Interval exchange maps and translation surfaces

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Introduction

Let T be a 2-dimensional torus equipped with a flat Riemannian metric and a vector field which is unitary and parallel for that metric. Then there exists a unique lattice $\Lambda \subset \mathbb{R}^2$ such that T is isometric to $\mathbb{R}^2/\Lambda$ and the vector field on T corresponds to the vertical vector field $\frac{\partial}{\partial y}$ on $\mathbb{R}^2/\Lambda$. The corresponding "Teichmüller space" (classification modulo diffeomorphisms isotopic to the identity) is thus $GL(2,\mathbb{R})$, viewed as the space of lattices equipped with a basis; the "moduli space" (classification modulo the full diffeomorphism group) is the homogeneous space $GL(2,\mathbb{R})/GL(2,\mathbb{Z})$, viewed as the space of lattices in $\mathbb{R}^2$.

The dynamics of the vertical vector field on $\mathbb{R}^2/\Lambda$ can be analyzed through the return map to a non vertical closed oriented geodesic S on $\mathbb{R}^2/\Lambda$; in the natural parameter on S which identifies S with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (after scaling time), the return map is a rotation $x \mapsto x + \alpha$ on $\mathbb{T}$ for some $\alpha \in \mathbb{T}$. When $\alpha \notin \mathbb{Q}/\mathbb{Z}$, all orbits are dense and equidistributed on $\mathbb{R}^2/\Lambda$: the rotation and the vectorfield are uniquely ergodic (which means that they have a unique invariant probability measure, in this case the respective normalized Lebesgue measures on S and $\mathbb{R}^2/\Lambda$).

In the irrational case, an efficient way to analyze the recurrence of orbits is to use the continuous fraction of the angle α. It is well-known that the continuous fraction algorithm is strongly related to the action of the 1-parameter diagonal subgroup in $SL(2,\mathbb{R})$ on the moduli space $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ of "normalized" lattices in $\mathbb{R}^2$. It is also important in this context that the discrete subgroup $SL(2,\mathbb{Z})$ of $SL(2,\mathbb{R})$ is itself a lattice, i.e. has finite covolume, but is not cocompact.

Our aim is to explain how every feature discussed so far can be generalized to higher genus surfaces. In the first ten sections, we give complete proofs of the basic facts of the theory, which owes a lot to the pionneering work of W. Veech [Ve1]-[Ve5], with significant contributions by M. Keane [Kea1][Kea2], H. Masur [Ma], G. Rauzy [Rau], A. Zorich [Zo2]-[Zo4], A.

Eskin, G.Forni [For1]-[For3] and many others. In the last four sections, we present without proofs some more advanced results in different directions.

The reader is advised to consult [Zo1] for an excellent and very complete survey on translation surfaces. See also [Y1] for a first and shorter version of these notes.

In Section 1 we give the definition of a translation surface, and introduce the many geometric structures attached to it. Section 2 explains how translation surfaces occur naturally in connection with billiards in rational polygonal tables. In Section 3, we introduce interval exchange maps, which occur as return maps of the vertical flow of a translation surface. We explain in Section 4 Veech's fundamental zippered rectangle construction which allow to obtain a translation surface from an interval exchange map and appropriate suspension data. The relation between interval exchange maps and translation surfaces is further investigated in Section 5, which concludes with Keane's theorem on the minimality of interval exchange maps with no connection. Section 6 introduces the Teichmüller spaces and the moduli spaces; the fundamental theorem of Masur and Veech on the finiteness of the canonical Lebesgue measure in normalized moduli space is stated. In Section 7, we introduce the Rauzy-Veech algorithm for interval exchange maps with no connection, which is a substitute for the continuous fraction algorithm. The basic properties of this algorithm are established. Invariant measures for interval exchange maps with no connection are considered in Section 8. In Section 9, the dynamics in parameter space are introduced, whose study lead ultimately to a proof of the Masur-Veech theorem. Almost sure unique ergodicity of interval exchange maps, a related fundamental result of Masur and Veech, is proven in Section 10.

In Section 11, we introduce the Kontsevich-Zorich cocycle, and present the related results of Forni and Avila-Viana. In section 12, we consider the cohomological equation for an interval exchange map and present the result of Marmi, Moussa and myself, which extend previous fundamental work of Forni. In Section 13, we present the classification of the connected components of the moduli space by Kontsevich and Zorich. In the last section, we discuss the exponential mixing of the Teichmüller flow proved by Avila, Gouezel and myself.

1 Definition of a translation surface

1.1 We start from the following combinational data :

- a compact orientable topological surface M of genus $g \geqslant 1$;
- a non-empty finite subset $\Sigma = \{A_1, \dots, A_s\}$ of M ;
- an associated family $\kappa = (\kappa_1, \dots, \kappa_s)$ of positive integers which should be seen as **ramification indices**.

Moreover we require (for reasons that will be apparent soon) that κ and g are related through

(1) $$2g - 2 = \sum_{i=1}^{s} (\kappa_i - 1) \ .$$

The classical setting considered in the introduction corresponds to $g = 1, s = 1, \kappa_1 = 1$.

Definition : A structure of translation surface on (M, Σ, K) is a maximal atlas ζ for $M - \Sigma$ of charts by open sets of $\mathbb{C} \simeq \mathbb{R}^2$ which satisfies the two following properties :

(i) any coordinate change between two charts of the atlas is locally a translation of $\mathbb{R}^2$;

(ii) for every $1 \leqslant i \leqslant s$, there exists a neighbourhood V_i of A_i, a neighbourhood W_i of 0 in $\mathbb{R}^2$ and a ramified covering $\pi : (V_i, A_i) \to (W_i, 0)$ of degree κ_i such that every injective restriction of π is a chart of ζ.

1.2 Because many structures on $\mathbb{R}^2$ are translation-invariant, a translation surface $(M, \Sigma, \kappa, \zeta)$ is canonically equipped with several auxiliary structures:

- a preferred orientation ; actually, one frequently starts with an **oriented** (rather than orientable) surface M and only considers those translation surface structures which are compatible with the preferred orientation ;
- a structure of Riemann surface ; this is only defined initially by the atlas ζ on $M - \Sigma$, but is easily seen to extend to M in a unique way : if V_i is a small disk around $A_i \in \Sigma, V_i - \{A_i\}$ is the κ_i- fold covering of $W_i - \{0\}$, with W_i a small disk around $0 \in \mathbb{C}$, hence is biholomorphic to $\mathbb{D}^*$;
- a flat metric on $M - \Sigma$; the metric exhibits a true singularity at each A_i such that $\kappa_i > 1$; the total angle around each $A_i \in \Sigma$ is $2\pi\kappa_i$;
- an area form on $M - \Sigma$, extending smoothly to M ; in the neighbourhood of $A_i \in \Sigma$, it takes the form $\kappa_i^2 (x^2 + y^2)^{\kappa_i - 1} dx \wedge dy$ in a natural system of coordinates ;
- the geodesic flow of the flat metric on $M - \Sigma$ gives rise to a 1-parameter family of constant unitary directional flows on $M - \Sigma$, containing in particular a vertical flow $\partial/\partial y$ and a horizontal flow $\partial/\partial x$.

We will be interested in the dynamics of these vector fields. By convention (and symmetry) we will generally concentrate on the vertical vector field.

1.3 Together with the complex structure on M, a translation surface structure ζ also provides an holomorphic (w.r.t that complex structure) 1-form ω, characterized by the property that it is written as dz in the charts of ζ. In particular, this holomorphic 1-form does not vanish on $M - \Sigma$. At a point $A_i \in \Sigma$, it follows from condition (ii) that ω has a zero of order $(\kappa_i - 1)$. The relation (1) between g and κ is thus a consequence of the Riemann-Roch formula.

We have just seen that a translation surface structure determine a complex structure on M and a holomorphic 1-form ω with prescribed zeros. Conversely, such data determine a translation surface structure ζ : the charts of ζ are obtained by local integration of the 1-form ω.

The last remark is also a first way to provide explicit examples of translation surfaces. Another very important way, that will be presented in Section 5, is by suspension of one-dimensional maps called interval exchange maps. A third way, which however only gives rise to a restricted family of translation surfaces, is presented in the next section.

2 The translation surface associated to a rational polygonal billiard

2.1 Let U be a bounded connected open subset in $\mathbb{R}^2 \simeq \mathbb{C}$ whose boundary is a finite union of line segments ; we say that U is a polygonal billiard table. We say that U is **rational** if the angle between any two segments in the boundary is commensurate with π.

The billiard flow associated to the billiard table U is governed by the laws of optics (or mechanics) : point particles move linearly at unit speed inside U, and reflect on the smooth parts of the boundary ; the motion is stopped if the boundary is hit at a non smooth point, but this only concerns a codimension one subset of initial conditions.

The best way to study the billiard flow on a rational polygonal billiard table is to view it as the geodesic flow on a translation surface constructed from the table ; this is the construction that we now explain.

2.2 Let $\widehat{U}$ be the **prime end compactification** of U : a point of $\widehat{U}$ is determined by a point z_0 in the closure $\overline{U}$ of U in $\mathbb{C}$ and a component of $B(z_0, \varepsilon) \cap U$ with ε small enough (as U is polygonal, this does not depend on ε if ε is small enough).

Exercise : Define the natural topology on $\widehat{U}$; prove that $\widehat{U}$ is compact, and that the natural map from U into $\widehat{U}$ is an homeomorphism onto a dense open subset of $\widehat{U}$.

Exercise : Show that the natural map from $\widehat{U}$ onto $\overline{U}$ is injective (and then a homeomorphism) iff the boundary of U is the disjoint union of finitely many polygonal Jordan curves.

A point in $\widehat{U} - U$ is **regular** if the corresponding sector in $B(z_0, \varepsilon) \cap U$ is flat ; the non regular points of $\widehat{U} - U$ are the vertices of $\widehat{U}$.

Exercise : Show that every component of $\widehat{U} - U$ is homeomorphic to a circle and contain at least two vertices. Show that there are only finitely many vertices.

A connected component of regular points in $\widehat{U} - U$ is a **side** of $\widehat{U}$. The closure in $\widehat{U}$ of a side C of $\widehat{U}$ is the union of C and two distinct vertices called the **endpoints** of C. A vertex is the endpoint of exactly two sides.

2.3 The previous considerations only depend on U being a polygonal billiard table ; we now assume that U is rational. For each side C of $\widehat{U}$, let $\sigma_C \in O(2, \mathbb{R})$ the orthogonal symmetry with respect to the direction of the image of C in $\overline{U} \subset \mathbb{R}^2$. Let G be the subgroup of $O(2, \mathbb{R})$ generated by the σ_C.

As U is rational, G is finite. More precisely, if N is the smallest integer such that the angle between any two sides of $\widehat{U}$ can be written as $\pi m/N$ for some integer m, G is a dihedral group of order $2N$, generated by the rotations of order N and a symmetry σ_C.

For any vertex $q \in \widehat{U}$, we denote by G_q the subgroup of G generated by σ_C and $\sigma_{C'}$, where C and C' are the sides of $\widehat{U}$ having q as endpoint ; if the angle of C and C' is $\pi\ m_q/N_q$ with $m_q \wedge N_q = 1, G_q$ is dihedral of order $2N_q$.

We now define a topological space M as the quotient of $\widehat{U} \times G$ by the following equivalence relation : two points $(z,g),(z',g')$ are equivalent iff $z = z'$ and moreover

- $g^{-1}g' = \mathbf{1}_G$ if $z \in U$;
- $g^{-1}g' \in \{\mathbf{1}_G, \sigma_C\}$ if z belongs to a side C of $\widehat{U}$;
- $g^{-1}g' \in G_z$ if z is a vertex of $\widehat{U}$.

We also define a finite subset Σ of M as the image in M of the vertices of $\widehat{U}$.

Exercise : Prove that M is a compact topological orientable surface.

To define a structure of translation surface on (M, Σ) (with appropriate ramification indices), we consider the following atlas on $M - \Sigma$.

- for each $g \in G$, we have a chart

$$U \times \{g\} \to \mathbb{R}^2$$
$$(z,g) \mapsto g(z) \ ;$$

- for each z_0 belonging to a side C of $\widehat{U}$, and each $g \in G$, let $\tilde{z}_0$ be the image of z_0 in $\overline{U}, \varepsilon$ be small enough, V be the component of $B(\tilde{z}_0, \varepsilon) \cap U$ corresponding to $z_0, \widehat{V}$ be interior of the closure of the image of V in $\widehat{U}$; we have a map

$$\widehat{V} \times \{g, g\ \sigma_c\} \to \mathbb{R}^2$$

  sending (z,g) to $g(z)$ and $(z, g\sigma_c)$ to $g(\tilde{\sigma}_c(z))$, where $\tilde{\sigma}_c$ is the **affine** orthogonal symmetry with respect to the line containing the image of C in $\mathbb{R}^2$. This map is compatible with the identifications defining M and defines a chart from a neighbourhood of (z,g) in M onto an open subset of $\mathbb{R}^2$.

One checks easily that the coordinate changes between the charts considered above are translations. One then completes this atlas to a maximal one with property (i) of the definition of translation surfaces.

Exercise : Let q be a vertex of $\widehat{U}$, and let $\pi m_q/N_q$ be the angle between the sides at q and G_q the subgroup of G as above. Show that property (ii) in the definition of a translation surface is satisfied at any point $(q, g\,G_q) \in \Sigma$, with ramification index m_q (independent of the coset $g\,G_q$ under consideration).

We have therefore defined the ramification indices κ_i at the points of Σ and constructed a translation surface structure on (M, Σ, κ).

2.4 The relation between the trajectories of the billiard flow on U and the geodesics on $M - \Sigma$ is as follows.

Let $z(t), 0 \leqslant t \leqslant T$ be a billiard trajectory ; let $t_1 < \ldots < t_N$ be the successive times in $(0, T)$ where the trajectory bounces on the sides of $\widehat{U}$ (by hypothesis, the trajectory does not go through a vertex, except perhaps at the endpoints 0 and T). Denote by C_i the side met at time t_i and define inductively $g_0, \ldots, g_N$ by

$$g_0 = \mathbf{1}_G \ ,$$
$$g_{i+1} = g_i \ \sigma_{C_{i+1}} \ .$$

For any $g \in G$, the formulas

$$z_g(t) = \begin{cases} (z(t), gg_0), & \text{for } 0 \leqslant t \leqslant t_1, \\ (z(t), gg_i), & \text{for } t_i \leqslant t \leqslant t_{i+1} \ (1 \leqslant i < N), \\ (z(t), gg_N), & \text{for } t_N \leqslant t \leqslant T, \end{cases}$$

define a geodesic path on M. Conversely, every geodesic path on M (contained in $M - \Sigma$ except perhaps for its endpoints) defines by projection on the first coordinate a trajectory of the billiard flow on U.

2.5 The left action

$$g_0(z, g) = (z, g_0\, g)$$

of G on $\widehat{U} \times G$ is compatible with the equivalence relation defining M and therefore defines a left action of G on M. The corresponding transformations of M are isometries of the flat metric of M but not isomorphisms of the translation surface structure (except for the identity !). The existence of such a large group of isometries explain why the translation surfaces constructed from billiard tables are special amongst general translation surfaces.

2.6 On the other hand, when a billiard table admits non trivial symmetries, this gives rise to isomorphisms of the translation surface structures. More precisely, let H be the subgroup of G formed of the $h \in G$ such that $h(U)$ is a translate $U + t_h$ of U. The group H acts on the left on M through the formula

$$h(z, g) = (h(z) - t_h, g\, h^{-1}),$$

which is compatible with the equivalence relation defining M. Each $h \in H$ acts through an isomorphism of the translation surface structure (permuting the points of Σ). This allows to consider the quotient under the action of H to get a reduced translation surface $(M', \Sigma', \kappa', \zeta')$ and a ramified covering from (M, Σ) onto (M', Σ').

2.7 To illustrate all this, consider the case where U is a regular n-gon, $n \geqslant 3$. The angle at each vertex is then $\pi \frac{n-2}{n}$.

Exercise : Show that $G = G_q$ for every vertex q and that G has order n if n is even, $2n$ if n is odd. Show that Σ has n points, each having ramification index $n-2$ if n is odd, $\frac{n-2}{2}$ if n is even.

