
- In the previous considerations, however, we have used the formulas computed by Alvarez-Gaumé and Witten that are valid in the limiting field theory with very specific couplings.
- The limiting field theory is type I supergravity.
- In type I supergravity there are additional couplings involving the Chern-Simon 3-form.
- Because of this, the two-form potential $C_2$ is changed by a gauge transformation.
- This gives an additional possibility to cancel the anomaly.
- Let us then look at type I supergravity.

- The action of type I supergravity is given by:

$$S = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} e^{-2\phi} \left[ R + 4\partial_\mu \phi \partial^\mu \phi - \frac{1}{12} (\tilde{F}_3)_{\mu_1\mu_2\mu_3} (\tilde{F}_3)^{\mu_1\mu_2\mu_3} \right] - \frac{1}{2g_{10}^2} \int d^{10}x \, e^{-\phi} tr \left( F_{2\mu_1\mu_2} F_2^{\mu_1\mu_2} \right)$$

- Local supersymmetry requires the presence of the Chern-Simon form $\omega_3$:

$$\tilde{F}_3 = F_3 - \frac{\kappa_{10}^2}{g_{10}^2} \omega_3 \quad ; \quad F_3 = dC_2 \;\; ; \;\; C_2 \equiv C_{2\mu\nu} \frac{dx^\mu \wedge dx^\nu}{2}$$

$$F_3 = \partial_\rho C_{2\mu\nu} \frac{dx^\rho \wedge dx^\mu \wedge dx^\nu}{6}$$

- The previous Lagrangian that has been determined by requiring local supersymmetry, contains two arbitrary coupling constants: the gravitational one $\kappa_{10}$ and the gauge one $g_{10}$.

- Since $\omega_3$ transforms as follows under a local gauge transformation:

$$\delta\omega_3 = \mathrm{tr}(d\Lambda dA)$$

the previous action is invariant under gauge transformations if $C_2$ also transforms as follows:

$$\delta C_2 = \frac{\kappa_{10}^2}{g_{10}^2} tr\,(\Lambda dA) \Longrightarrow \delta \tilde{F}_3 = 0$$

- The fact that $C_2$ transforms in such an unconventional way under a gauge transformation allows one to add a local counter-term whose gauge variation cancels the anomaly.

- The local counter-term is equal to:

$$W \sim \frac{g_{10}^2}{\kappa_{10}^2} C_2 \wedge tr(F^4)$$

- By performing on it a gauge transformation we get:

$$\delta W \sim \frac{g_{10}^2}{\kappa_{10}^2} \delta C_2 \wedge tr(F^4) = tr\left(\Lambda dA\right) \wedge tr(F^4)$$

- One finds that

$$I_{10}^1 + \delta W = 0$$

- No anomaly in type I supergravity.
- The cancellation of the anomaly occurs more naturally in type I string theory.
- Because of chiral fermions, one gets an anomaly due to the one-loop hexagonal planar diagram, that is equal to the one that we have computed before in field theory.

- But in string theory one has also a non-planar one-loop diagram.
- This contribution is also anomalous and such to cancel the anomaly that comes from the planar loop.
- In particular, the anomalous contribution in the non-planar loop comes from a closed string "tree diagram" where the field $C_2$ is exchanged.
- Although the anomaly has also been cancelled in the limiting type I supergravity by adding by hand a local counter-term, this cancellation appears more natural in string theory.
- In a way this cancellation is due to string effects.
- This follows from the fact that in string theory $\kappa_{10}$ and $g_{10}$ are given by:

$$\kappa_{10}^2 \sim (2\pi)^7 (\alpha')^4 \quad ; \quad g_{10}^2 \sim (2\pi)^7 (\alpha')^3 \Longrightarrow \frac{\kappa_{10}^2}{g_{10}^2} \sim \alpha'$$

- This means that ($c$ is a numerical constant )

$$\tilde{F}_3 = F_3 - c\alpha' \omega_3$$

# Conclusions

- The fact that type I string theory has chiral fermions and is free from gauge and gravitational anomalies generated in 1984 a lot of enthusiasm among particle physicists.
- A lot of them started to work again in string theory.
- Two new heterotic string theories with gauge groups $E_8 \times E_8$ and $SO(32)$ were constructed.
- They are called heterotic because the left movers are as in superstring, while the right movers are as in the bosonic string with $16 = 26 - 10$ directions compactified on a 16-torus.
- They are closed string theories, but, unlike type II theories, contain gauge bosons and their interaction unifies, in a consistent quantum theory, gauge theories with gravity.
- They are free from gauge and gravitational anomalies as it was anticipated by the anomaly calculation.
- All five consistent superstring theories are 10-dimensional and supersymmetric.

- In order to be consistent with experiments, we have to compactify six of the ten dimensions on a compact manifold.
- It was found that, by compactifying the six extra dimensions on a Calabi-Yau manifold, one is left with a $\mathcal{N} = 1$ four-dimensional supersymmetric theory that can have chiral fermions.
- In other words, one started to try to connect string theory with experiments.
- This is still going on today.