Conclude that the genus of M is $\frac{(n-1)(n-2)}{2}$ if n is odd, $(\frac{n}{2} - 1)^2$ if n is even.

Exercise : Show that the subgroup H of subsection 2.6. is equal to G if n is even, and is of index 2 if n is odd. Show that the reduced translation surface satisfies $\#\Sigma' = 2$ if $N - 2$ is divisible by 4, $\#\Sigma' = 1$ otherwise. Show that the corresponding ramification index is $n - 2$ if n odd, $\frac{(n-2)}{2}$ if n is divisible by 4, $\frac{(n-2)}{4}$ if $n - 2$ is divisible by 4. Conclude that the genus g' is $\frac{(n-1)}{2}$ if n is odd, $\frac{n}{4}$ if n is divisible by 4 , $\frac{(n-2)}{4}$ if $n - 2$ is divisible by 4.

3 Interval exchange maps : basic definitions

3.1 Let $(M, \Sigma, \kappa, \zeta)$ be a translation surface and let X be one of the non zero constant vector fields on $M - \Sigma$ defined by ζ.

Definitions An **incoming (resp. outgoing) separatrix** for X is an orbit of X ending (resp. starting) at a marked point in Σ. A **connection** is an orbit of X which is both an incoming and outgoing separatrix.

At a point $A_i \in \Sigma$, there are κ_i incoming separatrices and κ_i outgoing separatrices.

Let S be an open bounded geodesic segment in $M - \Sigma$, parametrized by arc length, and transverse to X. Consider the first return map T_S to S of the flow generated by the vectorfield X.

As X is area-preserving, the Poincaré recurrence theorem guarantees that the map T_S is defined on a subset D_{T_S} of S of full 1-dimensional Lebesgue measure. The domain D_{T_S} is open because S itself is open and the restriction of T_S to each component of D_{T_S} is a translation (because the flow of X is isometric). Also, the return time is constant on each component of D_{T_S}.

We now show that D_{T_S} has only finitely many components. Indeed, let $x \in S$ be an endpoint of some component J of D_{T_S}, and let t_J the return time to S of points in J. Either there exists $T \in (0, t_J)$ such that the orbit of X starting at x stops at time T at a point of Σ without having crossed S, or the orbit of X starting at x is defined up to time t_J and is at this moment at one of the endpoints of S, also without having crossed S. This leaves only a finite number of possibilities for x, which gives the finiteness assertion.

The return map T_S is thus an interval exchange map according to the following definition.

Definition Let $I \subset \mathbb{R}$ be a bounded open interval. An interval exchange map (i.e.m) T on I is a one-to-one map $T : D_T \to D_{T^{-1}}$ such that $D_T \subset I, D_{T^{-1}} \subset I, I - D_T$ and $I - D_{T^{-1}}$ are finite sets (with the same cardinality) and the restriction of T to each component of D_T is a translation onto some component of $D_{T^{-1}}$.

3.2 Markings, combinatorial data

Let $T : D_T \to D_{T^{-1}}$ be an interval exchange map. Let $d = \#\pi_0(D_T) = \#\pi_0(D_{T^{-1}})$. Then T realizes a bijection between $\pi_0(D_T)$ and $\pi_0(D_{T^{-1}})$. To keep track of the combinatorial data, in particular when we will consider below the Rauzy-Veech continuous fraction algorithm for i.e.m, it is convenient to give names to the components of D_T (and therefore through T also to those of $D_{T^{-1}}$). This is formalized as follow.

A **marking** for T is given by an alphabet $\mathcal{A}$ with $\#\mathcal{A} = d$ and a pair $\pi = (\pi_t, \pi_b)$ of one-to-one maps

$$\begin{matrix} \pi_t \\ \\ \pi_b \end{matrix} \quad \mathcal{A} \to \{1, \dots, d\}$$

such that, for each $\alpha \in \mathcal{A}$, the component of D_T in position $\pi_t(\alpha)$ (counting from the left) is sent by T to the component of $D_{T^{-1}}$ in position $\pi_b(\alpha)$. We summarize these combinatorial data by writing just

$$\begin{pmatrix} \pi_t^{-1}(1) & \dots & \pi_t^{-1}(d) \\ \pi_b^{-1}(1) & \dots & \pi_b^{-1}(d) \end{pmatrix}$$

expressing how the intervals which are exchanged appear before and after applying T.

Two markings $(\mathcal{A}, \pi_t, \pi_b), (\mathcal{A}', \pi'_t, \pi'_b)$ are equivalent if there exists a bijection $i : \mathcal{A} \to \mathcal{A}'$ with $\pi_t = \pi'_t \circ i, \pi_b = \pi'_b \circ i$.

Clearly T determines the marking up to equivalence.

3.3 Irreducible combinatorial data

We say that combinatorial data $(\mathcal{A}, \pi_t, \pi_b)$ are irreducible if for every $1 \leqslant k < d = \#\mathcal{A}$, we have

$$\pi_t^{-1}(\{1, \dots, k\}) \neq \pi_b^{-1}(\{1 \dots k\}).$$

The condition is invariant under equivalence of markings. We will always assume that the i.e.m under consideration satisfy this property. Otherwise, if we have

$$\pi_t^{-1}(\{1, \dots, k\}) = \pi_b^{-1}(\{1 \dots k\})$$

T is the juxtaposition of an i.e.m with k intervals and another with $d-k$, and the dynamics of T reduce to simpler cases.

3.4 Terminology and notations

Let $T : D_T \to D_{T^{-1}}$ be an i.e.m on an interval I ; let $(\mathcal{A}, \pi_t, \pi_b)$ a marking for T.

The points $u_1^t < u_2^t < \ldots < u_{d-1}^t$ of $I - D_T$ are called the **singularities** of T ; the points $u_1^b < u_2^b < \ldots < u_{d-1}^b$ of $I - D_{T^{-1}}$ are called the **singularities** of T^{-1}.

For each $\alpha \in \mathcal{A}$, we denote by I_α^t or just I_α the component of D_T in position $\pi_t(\alpha)$ (counting from the left), and by I_α^b its image by T which is also the component of $D_{T^{-1}}$ in position $\pi_b(\alpha)$.

We denote by λ_α the common length of I_α^t and I_α^b. The vector $\lambda = (\lambda_\alpha)_{\alpha \in \mathcal{A}}$ in $\mathbb{R}^{\mathcal{A}}$ is the **length vector** and will be considered as a **row vector**.

On the other hand, let δ_α be the real number such that $I_\alpha^b = I_\alpha^t + \delta_\alpha$. The vector $\delta = (\delta_\alpha)_{\alpha \in \mathcal{A}}$ is the **translation vector** and will be considered as a column vector.

The length vector and the translation vector are related through the obvious formulas

$$\delta_\alpha = \sum_{\pi_b(\beta) < \pi_b(\alpha)} \lambda_\beta - \sum_{\pi_t(\beta) < \pi_t(\alpha)} \lambda_\beta = \sum_\beta \Omega_{\alpha\beta} \lambda_\beta$$

where the antisymmetric matrix Ω is defined by

$$\Omega_{\alpha\beta} = \begin{cases} +1 & \text{if } \pi_b(\beta) < \pi_b(\alpha) \text{ and } \pi_t(\beta) > \pi_t(\alpha), \\ -1 & \text{if } \pi_b(\beta) > \pi_b(\alpha) \text{ and } \pi_t(\beta) < \pi_t(\alpha), \\ 0 & \text{otherwise.} \end{cases}$$

4 Suspension of i.e.m : the zippered rectangle construction

4.1 We have seen in subsection 3.1 that we come naturally to the definition of an interval exchange map by considering return maps for constant vector fields on translation surfaces.

Conversely, starting from an interval exchange map T, we will construct, following Veech [Ve2] a translation surface for which T appears as a return map of the vertical vector field. However, as the case of the torus for rotations already demonstrates, supplementary data such as return times are needed to specify uniquely the translation surface.

Let $T : D_T \to D_{T^{-1}}$ be an i.e.m on an interval I, equipped with a marking $(\mathcal{A}, \pi_t, \pi_b)$ as above.

A vector $\tau \in \mathbb{R}^{\mathcal{A}}$ is a **suspension vector** if it satisfies the following inequalities

$$(S_\pi) \qquad \sum_{\pi_t(\alpha)<k} \tau_\alpha > 0 \ , \qquad \sum_{\pi_b(\alpha)<k} \tau_\alpha < 0 \qquad \text{for all } 1 < k \leqslant d \ .$$

Define

$$\tau_\alpha^{can} = \pi_b(\alpha) - \pi_t(\alpha) \quad , \quad \alpha \in \mathcal{A} \ .$$

Then the vector τ^{can} satisfies (S_π) iff the combinatorial data are irreducible (an hypothesis that we will assume from now on). When the combinatorial data are not irreducible, no vector $\tau \in \mathbb{R}^{\mathcal{A}}$ satisfies (S_π).

4.2 A simple version of the construction

Let T as above ; we assume that the combinatorial data are irreducible and use the notations of subsection 3.4. Let also $\tau \in \mathbb{R}^{\mathcal{A}}$ be a suspension vector.

We will construct from these data a translation surface $(M, \Sigma, \kappa, \zeta)$. We first give a simple version of the construction that unfortunately is not valid for all values of the data.

We identify as usual $\mathbb{R}^2$ with $\mathbb{C}$ and set $\zeta_\alpha = \lambda_\alpha + i\tau_\alpha$ for $\alpha \in \mathcal{A}$.

Consider the "top" polygonal line connecting the points 0, $\zeta_{\pi_t^{-1}(1)}$, $\zeta_{\pi_t^{-1}(1)} + \zeta_{\pi_t^{-1}(2)}, \ldots,$ $\zeta_{\pi_t^{-1}(1)} + \zeta_{\pi_t^{-1}(2)} + \ldots + \zeta_{\pi_t^{-1}(d)}$ and the "bottom" polygonal line connecting the points $0, \zeta_{\pi_b^{-1}(1)}$, $\zeta_{\pi_b^{-1}(1)} + \zeta_{\pi_b^{-1}(2)}, \ldots, \zeta_{\pi_b^{-1}(1)} + \zeta_{\pi_b^{-1}(2)} + \ldots + \zeta_{\pi_b^{-1}(d)}$. Observe that both lines have the same endpoints and that, from the suspension condition (S_π), all intermediary points in the top (resp. bottom) line lie in the upper (resp. lower) half-plane.

When the two lines do not intersect except from their endpoints, their union is a Jordan curve and we can construct a translation surface as follows : denoting by W the closed polygonal disk bounded by the two lines, we identify for each $\alpha \in \mathcal{A}$ the ζ_α side of the top line with the ζ_α side of the bottom line through the appropriate translation and define M to be the topological space obtained from W with this identifications. The finite subset Σ is the image of the vertices of W.

Exercise : Check that M is indeed a compact oriented topological surface.

The atlas defining the translation surface structure is obvious : besides the identity map on the interior of W, we use charts defined on neighbourhoods of the interiors of the ζ_α sides which have been identified.

Condition (ii) in the definition of a translation surface and ramification indices will be discussed below.

This construction is very easy to visualize, and the non intersection condition is frequently satisfied : for instance when $\sum_\alpha \tau_\alpha = 0$ (in particular for $\tau = \tau^{can}$), or when $\lambda_{\pi_t^{-1}(d)} = \lambda_{\pi_b^{-1}(d)}$. Unfortunately, it is not always satisfied. For instance, taking for combinatorial data (with $\mathcal{A} = \{A, B, C, D\}$),

$$\pi = (\pi_t, \pi_b) = \begin{pmatrix} A & B & D & C \\ D & A & C & B \end{pmatrix},$$

we may have $\zeta_A = 1+i,\ \zeta_B = 3+3i,\ \zeta_C = \varepsilon + i,\ \zeta_D = 3-3i$ with $\varepsilon > 0$. Then the suspension condition (S_π) is satisfied but the two lines intersect non trivially when $0 < \varepsilon < 1$.

4.3 Zippered rectangles

Let $T, \lambda, \tau, \zeta = \lambda + i\tau$ as above. The length vector and the translation vector δ are related through.

$$\delta = \Omega^t \lambda\,.$$

We define

$$h = -\Omega^t \tau\ ,$$

$$\theta = \delta - ih = \Omega^t \zeta\,.$$

We consider here λ, τ as row vectors in $\mathbb{R}^{\mathcal{A}}$, ζ as a row vector in $\mathbb{C}^{\mathcal{A}}$, δ, h as column vectors in $\mathbb{R}^{\mathcal{A}}$ and θ as a column vector in $\mathbb{C}^{\mathcal{A}}$.

Exercise : Check that in the construction of subsection 4.2, the ζ_α side of the "top line" was identified to the ζ_α side of the "bottom line" through a translation by θ_α.

We observe that for all $\alpha \in \mathcal{A}$ we have

$$h_\alpha = \sum_{\pi_t\beta < \pi_t\alpha} \tau_\beta - \sum_{\pi_b\beta < \pi_b\alpha} \tau_\beta$$

and therefore, from the suspension condition (S_π) :

$$h_\alpha > 0\,.$$

Indeed, the first sum on the right-hand side is > 0 except if $\pi_t\alpha = 1$ when it is 0 and the second sum is < 0 except if $\pi_b\alpha = 1$ when it is 0. By irreducibility, we cannot have both $\pi_t\alpha = 1$ and $\pi_b\alpha = 1$.

Define the rectangles in $\mathbb{R}^2 = \mathbb{C}$:

$$R^t_\alpha = I^t_\alpha \times [0, h_\alpha]\ ,$$

$$R^b_\alpha = I^b_\alpha \times [-h_\alpha, 0] \ ,$$

Let $u^t_1 < u^t_2 < \ldots < u^t_{d-1}$ be the singularities of $T, u^b_1 < u^b_2 < \ldots < u^b_{d-1}$ those of T^{-1}. Write also $I = (u_0, u_d)$. Define, for $1 \leqslant i \leqslant d-1$:

$$S^t_i = \{u^t_i\} \times [0, \sum_{\pi_t \alpha \leqslant i} \tau_\alpha),$$

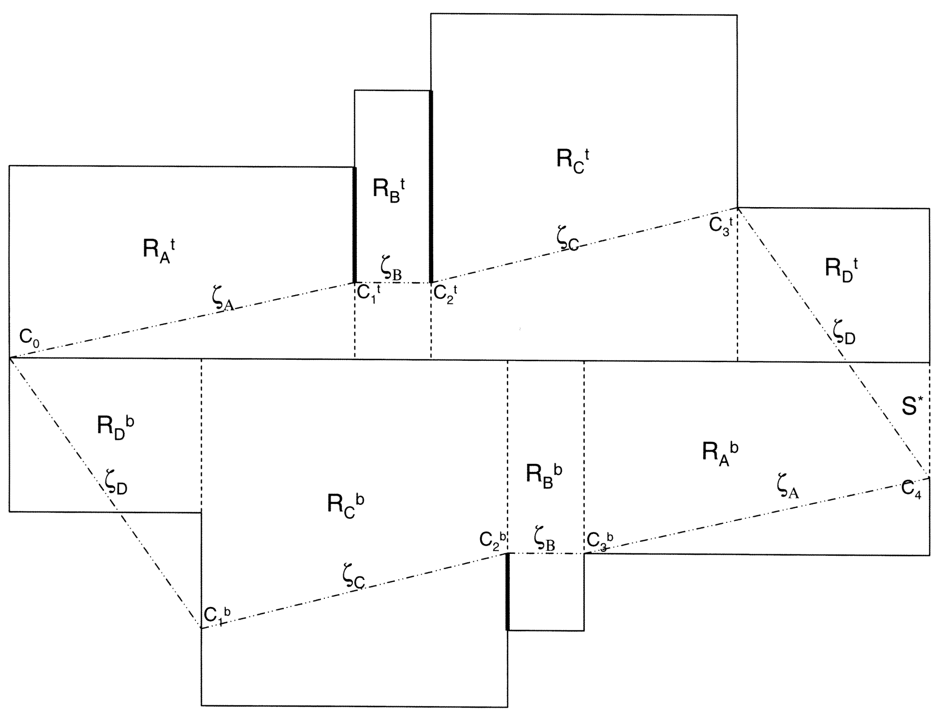
$$S^b_i = \{u^b_i\} \times (\sum_{\pi_b \alpha \leqslant i} \tau_\alpha, 0].$$

Define the points

$$C_0 = (u_0, 0), \qquad C_d = (u_d, \sum_\alpha \tau_\alpha),$$

$$C^t_i = C_0 + \sum_{\pi_t \alpha \leqslant i} \zeta_\alpha, \qquad C^b_i = C_0 + \sum_{\pi_b \alpha \leqslant i} \zeta_\alpha \ , \qquad \text{for } 0 < i < d.$$

Finally, let S^* be the closed vertical segment whose endpoints are $(u_d, 0)$ and C_d.



Let $\widehat{M}$ be the union of all the elements just defined : the R^t_α, R^b_α, $(\alpha \in \mathcal{A})$, S^t_i, S^b_i, $(0 < i < d)$, C_0, C_d, C^t_i, C^b_i, $(0 < i < d)$ and S^*.

We use translations by $\theta_\alpha, \alpha \in \mathcal{A}$ to identify some of these elements :

- We identify R^t_α and $R^b_\alpha = R^t_\alpha + \theta_\alpha$.
- We identify $C^t_{\pi_t(\alpha)}$ and $C^b_{\pi_b(\alpha)} = C^t_{\pi_t(\alpha)} + \theta_\alpha$, and also $C^t_{\pi_t(\alpha)-1}$ and $C^b_{\pi_b(\alpha)-1} = C^t_{\pi_t(\alpha)-1} + \theta_\alpha$; here, we have by convention $C^t_0 = C^b_0 = C_0, C^t_d = C^b_d = C_d$.
- finally, if $\Sigma_\alpha\, \tau_\alpha > 0$, we identify by $\theta_{\pi_b^{-1}(d)}$ the top part of $S^t_{\pi_t\pi_b^{-1}(d)}$ with S^* ; if $\Sigma_\alpha\, \tau_\alpha < 0$, we identify S^* with the bottom part of $S^b_{\pi_b\pi_t^{-1}(d)}$ by $\theta_{\pi_t^{-1}(d)}$.

We denote by M the topological space deduced from $\widehat{M}$ by these identifications. We denote by Σ the part of M which is the image of $\{C_0, C_d, C^t_i, C^b_i\}$.

One easily checks that M is compact and that $M - \Sigma$ is a topological orientable surface. Every point in $M - \Sigma$, except those in the image of S^* when $\Sigma\, \tau_\alpha \neq 0$, has a representative in the interior of $\widehat{M}$; for those points, a local continuous section of the projection from $\widehat{M}$ onto M provides a chart for the atlas defining the translation surface structure. We leave the reader provide charts around points in the image of S^*.

In the next section, we complete the construction by investigating the local structure at points in Σ : this means checking that M is indeed a topological surface, that condition (ii) in the definition of translation surfaces is satisfied, and computing the ramification indices.

Let us however observe right now that we have indeed a suspension for the i.e.m. T on I. The return map on the horizontal segment $I \times \{0\}$ (or rather its image in M) of the vertical vector field $\dfrac{\partial}{\partial y}$ is exactly T. The return time of I^t_α is equal to h_α.

4.4 Ramification indices

Let $\mathcal{C}$ the set $\{C^t_i, C^b_i\ ; 0 < i < d\}$ with $2d - 2$ elements ; turning around points of Σ in an anticlockwise manner, we define a "successor" map $\sigma : \mathcal{C} \to \mathcal{C}$:

- $\sigma(C^t_i) = C^b_{\pi_b\pi_t^{-1}(i+1)-1}$, except if $\pi_b\pi_t^{-1}(i+1) = 1$ in which case $\sigma(C^t_i) = C^b_{\pi_b\pi_t^{-1}(1)-1}$;
- $\sigma(C^b_j) = C^t_{\pi_t\pi_b^{-1}(j)}$ except if $\pi_t\pi_b^{-1}(j) = d$ in which case $\sigma(C^b_j) = C^t_{\pi_t\pi_b^{-1}(d)}$.

We see that σ is a permutation of $\mathcal{C}$, exchanging the C^t_i and the C^b_j. Therefore every cycle of σ has even length.

From the very definition of σ, points of Σ are in one-to-one correspondance with the cycles of σ. Moreover, one checks that small neighbourhoods of points of Σ are homeomorphic to disks, and that condition (ii) in the definition of a translation index is satisfied, the ramification index being half the length of the corresponding cycle.

Summing up :

- The number s of points in Σ is the number of cycles of the permutation σ.
- The ramification indices κ_j are the half lengths of the cycles ; in particular, we have

$$d-1=\sum_{j=1}^{s}\kappa_j\ .$$

If g is the genus of the compact surface M, we also must have

$$2g-2=\sum_{i=1}^{s}(\kappa_i-1)\ .$$

We therefore can relate d,g,s by

$$d=2g+s-1\ .$$

4.5 Homology and cohomology of M

Consider the homology groups $H_1(M,\mathbb{Z}), H_1(M-\Sigma,\mathbb{Z}), H_1(M,\Sigma,\mathbb{Z})$. The first one has rank $2g$, the last two have rank $2g+s-1=d$. They are related through maps

$$H_1(M-\Sigma,\mathbb{Z})\to H_1(M,\mathbb{Z})\to H_1(M,\Sigma,\mathbb{Z})$$

where the first map is onto and the second is injective.

The zippered rectangle construction provides natural bases for $H_1(M-\Sigma,\mathbb{Z})$ and $H_1(M,\Sigma,\mathbb{Z})$.

For $\alpha\in\mathcal{A}$, let $[\theta_\alpha]$ be the image in $H_1(M-\Sigma,\mathbb{Z})$ of a path joining in the interior of $\widehat{M}$ the center of R^t_α to the center of R^b_α ; and let $[\zeta_\alpha]$ be the image in $H_1(M,\Sigma,\mathbb{Z})$ of a path joining in $R^t_\alpha\cup\{C^t_{\pi_t(\alpha)-1},C^t_{\pi_t(\alpha)}\}$ the point $C^t_{\pi_t(\alpha)-1}$ to $C^t_{\pi_t(\alpha)}$ (if $\pi_t(\alpha)=d$ and $\Sigma_\alpha\tau_\alpha<0$, the path should be allowed to go through S^* also).

The intersection form establishes a duality between $H_1(M-\Sigma,\mathbb{Z})$ and $H_1(M,\Sigma,\mathbb{Z})$. Now we clearly have, for $\alpha,\beta\in\mathcal{A}$:

$$<[\theta_\alpha],[\zeta_\beta]>=\delta_{\alpha\beta}\ ,$$

which shows that $([\theta_\alpha])_{\alpha\in\mathcal{A}},([\zeta_\beta])_{\beta\in\mathcal{A}}$ are respectively bases of $H_1(M-\Sigma,\mathbb{Z}),H_1(M,\Sigma,\mathbb{Z})$ dual to each other.

Considering $[\theta_\alpha]$ as classes in $H_1(M,\mathbb{Z})$, the intersection form now reads :

$$< [\theta_\alpha], [\theta_\beta] >= \Omega_{\beta\alpha} \ ,$$

Indeed, writing $[\overline{\theta_\alpha}]$ for the image of $[\theta_\alpha]$ in $H_1(M,\Sigma,\mathbb{Z})$, we have

$$[\overline{\theta_\alpha}] = \sum_\beta \Omega_{\alpha\beta}[\zeta_\beta]$$

which shows in particular that

$$\text{rk}\ \Omega = 2g \ .$$

Going to cohomology, we have maps

$$H^1(M,\Sigma,\mathbb{Z}) \to H^1(M,\mathbb{Z}) \to H^1(M-\Sigma,\mathbb{Z})$$

(and similar maps with real and complex coefficients) where the first map is onto and the second is injective.

The holomorphic $1-$form ω associated to the translation surface structure determines by integration a class $[\omega] \in H^1(M,\Sigma,\mathbb{C})$ (this will be studied in more details and generality in section 6 below). One has

$$< [\omega], [\zeta_\alpha] >= \zeta_\alpha \ ,$$
$$< [\overline{\omega}], [\theta_\alpha] >= \theta_\alpha \ ,$$

where $[\overline{\omega}]$ is the image of $[\omega]$ in $H^1(M-\Sigma,\mathbb{C})$.

Therefore the vectors λ, τ can be considered as elements of $H^1(M,\Sigma,\mathbb{R})$, the vector $\zeta = \lambda + i\tau$ as an element of $H^1(M,\Sigma,\mathbb{C})$. The vectors δ, h can be considered as elements of $H^1(M-\Sigma,\mathbb{R})$; they actually belong to the image of $H^1(M,\mathbb{R})$ into $H^1(M-\Sigma,\mathbb{R})$ because they vanish on the kernel of the map from $H_1(M-\Sigma,\mathbb{Z})$ to $H_1(M,\mathbb{Z})$. Similarly, $\theta = \delta - ih$ belongs to the image of $H^1(M,\mathbb{C})$ into $H^1(M-\Sigma,\mathbb{C})$.

Finally, the area of the translation surface M is given by

$$A = \sum_\alpha \ \lambda_\alpha \ h_\alpha = \tau \ \Omega \ {}^t\lambda \ .$$

5 Representability, minimality, connections

5.1 We have seen in subsection 3.1 that for any translation surface, the return map of the vertical vector field on any horizontal segment is an interval exchange map. In the zippered rectangle construction, the horizontal segment $I \times \{0\}$ is wide enough to intersect all orbits of the vertical vector field.

Already in the case of the torus, when the vertical vectorfield has rational slope with respect to the lattice, it is clear that a short enough horizontal segment will not intersect all orbits. In higher genus, the same can happen even when the vertical vector field has no periodic orbits, as the following construction shows.

Let Λ_1, Λ_2 be two lattices in $\mathbb{R}^2$ with no non zero vertical vectors ; let $T_i = \mathbb{R}^2/\Lambda_i$; choose on each T_i two vertical segments $[A_i, B_i]$ of the same length. Slit T_i along $[A_i, B_i]$ and glue isometrically the left side of $[A_1, B_1]$ to the right side of $[A_2, B_2]$ and vice-versa. We obtain a compact oriented surface M of genus 2, with two marked points A (image of A_1, A_2) and B (image of B_1, B_2) ; the canonical translation surface structures on T_1, T_2 generate a translation surface structure on $(M, \{A, B\})$ with ramification indices $\kappa_A = \kappa_B = 2$. The vector field has no periodic orbit in view of the hypothesis on Λ_1, Λ_2 but obviously any small horizontal segment in T_1 not intersecting $[A_1, B_1]$ will only intersect the orbits of the vectorfield contained in T_1.

Even when an horizontal segment intersects all orbits of the vertical vectorfield, the number of intervals in the i.e.m obtained as return map depends on the segment.

Exercise For a torus with one marked point and a minimal vertical vectorfield, show that the return map on a horizontal segment starting at the marked point is an i.e.m with 2 or 3 intervals. Find necessary and sufficient conditions for the return map to have only 2 intervals.

5.2 In order to understand which translation surfaces can be obtained via the zippered rectangle construction, the following lemma is useful.

Let $(M, \Sigma, \kappa, \zeta)$ be a translation surface. Denote by (Φ_t^V), resp. (Φ_t^H), the flow of the vertical, resp. horizontal, vectorfield.

Let $x_0 \in M - \Sigma$ a point of period T for the vertical vectorfield.

Lemma *There exists a maximal open bounded interval J around 0 such that for $s \in J$, the vertical flow $\Phi_t^V(\Phi_s^H(x_0))$ is defined for all times $t \in \mathbb{R}$. One has*

$$\Phi_{t+T}^V(\Phi_s^H(x_0)) = \Phi_t^V(\Phi_s^H(x_0)),$$

for $s \in J, t \in \mathbb{R}$, and the map

$$J \times \mathbb{R}/T\mathbb{Z} \to M$$

$$(s,t) \mapsto \Phi_t^V(\Phi_s^H(x_0))$$

is injective. The compact set

$$Z^+ = \lim_{s \nearrow \sup J} \{\Phi_{[0,T]}^V(\Phi_s^H(x_0))\}$$

is a finite union of points of Σ *and vertical connections between them. The same holds for*

$$Z^- = \lim_{s \searrow \inf J} \{\Phi_{[0,T]}^V(\Phi_s^H(x_0))\}.$$

The image $\Phi_{[0,T]}^V(\Phi_J^H(x_0))$ is called the **cylinder** around the periodic orbit of x_0. Its boundary in M is $Z^+ \cup Z^-$.

Proof : Let J be an open bounded interval around 0 such that $\Phi_s^H(x_0)$ is defined for $s \in J$ and $\Phi_t^V(\Phi_s^H(x_0))$ is defined for all $t \in \mathbb{R}, s \in J$. Any J small enough will have this property. Moreover, we must have

$$\Phi_T^V(\Phi_s^H(x_0)) = \Phi_s^H(x_0)$$

for all $s \in J$ because the set of s with this property contains 0 and is open and closed in J. The map

$$J \times \mathbb{R}/T\mathbb{Z} \to M$$
$$(s,t) \mapsto \Phi_t^V(\Phi_s^H(x_0))$$

must be injective : if we had

$$\Phi_{t_0}^V(\Phi_{s_0}^H(x_0)) = \Phi_{t_1}^V(\Phi_{s_1}^H(x_0)),$$

then either $s_0 = s_1$, $0 < t_1 - t_0 < T$ would contradict that T is the minimal period of x_0 or $s_0 < s_1$ would imply that

$$\Phi_{[0,T]}^V(\Phi_{[s_0,s_1]}^H(x_0))$$

is open and closed in M, hence equal to M, contradicting that Σ is non empty.

The injectivity gives a bound on the length of J, namely

$$|J| \leqslant AT^{-1}$$

where A is the area of M. This bound means that there exists indeed a maximal bounded open interval with the required properties. The maximality in turn implies that the set Z^+ must meet Σ (otherwise $\Phi^H_{\sup J}(x_0)$ is defined and of period T for the vertical flow), and thus is a finite union of points of Σ and vertical connections between them. Similarly for Z^-. $\square$

5.3 Proposition *Let $(M, \Sigma, \kappa, \zeta)$ be a translation surface, and S be an open bounded horizontal segment in M. Assume that S meets every vertical connection (if any). Then, either every infinite half-orbit of the vertical vectorfield meets S, or there is a cylinder containing every (infinite) orbit of the vertical vectorfield not meeting S.*

Proof : Denote by T_S the return map of the vertical vectorfield to S, by Φ_t the flow of the vertical vectorfield. Let u^t be a singularity of T_S, J the component of the domain of T_S to the left of u^t, t_J the return time to S in J. For $0 \leqslant t \leqslant t^L(u^t) := t_J$, let

$$\Phi^L_t(u^t) = \lim_{x \nearrow u^t} \Phi_t(x).$$

In the same way we define a right-limit $\Phi^R_t(u^t), 0 \leqslant t \leqslant t^R(u^t)$, and, for a singularity u^b of T_S^{-1}, we define left and right limits $\Phi^L_t(u^b), \Phi^R_t(u^b)$ (for negative time intervals $0 \geqslant t \geqslant t^L(u^b), 0 \geqslant t \geqslant t^R(u^b)$ respectively).

Claim *The sets $X^L = [\ \bigcup_{u^t} \Phi^L_{[0,t^L(u^t)]}(u^t)\ ] \bigcup [\ \bigcup_{u^b} \Phi^L_{[t^L(u^b),0]}(u^b)\ ]$ and $X^R = [\ \bigcup_{u^t} \Phi^R_{[0,t^R(u^t)]}(u^t)\ ] \bigcup [\ \bigcup_{u^b} \Phi^R_{[t^R(u^b),0]}(u^b)\ ]$ are equal.*

Proof : Let u^t be a singularity of T_S. We prove that $\Phi^L_{[0,t^L(u^t)]}(u^t)$ is contained in X^R. The claim then follows by symmetry. We distinguish two cases.

a) Assume first that $\lim_{x \nearrow u^t} T_S(x)$ is not the right endpoint of S. Then, it is a singularity u^b of T_S^{-1}. As S meets every vertical connection, the set $\Phi^L_{[0,t^L(u^t)]}(u^t)$ contains exactly one point of Σ, say $\Phi^L_{t^*}(u^t)$. Then $\Phi^L_{[0,t^*]}(u^t)$ is equal to $\Phi^R_{[0,t^*]}(u^t)$, and $\Phi^L_{[t^*,t^L(u^t)]}(u^t)$ is contained in $\Phi^R_{[t^R(u^b),0]}(u^b)$.

b) Assume now that $\lim_{x \nearrow u^t} T_S(x) = u^*$ is the right endpoint of S. Then $u^b = \lim_{x \nearrow u^*} T_S(x)$ is a singularity of T_S^{-1}. Again, as S meets every vertical connection, the union $\Phi^L_{[0,t^L(u^t)]}(u^t) \cup \Phi^L_{[t^L(u^b),0]}(u^b)$ contains at most one point of Σ, and it is contained in $\Phi^R_{[0,t^R(u^t)]}(u^t) \cup \Phi^R_{[t^R(u^b),0]}(u^b)$. $\square$

End of proof of proposition : Let X be the union, over the components J of the domain pf T_S, of the $\Phi_{[0,t_J]}(J)$ (with t_J the return time to S on J) ; let $\widehat{X}$ be the union of X and $X^L = X^R$. As $X^L = X^R, \widehat{X} \cap (M - \Sigma)$ is open in $M - \Sigma$. There are now two possibilities.

a) the return map T_S does not coincide with the identity in the neighbourhood of either endpoint of S. Then, the set $\widehat{X}$ is easily seen to be also closed in M. Therefore $\widehat{X} = M$ and every infinite half-orbit of the vertical vectorfield meets S.

b) The return map T_S coincides with the identity in the neighbourhood of at least one of the endpoints of S. Let Y be the cylinder containing the corresponding periodic orbits. As the boundary of Y is made of vertical connections and points of Σ, it is contained in $\widehat{X}$. Then $\widehat{X} \cup Y$ must be equal to M and the second possibility in the statement of the proposition holds. □

5.4 Corollary *If the vertical vectorfield on a translation surface has no connection, it is minimal : every infinite half orbit is dense.*

Proof : Otherwise there exists an open bounded horizontal segment S which does not meet every infinite vertical half-orbit. By the proposition, there would exist a cylinder containing these orbits ; but this is also not possible, since the boundary of a cylinder contains a vertical connection. □

5.5 Corollary *Let $(M, \Sigma, \kappa, \zeta)$ be a translation surface and S be an open bounded horizontal segment. Assume that*
$(H1)$ *S meets every vertical connection (if any).*
$(H2)$ *The left endpoint of S is in Σ.*
$(H3)$ *The right endpoint of S either belongs to Σ, or to a vertical separatrix segment which does not meet S.*

Then the translation surface is isomorphic to the one constructed from the return map T_S by the zippered rectangle construction with appropriate suspension data.

Proof : Applying the proposition in 5.3, we see that the second possibility in the statement of the proposition is forbidden by the hypothesis $(H2)$ and therefore S meets every infinite half-orbit of the vertical vectorfield. Therefore, every ingoing separatrix of the vertical vectorfield meets S ; the intersection point which is closest (on the separatrix) to the marked point is a singularity of T_S and we obtain in this way a one-to-one map between ingoing separatrices of the vertical vectorfield and singularities of T_S ; in the same way, there is a natural one-to-one correspondence between outgoing separatrices and singularities of T_S^{-1}. The vertical lengths of the corresponding separatrices segments determine the suspension data. It is now a direct verification, which we leave to the reader, to check that our translation surface is indeed isomorphic to the one obtained from these suspension data by the zippered rectangle construction. □

5.6 Proposition *Let $(M, \Sigma, \kappa, \zeta)$ be a translation surface and let S_∞ be an outgoing separatrix of the horizontal vectorfield. If either the horizontal or the vertical vectorfield has no connection, then some initial segment S of S_∞ satisfies the hypotheses $(H1), (H2), (H3)$ of Corollary 5.5*

Proof : First assume that there is no vertical connection. Then any initial segment S of S_∞ satisfies $(H1)$ and $(H2)$.Let $\widetilde{S}$ be some initial segment of S_∞, and S' be some vertical separatrix ; as there is no vertical connection, S' is dense, and therefore intersects $\widetilde{S}$. Let B be the intersection point closest along S' to the point of Σ at the end of S' ; the initial segment S of S_∞ with right endpoint B satisfies $(H1)$, $(H2)$ and $(H3)$.

Assume now that there is no horizontal connection. Then S_∞ is dense. As there are only finitely many vertical connections, every initial segment S of S_∞ which is long enough satisfies $(H1)$, and also $(H2)$. Let S' be a short enough vertical separatrix segment ; if the initial segment $\widetilde{S}$ of S_∞ is long enough it will intersect S', but only after having met all vertical connections ; again we cut $\widetilde{S}$ at the intersection point with S' which is closest to the marked point at the end of S'. We get an initial segment S of S_∞ which satisfies $(H1)$, $(H2)$ and $(H3)$. □

5.7 We reformulate Corollary 5.4. in the context of i.e.m.

Definition A connection for an i.e.m. T on an interval I is a triple (m, u^t, u^b) where m is a non negative integer, u^t is a singularity of T, u^b is a singularity of T^{-1}, such that

$$T^m(u^b) = u^t .$$

Theorem (Keane [Kea1]) *If an i.e.m. has no connection, it is minimal : every half-orbit is dense.*

Proof : Choose suspension data, construct a translation surface by the zippered rectangle construction ; the vertical vectorfield has no connection because the i.e.m. does not have either ; thus it is minimal and the same holds for the i.e.m. □

5.8 In this context, the following result of Keane is also relevant.

Proposition *If the coordinates of the length vector of an i.e.m. are rationally independent, it has no connection.*

Proof : Choose suspension data, construct a translation surface by the zippered rectangle construction. We use the notations of 4.5. If the i.e.m. had a connection, the vertical vectorfield on the translation surface would have a connection which we could express as a linear combination $\Sigma\, n_\alpha[\zeta_\alpha]$ in $H_1(M, \Sigma, \mathbb{Z})$ with integer coefficients. Integrating against the holomorphic 1-form, we have $\Sigma\, n_\alpha\, \lambda_\alpha = 0$ but $\Sigma\, n_\alpha \tau_\alpha \neq 0$, a contradiction. □

Exercise For $d = 2$, T is minimal iff there is no connection, anf iff the lengths of the intervals are rationally independent. For $d \geqslant 3$, show that there exists T minimal but having a connection, and also T with no connection but lengths data rationally dependent.

6 The Teichmüller space and the Moduli space

6.1 The Teichmüller space

Let M be a compact orientable topological surface, Σ a finite non-empty subset, κ a set of ramification indices.

We denote by $\mathrm{Diff}(M, \Sigma)$ the group of homeomorphisms of M fixing each point of Σ, by $\mathrm{Diff}^+(M, \Sigma)$ the subgroup of index 2 formed of orientation preserving homeomorphisms, by $\mathrm{Diff}_0(M, \Sigma)$ the neutral component of $\mathrm{Diff}(M, \Sigma)$, by $\mathrm{Mod}(M, \Sigma)$ the **modular group** (or mapping class group) $\mathrm{Diff}(M, \Sigma)/\mathrm{Diff}_0(M, \Sigma)$, and by $\mathrm{Mod}^+(M, \Sigma)$ the subgroup (of index 2) $\mathrm{Diff}^+(M, \Sigma)/\mathrm{Diff}_0\ (M, \Sigma)$.

The group $\mathrm{Diff}(M, \Sigma)$ acts on the set of translation surface structures on (M, Σ, κ) : if $\zeta = (\varphi_\alpha)$ is an atlas defining such a structure, $f_*\zeta$ is the atlas $(\varphi_\alpha \circ f^{-1})$ (for $f \in \mathrm{Diff}(M, \Sigma)$).

Definition The Teichmüller space $Q(M, \Sigma, \kappa)$ is the set of orbits of the action of $\mathrm{Diff}_0(M, \Sigma)$ on the set of translation surface structures on (M, Σ, κ).

6.2 Topology on $Q(M, \Sigma, \kappa)$

We will fix once and for all a universal cover

$$p : (\widetilde{M}, *) \to (M, A_1)$$

where A_1 is the first point of Σ.

Given a translation surface structure ζ on (M, Σ, κ), we define an associated **developing map**

$$D_\zeta : (\widetilde{M}, *) \to (\mathbb{C}, 0)$$

by integrating from $*$ the 1-form $p^*\omega$, where ω is the holomorphic 1-form determined by ζ.

Conversely, the developing map determines ζ. The set of translation surface structures on (M, ζ, κ) can therefore be considered as a subset of $C(\widetilde{M}, \mathbb{C})$; we equip this set with the compact-open topology, the set of translation surface structures with the induced topology, and the Teichmüller space $Q(M, \Sigma, \kappa)$ with the quotient topology.

6.3 The period map

Let ζ be a translation surface structure on $(M, \Sigma, \kappa), \omega$ be the associated holomorphic 1-form, γ a relative homology class in $H_1(M, \Sigma, \mathbb{Z})$. As ω is closed, the integral $\int_\gamma \omega$ is well-defined. Moreover, if f is an homeomorphism in $\mathrm{Diff}_0(M, \Sigma)$, f acts trivially on $H_1(M, \Sigma, \mathbb{Z})$, therefore the map

$$\zeta \mapsto (\gamma \to \int_\gamma \omega)$$

is constant on orbits of $\mathrm{Diff}_0(M, \Sigma)$ and defines a map

$$\Theta : Q(M, \Sigma, \kappa) \to \mathrm{Hom}(H_1(M, \Sigma, \mathbb{Z}), \mathbb{C})$$

called the **period map**. Here, we will generally identify in the right-hand side $\mathrm{Hom}(H_1(M, \Sigma, \mathbb{Z}), \mathbb{C})$ with the cohomology group $H^1(M, \Sigma, \mathbb{C})$.

The importance of the period map lies in the following property.

Proposition *The period map is a local homeomorphism.*

The proposition will be proved in section 6.5

6.4 Action of $GL(2, \mathbb{R})$ on Teichmüller space

Let $\zeta = (\varphi_\alpha)$ be an atlas defining a translation surface structure on (M, Σ, κ), and let g be an element of $GL(2, \mathbb{R})$ acting on $\mathbb{R}^2 \simeq \mathbb{C}$.

Consider the atlas $g_*\zeta = (g \circ \varphi_\alpha)$; because the conjugacy of a translation by an element of $GL(2, \mathbb{R})$ is still a translation, the atlas $g_*\zeta$ defines another translation surface structure on (M, Σ, κ) and we have thus a left action of $GL(2, \mathbb{R})$ on the space of translation surface structures.

It is clear that this action commutes with the action of the group $\mathrm{Diff}(M, \Sigma)$. In particular, it defines a left action of $GL(2, \mathbb{R})$ on the Teichmüller space $Q(M, \Sigma, \kappa)$.

One easily checks that this action is continuous.

Regarding the period map Θ, the group $GL(2, \mathbb{R})$ acts on the right-hand side $\mathrm{Hom}(H_1(M, \Sigma, \mathbb{Z}), \mathbb{C})$ by acting on the target $\mathbb{C} = \mathbb{R}^2$. The period map is then covariant with respect to the actions of $GL(2, \mathbb{R})$ on the source and the image.

It is to be noted that the subgroup $SO(2, \mathbb{R})$ preserves some of the auxiliary structures associated to a translation surface structure : the complex structure is invariant, the holomorphic 1-form is replaced by a multiple of modulus 1, the flat metric is preserved as is the associated area. The group $SO(2, \mathbb{R})$ acts transitively on the set of constant unitary vectorfields ; therefore, every result proved for the vertical vectorfield is valid for a non constant unitary vectorfield. Actually, if we use the full action of $GL(2, \mathbb{R})$, we see that in section 5 we can replace the vertical and horizontal vectorfield by any two non-proportional constant vectorfields on the translation surface.

6.5 Proof of proposition 6.3

We first observe that the period map is continuous : this follows immediately from the definition of the topology on Teichmüller space. To study the properties of Θ in the neighbourhood of a point $[\zeta]$ in $Q(M, \Sigma, \kappa)$, we may assume that the translation structure ζ has no vertical connection ; otherwise, we could replace ζ by $R_*\zeta$ for some appropriate $R \in SO(2, \mathbb{R})$ and use the covariance of the period map.

Then we know that the translation surface structure ζ can be obtained by the zippered rectangle construction from some i.e.m. T on some interval I.

Because the conditions on the length data λ and the suspension data τ in the zippered rectangle construction are open, the period map, expressed locally by (λ, τ), is locally onto.

It remains to be seen that the period map is locally injective, with continuous inverse.

In the zippered rectangle construction, we will always assume (by choosing the horizontal separatrix S_∞ appropriately in proposition 5.6) that the first marked point A_1 of Σ is the left endpoint of the interval I. The surface M was obtained in section 4.3 from some explicitly defined subset $\widehat{M}$ of $\mathbb{C}$, depending only on π, λ and τ. We can lift $\widehat{M}$ to a (connected) subset $\widehat{M_\zeta}$ of $\widetilde{M}$ (with the left endpoint of I lifted to $*$) with the property that the developing map D_ζ is an homeomorphism from $\widehat{M_\zeta}$ onto $\widehat{M}$.

If ζ_0, ζ_1 are two translation surface structures close to ζ with the same image by the period map, the subset $\widehat{M}$ of $\mathbb{C}$ will be the same for ζ_0 and ζ_1. There will be a unique homeomorphism $h : \widehat{M_{\zeta_0}} \to \widehat{M_{\zeta_1}}$ such that $D_{\zeta_0} = D_{\zeta_1} \circ h$ on $\widehat{M_{\zeta_0}}$. It is easily checked that h extends uniquely as a homeomorphism of $(\widetilde{M}, *)$ still satisfying $D_{\zeta_0} = D_{\zeta_1} \circ h$, and that extension is the lift of an homeomorphism of M. This proves that $[\zeta_0] = [\zeta_1]$ in Teichmüller space. This proves local injectivity of the period map ; the continuity of local inverses is proven along the same lines and left to the reader. □

6.6 Geometric structures on Teichmüller space

First, we can use the locally injective restrictions of the period map as charts defining a structure of complex manifold of complex dimension $d = 2g + s - 1$.

This complex manifold will also be equipped with a canonical volume form. Indeed, we can normalize Lebesgue measure on $\mathrm{Hom}(H_1(M, \Sigma, \mathbb{Z}), \mathbb{R}^2)$ by asking that the lattice $\mathrm{Hom}(H_1(M, \Sigma, \mathbb{Z}), \mathbb{Z}^2)$ has covolume 1. We then lift by the period map this canonical volume to Teichmüller space.

6.7 Examples and remarks

Let us consider the case $g = s = 1$ of the torus T with a single marked point $\{A_1\}$. Fix a basis $[\zeta_1], [\zeta_2]$ for the homology group $H_1(T, \{A_1\}, \mathbb{Z})$.

In this case, the period map is injective and allows to identify the Teichmüller space with its image. The image of the period map is

$$Q(T, \{A_1\}, 1) = \{(\zeta_1, \zeta_2) \in (\mathbb{C}^*)^2, \zeta_2/\zeta_1 \notin \mathbb{R}\} .$$

The two components of $Q(T, \{A_1\}, 1)$ correspond to the two possible orientations. Restricting to Im $\zeta_2/\zeta_1 > 0$, the map $(\zeta_1, \zeta_2) \to \zeta_2/\zeta_1$ presents $Q(T, \{A_1\}, 1)$ as fibered over the upper half-plane $\mathbb{H}$ (representing the classical Teichmüller space of T) with fiber $\mathbb{C}^*$ (representing the choice of a non-zero holomorphic 1-form).

Remarks 1. For $g \geqslant 2$, the period map is not injective. Indeed, let γ be a loop on M which is homologous but not homotopic to 0. We assume that $\gamma \cap \Sigma = \emptyset$. Let then f be a Dehn twist along γ ; this can be constructed fixing each point of Σ and thus defining an element of Diff(M, Σ).

If ζ is any translation surface structure on (M, Σ, κ), $f_*\zeta$ and ζ will have the same image by the period map because f induces the identity on $H_1(M, \Sigma, \mathbb{Z})$. On the other hand, ζ and $f_*\zeta$ represent different points in Teichmüller space : indeed, we will see that $[f_*^n\zeta]$ goes to ∞ in Teichmüller space as n goes to $\pm\infty$ in $\mathbb{Z}$.

2. Regarding the relation to "classical" Teichmüller theory classifying the complex structures on compact surfaces, consider the two extremal cases.

Take first $s = 2g - 2, \kappa_1 = \kappa_2 = \ldots = \kappa_s = 2$; this means that the holomorphic 1-form associated with the translation surface structure has only simple zeros, the generic situation for an holomorphic 1-form. The Teichmüller space $Q(M, \Sigma, \kappa)$ of dimension $2g+s-1 = 4g-3$ is fibered over the "classical" Teichmüller space of dimension $3g - 3$; the fiber of dimension g corresponds to the choice of the holomorphic 1-form (which form a g-dimensional vector space; however, one has to exclude the zero form and those having multiple zeros).

Consider now the case $s = 1$, $\kappa_1 = 2g - 2$; this means that the holomorphic 1-form has a single zero of maximal multiplicity ; when $g \geqslant 3$, not all Riemann surfaces of genus g admit such an holomorphic 1-form. Indeed the Teichmüller space has dimension $2g + s - 1 = 2g$ and the scaling of the holomorphic 1-form corresponds to 1 dimension, hence $Q(M, \Sigma, \kappa)$ is fibered over a subvariety of "classical" Teichmüller space of codimension $\geqslant g - 2$.

6.8 Normalizations

6.8.1 Normalization of orientation

It is generally convenient to fix an orientation of the orientable topological surface M and then to consider only those translation surface structures ζ on (M, Σ, κ) which are compatible with the given orientation. The groups Diff$^+(M, \Sigma)$ and $GL^+(2, \mathbb{R})$ act on this subset. We denote by $Q^+(M, \Sigma, \kappa)$ the corresponding subset of Teichmüller space.

6.8.2 Normalization of area

Given a translation surface structure ζ compatible with a chosen orientation, let $A(\zeta)$ be the surface of M for the area-form (on $M - \Sigma$) induced by ζ. It is clear that the function A is invariant under the action of $\mathrm{Diff}^+(M, \Sigma)$ and therefore induces a function still denoted by A on the Teichmüller space $Q^+(M, \Sigma, \kappa)$.

We will write $Q^{(1)}(M, \Sigma, \kappa)$ for the locus $\{A = 1\}$ in $Q^+(M, \Sigma, \kappa))$. As A is a smooth submersion, $Q^{(1)}(M, \Sigma, \kappa)$ is a codimension 1 real-analytic submanifold of $Q^+(M, \Sigma, \kappa)$.

If $[\zeta] \in Q^+(M, \Sigma, \kappa)$ and $g \in GL^+(2, \mathbb{R})$, we have

$$A(g_*[\zeta]) = \det g \; A([\zeta]) \; .$$

In particular, $Q^{(1)}(M, \Sigma, \kappa)$ is invariant under the action of $\mathrm{Diff}^+(M, \Sigma)$ and $SL(2, \mathbb{R})$.

Let μ be the canonical volume form on $Q^+(M, \Sigma, \kappa)$. We write

$$\mu = \mu_1 \wedge \; \frac{dA}{A} \; ;$$

then μ_1 induces on $Q^{(1)}(M, \Sigma, \kappa)$ a canonical volume form which is invariant under the action of $\mathrm{Diff}^+(M, \Sigma)$ and $SL(2, \mathbb{R})$.

6.9 The moduli space

The discrete group $\mathrm{Mod}(M, \Sigma)$ acts continuously on the Teichmüller space $Q(M, \Sigma, \kappa)$.

Definition The moduli space is the quotient

$$\mathcal{M}(M, \Sigma, \kappa) := Q(M, \Sigma, \kappa) \; / \; \mathrm{Mod}(M, \Sigma) \; .$$

The normalized moduli space is the quotient

$$\mathcal{M}^{(1)}(M, \Sigma, \kappa) := Q^{(1)}(M, \Sigma, \kappa) \; / \; \mathrm{Mod}^+(M, \Sigma) \; .$$

The action of the modular group $\mathrm{Mod}(M, \Sigma)$ on $Q(M, \Sigma, \kappa)$ is proper but not always free, as we explain below. This means that the moduli space is an orbifold (locally the quotient of a manifold by a finite group) but not (always) a manifold.

To see that the action is proper, consider as above a universal cover $p : (\widetilde{M}, *) \to (M, A_1)$. Let $\widetilde{\Sigma} = p^{-1}(\Sigma)$. Given a translation surface structure ζ, we can lift the flat metric defined by ζ to $\widetilde{M}$ and consider the distance d_ζ on $\widetilde{\Sigma}$ induced by this metric (as length of shortest path). It is clear that this distance only depends on the class of ζ in Teichmüller space. If ζ, ζ' are two translation surface structures, the distances $d_\zeta, d_{\zeta'}$ on $\widetilde{\zeta}$ are quasiisometric : there exists $C \geqslant 1$ such that

$$C^{-1} d_\zeta(B,B') \leqslant d_{\zeta'}(B,B') \leqslant C\ d_\zeta(B,B')$$

for all $B, B' \in \widetilde{\Sigma}$. We write $C(\zeta,\zeta')$ for the best constant C.

Exercise Prove that a subset $X \subset Q(M,\Sigma,\kappa)$ is relatively compact iff (given any $\zeta_0 \in Q(M,\Sigma,\kappa)$) the quantities $C(\zeta,\zeta_0), \zeta \in X$, are bounded.

The distances d_ζ have the property that any ball of finite radius only contain finitely many points. On the other hand, the modular group $\mathrm{Mod}(M,\Sigma)$ acts on $\widetilde{\Sigma}$, and there exists a finite subset $\widetilde{\Sigma}_0$ of $\widetilde{\Sigma}$ such that, for any finite subset $\widetilde{\Sigma}_1$ of $\widetilde{\Sigma}$, the set $\{g \in \mathrm{Mod}(M,\Sigma), g(\widetilde{\Sigma}_0) \subset \widetilde{\Sigma}_1\}$ is finite. Using the compactness criterion given by the exercise, it is easy to conclude that the action is proper.

To see that the action is not always free, it is sufficient to construct a translation surface with a non trivial group of automorphisms.

Start with an integer $k \geqslant 2$ and k copies of the same translation torus T with two marked points A, B. Denote by T_i, A_i, B_i the i^{th} copy, $1 \leqslant i \leqslant k$. Slit T_i along a geodesic segment A_iB_i (the same for all i). For each i, glue isometrically the left side of A_iB_i in T_i to the right side of $A_{i+1}B_{i+1}$ (with $(T_{k+1}, A_{k+1}, B_{k+1}) = (T_1, A_1, B_1)$). One obtains a translation surface of genus k with 2 marked points of ramification index k and an obvious automorphism group cyclic of order k.

6.10 Marked translation surfaces and marked moduli space

From the point of view of the zippered rectangles construction, it is more convenient to consider translation surfaces with an additional marking.

Indeed, if the construction starts from an i.e.m. T on an interval I, we have said above that we always take the left endpoint of I as the first marked point A_1 of the set Σ on the surface M. But the interval I itself appears on the surface as an outgoing separatrix of the horizontal vector field.

Definition A marked translation surface is a translation surface (M,Σ,κ,ζ) with a marked outgoing horizontal separatrix coming out of A_1.

Obviously, we require that an isomorphism between marked translation surfaces should respect the marked horizontal separatrices. We can then define a Teichmüller space $\widetilde{Q}(M,\Sigma,\kappa)$ of marked translation surfaces. It is a κ_1-fold cover of $Q(M,\Sigma,\kappa)$, because there are κ_1 possible choices for an horizontal separatrix out of A_1. In particular, when $\kappa_1 = 1$, the marking is automatic and $\widetilde{Q}(M,\Sigma,\kappa) = Q(M,\Sigma,\kappa)$.

On the other hand, it is quite obvious that a marked translation surface cannot have an automorphism distinct from the identity. Therefore, the modular group $\mathrm{Mod}(M, \Sigma)$ acts freely on $\widetilde{Q}(M, \Sigma, \kappa)$ and the quotient space, that we denote by $\widetilde{\mathcal{M}}(M, \Sigma, \kappa)$, is now a complex manifold. This moduli space is a κ_1-fold ramified covering of the moduli space $\mathcal{M}(M, \Sigma, \kappa)$. Normalizing orientation and area gives a codimension 1 real-analytic submanifold $\widetilde{\mathcal{M}}^{(1)}(M, \Sigma, \kappa)$.

6.11 In the following sections, we will present the proofs of the following results, obtained independently by H. Masur [Ma] and W. Veech [Ve2].

Theorem 1 *Almost all i.e.m. are uniquely ergodic.*

The combinatorial data are here fixed and "almost all" refer to the choice of length data according to Lebesgue measure.

Theorem 2 *The normalized moduli space $\widetilde{\mathcal{M}}^1(M, \Sigma, \kappa)$ has finite volume. The action of the group $SL(2, \mathbb{R})$ on it is ergodic.*

We will follow the approach of W. Veech [Ve5]. The **Teichmüller flow** on the moduli space $\widetilde{\mathcal{M}}^{(1)}(M, \Sigma, \kappa)$ is the restriction of the action of $SL(2, \mathbb{R})$ to the 1-parameter diagonal subgroup $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$. The ergodicity of the action will follow from the ergodicity of this flow (stronger properties of this flow will be presented in later sections).

Let us consider what happens in the simple case $g = s = 1$. Then, the normalized Teichmüller space is $Q^{(1)}(M, \Sigma, \kappa) = SL(2, \mathbb{R})$, the modular group $\mathrm{Mod}^+(M, \Sigma)$ is $SL(2, \mathbb{Z})$, the normalized moduli space is the space of normalized lattices $SL(2, \mathbb{R}) \,/\, SL(2, \mathbb{Z})$ which has unit area and on which $SL(2, \mathbb{R})$ obviously acts transitively. The Teichmüller flow is essentially the geodesic flow on the modular surface. It is well known that this flow is closely related to the classical continuous fraction algorithm. G. Rauzy and W. Veech, introduced a renormalization algorithm for i.e.m., later refined by A. Zorich, which plays the role of the classical continuous fraction algorithm for more than 2 intervals. This will be the subject of the next sections.

7 The Rauzy-Veech algorithm

7.1 The aim of the Rauzy-Veech algorithm ([Rau],[Ve1],[Ve2]), to be defined below, is to understand the dynamics of an i.e.m. by looking at the return map on shorter and shorter intervals. What makes this general "renormalization" method available is the fact that the return maps are still i.e.m. with bounded combinatorial complexity : actually, by choosing the small intervals carefully, they have the same number of singularities than the i.e.m. we started with.

7.2 Definition of one step of the algorithm

Let T be an i.e.m. on an interval I, with irreducible combinatorial data $(\mathcal{A}, \pi_t, \pi_b)$.

Let $d = \#\mathcal{A}$; let $u_1^t < \ldots < u_{d-1}^t$ be the singularities of T, $u_1^b < \ldots < u_{d-1}^b$ be the singularities of T^{-1}.

The step of the algorithm is defined for T if $u_{d-1}^t \neq u_{d-1}^b$. Observe that if $u_{d-1}^t = u_{d-1}^t$, then $(0, u_{d-1}^t, u_{d-1}^b)$ is a **connection** for T (see subsection 5.7).

When $u_{d-1}^t \neq u_{d-1}^b$, we define $\widetilde{I}$ to be the open interval with the same left endpoint than I and right endpoint equal to $\max(u_{d-1}^t, u_{d-1}^b)$.
Let $\widetilde{T}$ be the return map of T to $\widetilde{I}$.
To understand $\widetilde{T}$, let us introduce the letters α_t, α_b satisfying $\pi_t(\alpha_t) = \pi_b(\alpha_b) = d$ which correspond to the intervals at the right of I before and after applying T. The hypothesis $u_{d-1}^t \neq u_{d-1}^b$ corresponds to $\lambda_{\alpha_t} \neq \lambda_{\alpha_b}$. We distinguish two cases.

1) $u_{d-1}^b > u_{d-1}^t \Longleftrightarrow \lambda_{\alpha_t} > \lambda_{\alpha_b}$.

We say that α_t is the **winner** and α_b is the **loser** of this step of the algorithm, and that the step is of **top** type.

We have in this case

$$\widetilde{T}(x) = \begin{cases} T(x) & \text{if} \quad x \notin I_{\alpha_b}^t \\ T^2(x) & \text{if} \quad x \in I_{\alpha_b}^t \end{cases}$$

We use the same alphabet to label the intervals of $\widetilde{T}$; we define :

$$\begin{aligned}
&\widetilde{I}_\alpha^t = I_\alpha^t \quad \text{for} \quad \alpha \neq \alpha_t \ , \\
&\widetilde{I}_{\alpha_t}^t = I_{\alpha_t}^t \cap \widetilde{I} = (u_{d-1}^t, u_{d-1}^b) \ , \\
&\widetilde{I}_\alpha^b = I_\alpha^b \quad \text{for} \quad \alpha \neq \alpha_b, \alpha_t \ , \\
&\widetilde{I}_{\alpha_b}^b = T(I_{\alpha_b}^b) \ , \\
&\widetilde{I}_{\alpha_t}^b = I_{\alpha_t}^b / \widetilde{I}_{\alpha_b}^b \ .
\end{aligned}$$

The new length data are given by

$$\widetilde{\lambda}_\alpha = \begin{cases} \lambda_\alpha & \text{if} \quad \alpha \neq \alpha_t \\ \lambda_{\alpha_t} - \lambda_{\alpha_b} & \text{if} \quad \alpha = \alpha_t \ . \end{cases}$$

The new combinatorial data are given by

$$\widetilde{\pi}_t = \pi_t \ ;$$

$$\widetilde{\pi}_b(\alpha) = \begin{cases} \pi_b(\alpha) & \text{if} \quad \pi_b(\alpha) \leqslant \pi_b(\alpha_t), \\ \pi_b(\alpha_t) + 1 & \text{if} \quad \alpha = \alpha_b \ , \\ \pi_b(\alpha) + 1 & \text{if} \quad \pi_b(\alpha_t) < \pi_b(\alpha) < d \ . \end{cases}$$

2) $u^t_{d-1} > u^b_{d-1} \Longleftrightarrow \lambda_{\alpha_b} > \lambda_{\alpha_t}$.

We now say that α_b is the winner, α_t the loser, and the step is of **bottom** type. We have

$$\widetilde{T}^{-1}(x) = \begin{cases} T^{-1}(x) & \text{if} \quad x \notin I^b_{\alpha^t} \\ T^{-2}(x) & \text{if} \quad x \in I^b_{\alpha^t} \end{cases}$$

(we could also write the formulas for $\widetilde{T}$; we prefer to write them for $\widetilde{T}^{-1}$ in order to keep more obvious the bottom/top time symmetry of the setting). The new labelling is

$$\begin{aligned} &\widetilde{I}^b_\alpha = I^b_\alpha \quad \text{for} \quad \alpha \neq \alpha_b \ , \\ &\widetilde{I}^b_{\alpha_b} = I^b_{\alpha_b} \cap \widetilde{I} = (u^b_{d-1}, u^t_{d-1}) \ , \\ &\widetilde{I}^t_\alpha = I^t_\alpha \quad \text{for} \quad \alpha \neq \alpha_t, \alpha_b \ , \\ &\widetilde{I}^t_{\alpha_t} = T^{-1}(I^t_{\alpha_t}) \ , \\ &\widetilde{I}^t_{\alpha_b} = I^t_{\alpha_b} / \widetilde{I}^t_{\alpha_t} \ . \end{aligned}$$

The new length data are given by

$$\widetilde{\lambda}_\alpha = \begin{cases} \lambda_\alpha & \text{if} \quad \alpha \neq \alpha_b \\ \lambda_{\alpha_b} - \lambda_{\alpha_t} & \text{if} \quad \alpha = \alpha_b \ . \end{cases}$$

The new combinatorial data are given by

$$\widetilde{\pi}_b = \pi_b \ ;$$

$$\widetilde{\pi}_t(\alpha) = \begin{cases} \pi_t(\alpha) & \text{if} \quad \pi_t(\alpha) \leqslant \pi_t(\alpha_t), \\ \pi_t(\alpha_b) + 1 & \text{if} \quad \alpha = \alpha_t \ , \\ \pi_t(\alpha) + 1 & \text{if} \quad \pi_t(\alpha_b) < \pi_t(\alpha) < d \ . \end{cases}$$

Exercise Show that the combinatorial data $(\widetilde{\pi}_t, \widetilde{\pi}_b)$ for $\widetilde{T}$ are irreducible.

Exercise Show that if T has no connection, then $\widetilde{T}$ also has no connection.

This means that for i.e.m. with no connections, it is possible to iterate indefinitely the algorithm ; the converse is also true, see below.

Exercise Check that the return map of T on an interval I' with the same left endpoint than I and $|\widetilde{I}| < |I'| < |I|$ is an i.e.m with $d+1$ intervals.

In the case of 2 intervals, there is only one possible set of irreducible combinatorial data and the algorithm is given by

$$(\lambda_A, \lambda_B) \longmapsto \begin{cases} (\lambda_A - \lambda_B, \lambda_B) & \text{if} \quad \lambda_A > \lambda_B, \\ (\lambda_A, \lambda_B - \lambda_A) & \text{if} \quad \lambda_B > \lambda_A \ . \end{cases} ,$$

the iteration of which gives the classical continued fraction algorithm.

7.3 Rauzy diagrams

Let $\mathcal{A}$ be an alphabet. For irreducible combinatorial data $\pi = (\pi_t, \pi_b)$, we have defined in the last section new combinatorial data $\widetilde{\pi} = (\widetilde{\pi}_t, \widetilde{\pi}_b)$ depending only on (π_t, π_b) and the type (top or bottom) of the step ; we write $\widetilde{\pi} = R_t(\pi)$ or $\widetilde{\pi} = R_b(\pi)$ accordingly.

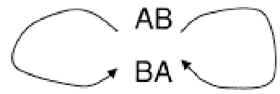
A **Rauzy class** on the alphabet $\mathcal{A}$ is a set of irreducible combinatorial data $\pi = (\pi_t, \pi_s)$ which is invariant under both R_t and R_b and minimal with this property. The associated **Rauzy diagram** has the elements of this set as vertices. The arrows of the diagram join a vertex to its images by R_t and R_b and are of **top** and **bottom** type accordingly.

The **winner** of an arrow of top type (resp. bottom type) starting at (π_t, π_b) is the letter α_t (resp. α_b) such that $\pi_t(\alpha_t) = d$ (resp. $\pi_b(\alpha_b) = d$). The **loser** is the letter α_b (resp. α_t) such that $\pi_b(\alpha_b) = d$ (resp. $\pi_t(\alpha_t) = d$).

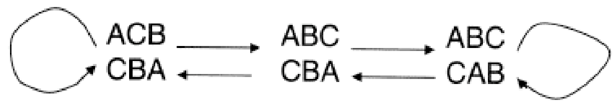
Exercise Show that the maps R_t, R_b are invertible and that each vertex is therefore the endpoint of exactly one arrow of top type and an arrow of bottom type.

Exercise Let γ, γ' be arrows in a Rauzy diagram of the same type such that the endpoint of γ is the starting point of γ' ; show that γ, γ' have the same winner.

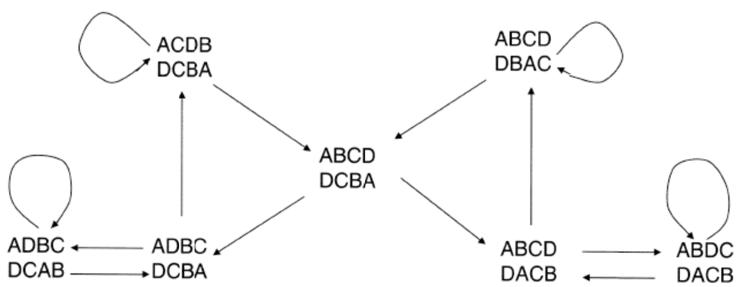
For $d = 2$ or 3, there is, up to equivalence, only one Rauzy diagram pictured below.
For $d = 4$, there are two non-equivalent Rauzy diagrams pictured below. They correspond respectively (see next section) to the cases $g = 2, s = 1$ and $g = 1, s = 3$.



g=1, s=1,d=2



g=1,s=2,d=3



g=2,s=1,d=4

7.4 The basic step for suspensions

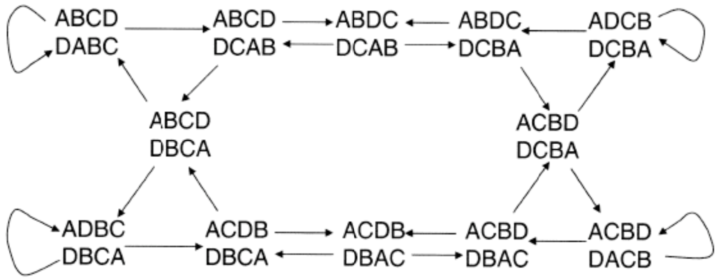
Recall from section 4.1 that for combinatorial data $\pi = (\pi_t, \pi_b)$, suspension data $(\tau_\alpha)_{\alpha\in\mathcal{A}}$ must satisfy

$$(S_\pi) \qquad \sum_{\pi_t(\alpha)<k} \tau_\alpha > 0 \ , \qquad \sum_{\pi_b(\alpha)<k} \tau_\alpha < 0 \qquad \text{for all } 1 < k \leqslant d \ .$$

We denote by Θ_π the convex open cone in $\mathbb{R}^{\mathcal{A}}$ defined by these inequalities.

The main reason to consider Θ_π is the following property. Set $\widetilde{\pi} = R_t(\pi)$. Define also, for $\tau \in \mathbb{R}^{\mathcal{A}}$

$$\widetilde{\tau}_\alpha = \begin{cases} \tau_\alpha & \text{if } \ \alpha \neq \alpha_t \\ \tau_{\alpha_t} - \tau_{\alpha_b} & \text{if } \ \alpha = \alpha_t \ . \end{cases}$$



g=1,s=3,d=4

where $\pi_t(\alpha_t) = \pi_b(\alpha_b) = d$.

Lemma *The linear map* $\tau \to \widetilde{\tau}$ *sends* Θ_π *onto* $\Theta_{\widetilde{\pi}} \cap \{\sum_\alpha \widetilde{\tau}_\alpha < 0\}$.

There is a symmetric statement exchanging top and bottom.

Proof: Let $\tau \in \Theta_\pi$. As $\widetilde{\pi}_t = \pi_t$, and $\widetilde{\tau}_\alpha = \tau_\alpha$ for $\pi_t(\alpha) < d$, the first half of the conditions for $(S_{\widetilde{\pi}})$ are satisfied. Let $\ell = \pi_b(\alpha_t)$; for $k \leqslant \ell$, we have

$$\sum_{\widetilde{\pi}_b(\alpha)<k} \widetilde{\tau}_\alpha = \sum_{\pi_b(\alpha)<k} \widetilde{\tau}_\alpha = \sum_{\pi_b(\alpha)<k} \tau_\alpha < 0 \ .$$

Next we have

$$\sum_{\widetilde{\pi}_b(\alpha)\leqslant\ell} \widetilde{\tau}_\alpha = \sum_{\pi_b(\alpha)<\ell} \tau_\alpha + \tau_{\alpha_t} - \tau_{\alpha_b} = \sum_{\pi_b(\alpha)<\ell} \tau_\alpha - \sum_{\pi_t(\alpha)<d} \tau_\alpha + \sum_{\pi_b(\alpha)<d} \tau_\alpha < 0 \ ,$$

and for $\ell < k \leqslant d$

$$\sum_{\widetilde{\pi}_b(\alpha)\leqslant k} \widetilde{\tau}_\alpha = \sum_{\pi_b(\alpha)\leqslant k-1} \tau_\alpha < 0 \ .$$

Conversely, let $\widetilde{\tau} \in \Theta_{\widetilde{\pi}} \cap \{\sum \widetilde{\tau}_\alpha < 0\}$. Again the first half of (S_π) is satisfied. For the second half, we have

$$\sum_{\pi_b(\alpha)<k} \tau_\alpha = \begin{cases} \sum_{\widetilde{\pi}_b(\alpha)<k} \tau_\alpha & \text{if} \quad 1 < k \leqslant \ell, \\ \sum_{\widetilde{\pi}_b(\alpha)\leqslant k} \tau_\alpha & \text{if} \quad \ell < k \leqslant d \ . \end{cases}$$

Thus, condition (S_π) is satisfied. □

Let then T be an i.e.m. on an interval I, with (irreducible) combinatorial data $\pi = (\pi_t, \pi_b)$ on an alphabet $\mathcal{A}$. Assume that the condition $\lambda_{\alpha_t} \neq \lambda_{\alpha_b}$ (with $\pi_t(\alpha_t) = \pi_b(\alpha_b) = d$) for one step of the algorithm is satisfied. Let $\tau \in \Theta_\pi$ be suspension data satisfying the required conditions (S_π).

If the step is of top type, we define

$$\widetilde{\tau}_\alpha = \begin{cases} \tau_\alpha & \text{if} \quad \alpha \neq \alpha_t \\ \tau_{\alpha_t} - \tau_{\alpha_b} & \text{if} \quad \alpha = \alpha_t \ . \end{cases}$$

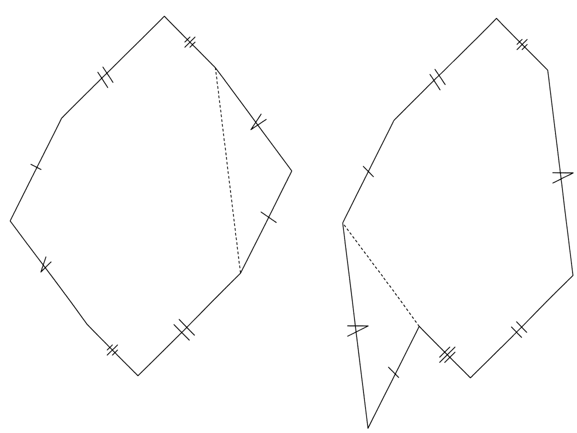
If the step is of bottom type, we define

$$\widetilde{\tau}_\alpha = \begin{cases} \tau_\alpha & \text{if} \quad \alpha \neq \alpha_b \\ \tau_{\alpha_b} - \tau_{\alpha_t} & \text{if} \quad \alpha = \alpha_b \ . \end{cases}$$

(The formulas are the same than for the length data).

We have explained in Section 4 how to construct a translation surface $M(\pi, \lambda, \tau)$ from the given data by the zippered rectangle construction. Writing $\widetilde{\pi} = R_t(\pi)$ or $R_b(\pi)$ according to the type of the step and writing $\widetilde{\lambda}$ for the length data of $\widetilde{T}$ as above, we construct another translation surface $M(\widetilde{\pi}, \widetilde{\lambda}, \widetilde{\tau})$ from these new data.

An easily checked but fundamental observation is that $M(\pi, \lambda, \tau)$ and $M(\widetilde{\pi}, \widetilde{\lambda}, \widetilde{\tau})$ are **canonically isomorphic**. This is best seen by contemplating the picture below :



$M(\pi, \lambda, \tau)$ $\qquad\qquad$ $M(\widetilde{\pi}, \widetilde{\lambda}, \widetilde{\tau})$

The canonical bases of the homology groups $H_1(M, \Sigma, \mathbb{Z})$, $H_1(M - \Sigma, \mathbb{Z})$ are related as follows : If α_0 is the winner and α_1 is the loser of the step of the algorithm, one has, with the notations of section 4.5

$$[\widetilde{\zeta_\alpha}] \quad = [\zeta_\alpha] \qquad \qquad \text{if } \alpha \neq \alpha_0 \ ,$$

$$
\begin{aligned}
[\widetilde{\zeta}_{\alpha_0}] &= [\zeta_{\alpha_0}] - [\zeta_{\alpha_1}] \ , \\
[\widetilde{\theta}_{\alpha}] &= [\theta_{\alpha}] && \text{if } \alpha \neq \alpha_1 \ , \\
[\widetilde{\theta}_{\alpha_1}] &= [\theta_{\alpha_1}] + [\theta_{\alpha_0}] \ .
\end{aligned}
$$

7.5 Formalism for the iteration of the algorithm

Given an i.e.m. T_0 on an interval $I^{(0)}$ with no connection and irreducible combinatorial data $(\mathcal{A}, \pi^{(0)})$, the iteration of the Rauzy-Veech algorithm will produce a sequence of i.e.m. T_n on shorter and shorter intervals $I^{(n)}$ with combinatorial data $\pi^{(n)}$ (on the same alphabet $\mathcal{A}$). The sequence $(\pi^{(n)})_{n \geqslant 0}$ represents an infinite path in the Rauzy diagram $\mathcal{D}$ containing $\pi^{(0)}$ which is determined by its starting vertex $\pi^{(0)}$ and the types of the successive arrows.

To relate the length vectors and the translation vectors, as well as the suspension data that we could associate to the i.e.m., we introduce the following matrices in $SL(\mathbb{Z}^{\mathcal{A}})$.

Let γ be an arrow of $\mathcal{D}$, with winner α and loser β. We define

$$B_\gamma = \mathbb{I} + E_{\beta\alpha}$$

where $\mathbb{I}$ is the identity matrix and $E_{\beta\alpha}$ is the elementary matrix with only one non-zero coefficient, equal to 1, in position $\beta\,\alpha$. We extend the definition to a path $\underline{\gamma} = (\gamma_1, \cdots, \gamma_n)$ defining

$$B_{\underline{\gamma}} = B_{\gamma_n} \cdots B_{\gamma_1}.$$

The matrices $B_{\underline{\gamma}}$ belong to $SL(\mathbb{Z}^{\mathcal{A}})$ and have nonnegative coefficients.

For $n \geqslant 0$, let $\lambda^{(n)}$ be the length vector for T_n (considered as a row vector), let $\delta^{(n)}$ be the translation vector (considered as a column vector) ; for $m \leqslant n$, let $\gamma(m,n)$ the finite path in $\mathcal{D}$ from $\pi^{(m)}$ to $\pi^{(n)}$ determined by the algorithm. The following formulas are trivially checked when $n = m+1$ and then extended by functoriality :

$$\lambda^{(m)} = \lambda^{(n)} \ B_{\gamma(m,n)} \ ,$$

$$\delta^{(n)} = B_{\gamma(m,n)} \ \delta_{(m)} \ .$$

The following interpretation of the coefficients of the matrices $B_{\gamma(m,n)}$ is also immediately checked by induction on $n-m$: for $\alpha, \beta \in \mathcal{A}$, the coefficient of $B_{\gamma(m,n)}$ in position $\alpha\beta$ is the time spent in $I_\beta^{(m)}$ by a point in $I_\alpha^{(n)}$ under iteration by T_m before coming back to $I^{(n)}$. In particular, the sum over β of the row of the matrix of index α gives the return time under T_m of $I_\alpha^{(n)}$ in $I^{(n)}$.

7.6 Symplecticity of B_γ

Let $\underline{\gamma}$ be a finite path in a Rauzy diagram $\mathcal{D}$, starting at a vertex π and ending at a vertex π'. Let $\Omega_\pi, \Omega_{\pi'}$ be the matrices associated to π, π' as in subsection 3.4. We have seen in subsection 4.5 that $rk\ \Omega_\pi = rk\ \Omega_{\pi'} = 2g$, where g is the genus of the translation surface obtained by the zippered rectangle construction from any vertex in $\mathcal{D}$ (and any choice of length and suspension data).

From the relation between length and translation vectors given in subsection 3.4 and in the last section, we obtain

$$\Omega_{\pi'} = B_{\underline{\gamma}}\, \Omega_\pi\, {}^t B_{\underline{\gamma}}\ .$$

From this we see that :

- $B_{\underline{\gamma}}^{-1}$, acting on row vectors, sends the kernel of Ω_π onto the kernel of $\Omega_{\pi'}$;
- $B_{\underline{\gamma}}$, acting on column vectors, sends the image of Ω_π onto the image of $\Omega_{\pi'}$;
- if we equip the quotients $\mathbb{R}^{\mathcal{A}}/\mathrm{Ker}\,\Omega_\pi \ \simeq \mathrm{Im}\Omega_\pi$, $\mathbb{R}^{\mathcal{A}}/\mathrm{Ker}\ \Omega_{\pi'} \simeq \mathrm{Im}\Omega'_\pi$ of the symplectic structures determined by $\Omega_\pi, \Omega_{\pi'}$ respectively, then $B_{\underline{\gamma}}$ (acting on column vectors) is symplectic w.r.t these structures.

Proposition *One can choose, for each vertex π of $\mathcal{D}$, a basis of row vectors for $Ker\,\Omega_\pi$ such that, for all $\underline{\gamma} : \pi \to \pi'$, the matrix of the restriction of $B_{\underline{\gamma}}^{-1}$ w.r.t. the selected bases of $Ker\,\Omega_\pi$, $Ker\,\Omega_{\pi'}$ is the identity.*

In particular, if $\underline{\gamma}$ is a loop at π, the restriction of $B_{\underline{\gamma}}^{-1}$ to the kernel of Ω_π is the identity.

Proof : We construct, for each vertex π of $\mathcal{D}$, an isomorphism i_π from $\mathrm{Ker}\,\Omega_\pi$ onto the same subspace K of $\mathbb{R}^{\mathcal{A}}$, such that $i_{\pi'} \circ {}^t B_\gamma^{-1} = i_\pi$ for any arrow $\gamma : \pi \to \pi'$. Choosing a basis for K and transferring it to each $\mathrm{Ker}\,\Omega_\pi$ via i_π then achieves the required property.

For $0 \leqslant k < d$, let u^t_k, u^b_k be the linear forms on the space of row vectors defined by

$$u^t_k(\lambda) = \sum_{\pi_t \alpha \leqslant k} \lambda_\alpha\ ,$$

$$u^b_k(\lambda) = \sum_{\pi_b \alpha \leqslant k} \lambda_\alpha\ .$$

For each vertex π, define linear maps i^t_π, i^b_π of $\mathbb{R}^{\mathcal{A}}$ into itself by

$$i^t_\pi(\lambda) = (u^t_{\pi^t(\alpha)-1})_{\alpha\in\mathcal{A}}, \qquad\qquad i^b_\pi(\lambda) = (u^b_{\pi^b(\alpha)-1})_{\alpha\in\mathcal{A}}.$$

Then the map $(i^t_\pi, i^b_\pi) : \mathbb{R}^{\mathcal{A}} \to \mathbb{R}^{\mathcal{A}} \times \mathbb{R}^{\mathcal{A}}$ is injective and $\mathrm{Ker}\,\Omega_\pi$ is the inverse image by this map of the diagonal of $\mathbb{R}^{\mathcal{A}} \times \mathbb{R}^{\mathcal{A}}$. Let K_π be the image of $\mathrm{Ker}\,\Omega_\pi$ by i^t_π; it is also the image by i^b_π. Let i_π be the common restriction of i^t_π, i^b_π to $\mathrm{Ker}\,\Omega_\pi$.
When we perform a single step of the algorithm, corresponding to an arrow $\gamma : \pi \to \pi'$, of top type for instance, the λ_α with $\pi_t(\alpha) < d$ and π_t itself do not change, hence the u^t_k for $0 \leqslant k < d$

stay the same. This means that $K_\pi = K_{\pi'}$ and $i_{\pi'} \circ {}^tB_\gamma^{-1} = i_\pi$. □

7.7 Complete paths

Definitions A (finite) path in a Rauzy diagram is **complete** if every letter in $\mathcal{A}$ is the winner of at least one arrow in the path. An infinite path in a Rauzy diagram is **∞-complete** if every letter in $\mathcal{A}$ is the winner of infinitely many arrows in the path. Equivalently, an **∞-complete** path is one can be written as the concatenation of infinitely many complete paths.

This is a relevant notion because of the following characterization of paths associated to an i.e.m.

Proposition 1 *An infinite path in a Rauzy diagram is associated to some i.e.m. iff it is ∞-complete.*

We prove first that a path associated to an i.e.m. is ∞-complete, then an important auxiliary result, and then that an ∞-complete path is associated to some i.e.m .

Proof Let $\mathcal{A}'$ be the set of letters which are the winners of at most finitely many arrows in the path γ_T associated to an i.e.m. $T = T_0$.

Let $(T_n)_{n\geqslant 0}$ be the sequence of i.e.m. obtained from T by iterating the Rauzy-Veech algorithm, $\lambda^{(n)}, \pi^{(n)}$ the length and combinatorial data of T_n.

There exists n_0 such that no letter in $\mathcal{A}'$ is a winner for $n \geqslant n_0$. Then the lengths $\lambda_\alpha^{(n)}$ for $\alpha \in \mathcal{A}'$, $n \geqslant n_0$, are independent of n.

At each step, the length of the loser is subtracted from the length of the winner. As lengths are always positive, there must exist $n_1 \geqslant n_0$ such that no letter in $\mathcal{A}'$ is a loser for $n \geqslant n_1$. This means that, for $\alpha \in \mathcal{A}'$, both $\pi_t^{(n)}(\alpha)$ and $\pi_b^{(n)}(\alpha)$ are non-decreasing with n for $n \geqslant n_1$, hence there exists $n_2 \geqslant n_1$ such that these quantities are independent of n for $n \geqslant n_2$.

Let $\alpha \in \mathcal{A}', \beta \in \mathcal{A} - \mathcal{A}'$. We claim that $\pi_t^{(n_2)}(\alpha) < \pi_t^{(n_2)}(\beta)$ and $\pi_b^{(n_2)}(\alpha) < \pi_b^{(n_2)}(\beta)$. As $\mathcal{A} - \mathcal{A}'$ is not empty and $\pi^{(n_2)}$ is irreducible, this implies that $\mathcal{A}'$ is empty, and therefore γ_T is ∞-complete.

Assume by contradiction, for instance, that $\pi_t^{(n_2)}(\beta) < \pi_t^{(n_2)}(\alpha)$. We have $\pi_t^{(n)}(\alpha) = \pi_t^{(n_2)}(\alpha)$ for $n \geqslant n_2$, hence also $\pi_t^{(n)}(\beta) = \pi_t^{(n_2)}(\beta)$ for $n \geqslant n_2$. Thus β is not the winner of an arrow of top type for $n \geqslant n_2$. As $\beta \in \mathcal{A} - \mathcal{A}'$, β is the winner of an arrow of bottom type for some $n \geqslant n_2$, which gives

$$\pi_t^{(n+1)}(\alpha) = \pi_t^{(n)}(\alpha) + 1,$$

a contradiction. The claim is proved; this completes the proof of the first part of the proposition.

Before proving the second half of Proposition 1, we give some Corollaries of the first half.

Corollary 1 *The length of the interval $I^{(n)}$ on which T_n acts goes to zero as n goes to $+\infty$.*

Proof : Each length $\lambda_\alpha^{(n)}$ is a non-increasing function of n hence has a limit $\lambda_\alpha^{(\infty)}$. Let $\varepsilon > 0$, n_0 such that $\lambda_\alpha^{(n)} \leqslant \lambda_\alpha^{(\infty)} + \varepsilon$ for all $n \geqslant n_0$, $\alpha \in \mathcal{A}$.
Let $\beta \in \mathcal{A}$. There exists $n_1 > n_0$ such that β is the winner of the arrow of index $n_1 - 1$ but not of the next arrow of index n_1. Then β is the loser of the arrow of index n_1. Let α be the winner of this arrow. We have

$$\lambda_\alpha^{(n_1)} = \lambda_\alpha^{(n_1-1)} - \lambda_\beta^{(n_1-1)} ,$$

hence $\lambda_\beta^{(\infty)} \leqslant \lambda_\beta^{(n_1-1)} \leqslant \varepsilon$. As ε is arbitrary, we have $\lambda_\beta^{(\infty)} = 0$ for all $\beta \in \mathcal{A}$. □

Corollary 2 *The Rauzy-Veech algorithm stops iff the i.e.m. has a connection.*

Proof : We already know that the algorithm does not stop if the i.e.m. has no connection. Assume that T has a connection (m, u^t, u^b) ; here u^t is a singularity of T, u^b a singularity of T^{-1} and m is a nonnegative integer such that $T^m(u^b) = u^t$. Assume that one can apply the algorithm once to get an i.e.m. $\widetilde{T}$ on an interval $\widetilde{I}$; the intersection $\{u^b, T(u^b), \ldots T^m(u^b) = u^t\} \cap \widetilde{I}$ will produce a connection $(\widetilde{m}, \widetilde{u}^t, \widetilde{u}^b)$ for $\widetilde{T}$ with $\widetilde{m} \leqslant m$, and $\widetilde{m} = m$ iff $\{u^b, T(u^b), \ldots, T^m(u^b)\} \subset \widetilde{I}$. When we iterate the algorithm, the length of the interval goes to zero unless the algorithm stops ; this must therefore happen at some point. □

Proposition 2 ([MsMmY],[Y1]) *Let $\underline{\gamma}$ be a finite path in a Rauzy diagram that can be written as the concatenation of $2d-3$ complete paths (where $d = \#\mathcal{A}$). Then all coefficients of $B_{\underline{\gamma}}$ are positive.*

Proof : Write $\underline{\gamma} = \underline{\gamma}_1 * \ldots * \underline{\gamma}_{2d-3}$, with each $\underline{\gamma}_i$ complete, and let $\underline{\gamma}(i) = \underline{\gamma}_1 * \ldots * \underline{\gamma}_i$. Recall that the diagonal coefficients of $B_{\underline{\gamma}}$ (for any path $\underline{\gamma}$) are always positive. It is therefore sufficient to show that, for any distinct letters α_1, α_0 in $\mathcal{A}$, we have $(B_{\underline{\gamma}(i)})_{\alpha_0\alpha_1} > 0$ for some i.
As α_1 is the winner of an arrow in $\underline{\gamma}_1$, the loser of which we call α_2, we have

$$(B_{\underline{\gamma}(1)})_{\alpha_2\alpha_1} > 0.$$

When $d = 2$, we must have $\alpha_2 = \alpha_0$ and the result is achieved. Assume $d > 2$ and $\alpha_2 \neq \alpha_0$. Because $\underline{\gamma}_2$ and $\underline{\gamma}_3$ are complete, there exists in $\underline{\gamma}_2 * \underline{\gamma}_3$ an arrow with winner $\alpha_3 \neq \alpha_1, \alpha_2$ immediately followed by an arrow with winner α_1 or α_2. This leads to

$$(B_{\underline{\gamma}(3)})_{\alpha_3\alpha_1} > 0 .$$

If $d = 3$, we must have $\alpha_0 = \alpha_3$ and we are done. If $d > 3$ and $\alpha_3 \neq \alpha_0$, we go on in the same way : there exists in $\underline{\gamma}_4 * \underline{\gamma}_5$ an arrow with winner $\alpha_4 \neq \alpha_1, \alpha_2, \alpha_3$ immediately followed by an arrow with winner α_1, α_2 or α_3. This leads to

$$(B_{\underline{\gamma}(5)})_{\alpha_4 \alpha_1} > 0 \ .$$

and we go on ... □

End of proof of Proposition 1 : We want to show that if an infinite path γ can be written as the concatenation $\underline{\gamma}_1 * \underline{\gamma}_2 * \underline{\gamma}_3 \dots$ of complete paths, then γ is associated to some i.e.m. with no connection by the Rauzy-Veech algorithm.

Define the convex open cone

$$\mathcal{C}_n = (\mathbb{R}_+^*)^{\mathcal{A}} B_{\underline{\gamma}_n} B_{\underline{\gamma}_{n-1}} \dots B_{\underline{\gamma}_1}$$

This is the set of length data (for i.e.m having the starting point of γ as combinatorial data) which lead to a path starting with $\underline{\gamma}_1 * \dots * \underline{\gamma}_n$. The set of length data corresponding to γ is therefore

$$\mathcal{C}(\gamma) = \bigcap_{n \geqslant 0} \mathcal{C}_n \ .$$

By Proposition 2, the closure of $\mathcal{C}_{n+2d-3}$ is contained in $\mathcal{C}_n \cup \{0\}$. Therefore

$$\{0\} \cup \mathcal{C}(\gamma) = \bigcap_{n \geqslant 0} \overline{\mathcal{C}}_n \ .$$

which shows that $\mathcal{C}(\gamma)$ is not empty. □

We will describe more precisely $\mathcal{C}(\gamma)$ in the next section.

8 Invariant measures

8.1 Invariant measures and topological conjugacy

Let T be an i.e.m on an interval I, with combinatorial data $\pi = (\pi_t, \pi_b)$ on an alphabet $\mathcal{A}$. We assume that T has no connection and denote by $\gamma = \gamma_T$ the infinite path associated to T in the Rauzy diagram $\mathcal{D}$ of π.

Let $\mathcal{C}(\gamma)$ be the convex cone considered above ; its elements are the length data of the i.e.m with combinatorial data π which have no connection and γ as associated path.

Let $\mathcal{M}(T)$ be the set of finite measures on I invariant under T.

The sets $\mathcal{C}(\gamma)$ and $\mathcal{M}(T)$ are in one-to-one correspondence as follows.

Let $\lambda \in \mathcal{C}(\gamma)$ and let T_λ be an i.e.m with these length data (and combinatorial data π) on an interval I_λ. Let u (resp. u_λ) be the largest singularity of T^{-1} (resp. T_λ^{-1}). The sets $(T^n(u))_{n\geqslant 0}$ and $(T_\lambda^n(u_\lambda))_{n\geqslant 0}$ are dense in I and I_λ respectively because T and T_λ are minimal, having no connection. The bijection

$$H : T_\lambda^n(u_\lambda) \longmapsto T^n(u)$$

is increasing because T and T_λ have the same path for the Rauzy-Veech algorithm. Therefore H extends uniquely to an homeomorphism from I_λ onto I, which obviously satisfies

$$H \circ T_\lambda = T \circ H$$

Thus, T_λ and T are topologically conjugated. The image under H_* of the Lebesgue measure on I_λ is a measure on I (of total mass $|I_\lambda|$) which is invariant under T.

Conversely, let μ be a finite measure invariant under T. We set, for $\alpha \in \mathcal{A}$

$$\lambda_\alpha = \mu(I_\alpha^t) = \mu(I_\alpha^b) \ .$$

We also define, for $x \in I$

$$K(x) = \mu(\{y \in I \ ; \ y < x\}) \ .$$

As T is minimal, μ has no atom and the support of μ is I ; therefore, K is an homeomorphism from I onto $(0, \mu(I)) =: I_\mu$.

Define then

$$T_\mu = K \circ T \circ K^{-1} \ .$$

Then T_μ preserves the Lebesgue measure and it is easy to check that T_μ is an i.e.m on I_μ with combinatorial data π and length data λ.

It is immediate to check that the two maps $\mathcal{C}(\gamma) \to \mathcal{M}(T)$, $\mathcal{M}(T) \to \mathcal{C}(\gamma)$ just defined are inverse to each other.

8.2 Number of invariant ergodic probability measures.

Let T be an i.e.m on an interval I. Let g be the genus of the translation surfaces that can be constructed from T by the zippered rectangle construction. Let $\mathcal{M}(T)$ be the cone of finite invariant measures for T, which can be identified with the cone $\mathcal{C}(\gamma)$ determined by the infinite path γ associated to T by the Rauzy-Veech algorithm.

Proposition *The cone $\mathcal{C}(\gamma) \cup \{0\}$ is a closed simplicial cone of dimension $\leqslant g$. The number of invariant ergodic probability measures is therefore $\leqslant g$.*

Proof : We have seen in the second part of the proof of Proposition 1 in Section 7.7 that $\mathcal{C}(\gamma) \cup \{0\}$ is a closed cone. That this closed cone is simplicial follows from the identification of $\mathcal{C}(\gamma)$ with $\mathcal{M}(T)$: extremal rays of $\mathcal{C}(\gamma)$ correspond to ergodic invariant probability measures and invariant probability measures can be written in a unique way as convex combination of ergodic ones.

It remains to be seen that the subspace E of $\mathbb{R}^{\mathcal{A}}$ generated by $\mathcal{C}(\gamma)$ has dimension $\leqslant g$. Let $(\mathcal{A}, \pi)$ be combinatorial data for T, let Ω be the corresponding antisymmetric matrix.

We first claim that $E \cap \operatorname{Ker} \Omega = \{0\}$. Indeed, let $v, v' \in \mathcal{C}(\gamma)$ such that $v - v' \in \operatorname{Ker} \Omega$. Write $\gamma(n)$ for the initial part of γ of length n. According to the Proposition in Section 7.6, the vector $(v-v')B_{\gamma(n)}^{-1}$ depends only on the endpoint of $\gamma(n)$. On the other hand, from Corollary 1 in Section 7.7, we have that $v\ B_{\gamma(n)}^{-1}$ and $v'\ B_{\gamma(n)}^{-1}$ go to zero. Hence $v = v'$, proving the claim.

We now show that the image of E in $\mathbb{R}^{\mathcal{A}} / \operatorname{Ker} \Omega$ is isotropic for the symplectic form determined by Ω. Otherwise, there would exist $v, v' \in \mathcal{C}(\gamma)$ with

$$v\, \Omega\, {}^t v' > 0\ .$$

Again, $v\, B_{\gamma(n)}^{-1}, v'\, B_{\gamma(n)}^{-1}$ go to zero. But according to Section 7.6 we have

$$v\, B_{\gamma(n)}^{-1}\ \Omega_n\ {}^t B_{\gamma(n)}^{-1}\ {}^t v' = v\, \Omega\, {}^t v'\ ,$$

where Ω_n is the matrix associated to the endpoint of $\gamma(n)$. This gives a contradiction ; as $\operatorname{rk} \Omega = 2g$, we conclude that $\dim E \leqslant g$. $\square$

In the next Section, we see that the bound in the proposition is optimal. However, as mentioned in Section 6.11, a theorem of Masur and Veech guarantees that $\mathcal{C}(\gamma)$ is a ray for almost all i.e.m.

8.3 Examples of non uniquely ergodic i.e.m. [Kea1],[KeyNew]

We will construct in a Rauzy diagram of genus g an infinite path γ which is an infinite concatenation of complete paths but has the property that the subspace generated by $\mathcal{C}(\gamma)$ has dimension g.

Let $d \geqslant 2$. Define $\mathcal{A}^{(d)} = \{1, \ldots, d\}$ and

$$\pi_t^{(d)}(k) = k\ ,\ \pi_b^{(d)}(k) = d + 1 - k\ ,$$

for $1 \leqslant k \leqslant d$. Let $\mathcal{R}(d)$ be the Rauzy class for $\pi^{(d)} = (\pi_t^{(d)}\ ,\ \pi_b^{(d)})$, $\mathcal{D}(d)$ the associated Rauzy diagram. From Section 4.4 , we check that the translation surfaces constructed from these combinatorial data through the zippered rectangle construction satisfy :

- if d is even, $d = 2g, s = 1, k_1 = 2g - 1$;
- if d is odd, $d = 2g + 1\ ,\ s = 2\ ,\ k_1 = k_2 = g$.

The diagrams $\mathcal{D}(d)$ for $d = 2, 3, 4$ have been pictured in Section 7.3. Their structure can be described as follows.

There is a canonical involution i of $\mathcal{D}(d)$ defined on vertices by $i(\pi) = \widehat{\pi}$ with

$$\widehat{\pi}_t(k) = \pi_b(d+1-k) \ ,$$

$$\widehat{\pi}_b(k) = \pi_t(d+1-k) \ .$$

The unique fixed point of i is $\pi^{(d)}$, and i changes the type of arrows from top to bottom and back. If one defines

$$\mathcal{D}_t(d) = \{\pi \in \ \mathcal{R}(d) \ , \ \pi_t(2) = 2\}$$
$$\mathcal{D}_b(d) = \{\pi \in \ \mathcal{R}(d) \ , \ \pi_b(d-1) = 2\}$$

then $i(\mathcal{D}_t(d)) = \mathcal{D}_b(d)$, $i(\mathcal{D}_b(d)) = \mathcal{D}_t(d)$, $\mathcal{D}_t(d) \cap \mathcal{D}_b(d) = \{\pi^{(d)}\}$ and any arrow has both endpoints in $\mathcal{D}_t(d)$ or both endpoints in $\mathcal{D}_b(d)$. Moreover, if one defines, for $3 \leqslant k \leqslant d$

$$\mathcal{D}_{b,k}(d) = \{\pi \in \mathcal{R}(d) \ ; \ \pi_b(d-1) = 2 \ , \pi_t(k) = 2\} \ ,$$

then $\mathcal{D}_{b,k}(d)$ is isomorphic to $\mathcal{D}_t(k-1)$ through an isomorphism which respects type, winner and loser.

A cycle of length $d-1$ of arrows of bottom type starting at $\pi^{(d)}$ connects together the vertex in $\mathcal{D}_{b,k}(d)$ corresponding to $\pi^{(k-1)}$ in $\mathcal{D}_t(k-1)$.

Let us now assume that $d = 2g$ is even. Consider, for positive integers $m_1, \ldots, m_g$, the loop $\gamma(m_1, \ldots, m_g)$ at $\pi^{(d)}$ in $\mathcal{D}(d)$ whose successful winners are (in exponential notation for repetition)

$$(1^{d-2}2^{m_1}1)d^2 1(d^{d-4}4^{m_2}3)d^2 \ldots ((d-3)^2(d-2)^{m_{g-1}}(d-3))d^2(d-1)^{m_g} \ .$$

This is a complete loop in $\mathcal{D}(d)$.

Assume that $0 \ll m_1 \ll m_2 \ldots \ll m_g$ and let $e_1, \ldots, e_d$ be the canonical basis of $\mathbb{R}^d$. One checks that

- $e_1 B_\gamma$ and $e_2 B_\gamma$ have size $\sim m_1$ in the approximate direction of $f_1 := e_2$;
- $e_3 B_\gamma$ and $e_4 B_\gamma$ have size $\sim m_2$ in the approximate direction of $f_2 := e_4 + e_1$;
- $e_5 B_\gamma$ and $e_6 B_\gamma$ have size $\sim m_3$ in the approximate direction of $f_3 := e_6 + e_3 + 2e_1$;

$\vdots$

- $e_{d-3} B_\gamma$ and $e_{d-2} B_\gamma$ have size $\sim m_{g-1}$ in the approximate direction of $f_{g-1} := e_{d-2} + e_{d-5} + \ldots + 2^{g-3} e_1$;
- $e_{d-1} B_\gamma$ and $e_d B_\gamma$ have size $\sim m_g$ in the approximate direction of $f_g := e_{d-1} + e_{d-3} + \ldots + 2^{g-2} e_1$.

Observe that $f_1, \dots, f_g$ are linearly independent.

Now take a sequence $(m_\ell)_{\ell>0}$ increasing very fast, define

$$\gamma_i = \gamma(m_{ig+1}, m_{ig+2}, \dots, m_{ig+g-1}),$$
$$\gamma(i) = \gamma_0 * \gamma_1 \dots * \gamma_{i-1},$$
$$\gamma = \gamma_0 * \gamma_1 * \dots$$

One checks that for all $i > 0$ and $1 \leqslant k \leqslant g$, the vectors $e_{2k-1}\ B_{\gamma(i)}$ and $e_{2k}\ B_{\gamma(i)}$ have approximate directions f_k ; more precisely, as $i \to \infty$, their directions converge to the same limit $f_k(\infty)$ which can be chosen arbitrarily close to f_k. In particular, if the sequence $(m_\ell)_{\ell>0}$ increases fast enough, the limit directions $f_k(\infty), 1 \leqslant k \leqslant g$, are linearly independent, which implies that the vector space spanned by $\mathcal{C}(\gamma)$ has dimension g.

9 Rauzy-Veech dynamics and Teichmüller flow

We establish in this section a relation between the Rauzy-Veech continued fraction algorithm and the Teichmüller flow on the moduli space $\mathcal{M}(M, \Sigma, \kappa)$ that generalizes the classical case of the usual continued fraction and the geodesic flow on the modular surface.

This will also exhibit the moduli space in a form which allows to check that its volume is finite.

Throughout this section, we fix an alphabet $\mathcal{A}$, a Rauzy class $\mathcal{R}$ and denote by $\mathcal{D}$ the associated Rauzy diagram.

9.1 Rauzy-Veech dynamics

With

$$\Delta = \{\lambda \in \mathbb{R}^{\mathcal{A}};\ \lambda_\alpha > 0,\ \ \forall\ \alpha\ \in\ \mathcal{A}\ \},$$

we set

$$\Delta(\mathcal{D}) = \mathcal{R} \times \mathbb{P}(\Delta).$$

We denote by $V_+ : \Delta(\mathcal{D}) \to \Delta(\mathcal{D})$ the map induced by one step of the Rauzy-Veech algorithm. More precisely, let $\gamma : \pi \to \pi'$ be an arrow of $\mathcal{D}$. Let α_0 be the winner of γ and let α_1 be the loser of γ. Define

$$\Delta_\gamma = \{\lambda \in \Delta;\ \lambda_{\alpha_0} > \lambda_{\alpha_1}\ \}.$$

Then the domain of V_+ is the union, over all arrows γ, of the $\{\pi\} \times \mathbb{P}(\Delta_\gamma)$ and the restriction of V_+ to this set is induced by

$$(\pi, \lambda) \mapsto (\pi', \lambda B_\gamma^{-1}).$$

Each simplex in $\Delta(\mathcal{D})$ (identified by a vertex π of $\mathcal{D}$) contains two components of the domain of V_+ (associated to the two arrows starting at π), each being sent to a full simplex of $\Delta(\mathcal{D})$ (corresponding to the endpoint of the arrow). The map V_+ is therefore essentially 2-to-1.

Introducing the suspension variables τ leads to a map V which is essentially the natural extension of V_+. Let

$$S(\mathcal{D}) = \bigsqcup_{\mathcal{R}} (\{\pi\} \times \mathbb{P}(\Delta) \times \mathbb{P}(\Theta_\pi)),$$

where we recall from subsection 7.4 that

$$\Theta_\pi = \{\tau \in \mathbb{R}^{\mathcal{A}};\ \sum_{\pi_t(\alpha)<k} \tau_\alpha > 0,\quad \sum_{\pi_b(\alpha)<k} \tau_\alpha < 0,\ \forall\, 1 < k \leqslant d\}.$$

For an arrow $\gamma : \pi \to \pi'$ of $\mathcal{D}$, we set

$$\Theta_\gamma = \{\tau \in \Theta_{\pi'};\ \epsilon \sum_\alpha \tau_\alpha > 0\},$$

where $\epsilon = -1$ (resp.$\epsilon = +1$) if γ is of top type (resp. bottom type). Define then

$$S_\gamma(\mathcal{D}) = \{\pi\} \times \mathbb{P}(\Delta_\gamma) \times \mathbb{P}(\Theta_\pi),$$
$$S^\gamma(\mathcal{D}) = \{\pi'\} \times \mathbb{P}(\Delta) \times \mathbb{P}(\Theta_\gamma).$$

The domain of $V : S(\mathcal{D}) \to S(\mathcal{D})$ is the (disjoint) union, over all arrows of $\mathcal{D}$, of the $S_\gamma(\mathcal{D})$; the image of V is the (disjoint) union of the $S^\gamma(\mathcal{D})$, the restriction of V to $S_\gamma(\mathcal{D})$ sends this set in a one-to-one way onto $S^\gamma(\mathcal{D})$ through the map induced by

$$(\pi, \lambda, \tau) \mapsto (\pi', \lambda B_\gamma^{-1}, \tau B_\gamma^{-1}).$$

The map V is therefore, up to codimension one sets, invertible.

9.2 Rauzy diagrams and Teichmüller spaces

Let π be an element of $\mathcal{R}$. Recall that the canonical length and suspension data are given by

$$\lambda_\alpha^{can} = 1, \qquad \tau_\alpha^{can} = \pi_b(\alpha) - \pi_t(\alpha), \qquad \forall\, \alpha \in \mathcal{A}.$$

With these data, we construct (using the zippered rectangle construction of subsection 4.3, or the simplified version of subsection 4.2) a translation surface $(M_\pi, \Sigma_\pi, \kappa_\pi, \zeta_\pi)$.

On the other hand, starting from data $(\lambda, \tau) \in \Delta \times \Theta_\pi$, the zippered rectangle construction produces a translation surface which is a deformation of $(M_\pi, \Sigma_\pi, \kappa_\pi, \zeta_\pi)$, i.e homeomorphic to $(M_\pi, \Sigma_\pi, \kappa_\pi)$ through an homeomorphism whose isotopy class is canonically defined. We therefore obtain a canonical embedding

$$i_\pi : \Delta \times \Theta_\pi \longrightarrow \widetilde{Q}(M_\pi, \Sigma_\pi, \kappa_\pi)$$

in the marked Teichmüller space. This is an embedding because it is a local section of the period map.

Let now $\gamma : \pi \to \pi'$ be an arrow of $\mathcal{D}$. The data $\lambda = \lambda^{can} B_\gamma, \tau = \tau^{can}$ produce a translation surface $(M_\pi, \Sigma_\pi, \kappa_\pi, \zeta_\pi^0)$; the data $\lambda = \lambda^{can}, \tau = \tau^{can} B_\gamma^{-1}$ with the combinatorial data

π' produce a translation surface $(M_{\pi'}, \Sigma_{\pi'}, \kappa_{\pi'}, \zeta^1_{\pi'})$. As observed in subsection 7.4, these two translation surfaces are canonically isomorphic. This means that there exists an homeomorphism between the topological surfaces $(M_\pi, \Sigma_\pi, \kappa_\pi)$ and $(M_{\pi'}, \Sigma_{\pi'}, \kappa_{\pi'})$ whose isotopy class is canonically defined by γ. This leads to a canonical homeomorphism

$$j_\gamma : \widetilde{Q}(M_\pi, \Sigma_\pi, \kappa_\pi) \longrightarrow \widetilde{Q}(M_{\pi'}, \Sigma_{\pi'}, \kappa_{\pi'})$$

between marked Teichmüller spaces.

Let us observe that the isomorphic translation surfaces $(M_\pi, \Sigma_\pi, \kappa_\pi, \zeta^0_\pi)$, $(M_{\pi'}, \Sigma_{\pi'}, \kappa_{\pi'}, \zeta^1_{\pi'})$ above define a point in

$$i_\pi(\Delta_\pi \times \Theta_\pi) \cap j_\gamma^{-1}(i_{\pi'}(\Delta_{\pi'} \times \Theta_{\pi'})).$$

As a consequence the union

$$i_\pi(\Delta_\pi \times \Theta_\pi) \cup j_\gamma^{-1}(i_{\pi'}(\Delta_{\pi'} \times \Theta_{\pi'}))$$

is a connected subset of $\widetilde{Q}(M_\pi, \Sigma_\pi, \kappa_\pi)$.

We introduce the groupoid $\Gamma(\mathcal{D})$ of paths in the **non-oriented** Rauzy diagram $\widetilde{\mathcal{D}}$: the vertices of $\widetilde{\mathcal{D}}$ are those of $\mathcal{D}$ (i.e the elements of the Rauzy class $\mathcal{R}$) but for each arrow $\gamma : \pi \to \pi'$ in $\mathcal{D}$ we have two arrows $\gamma^+ : \pi \to \pi'$ and $\gamma^- : \pi' \to \pi$ in $\widetilde{\mathcal{D}}$. The groupoid $\Gamma(\mathcal{D})$ is the groupoid of oriented paths in $\widetilde{\mathcal{D}}$, quotiented out by the cancellation rules $\gamma^+ * \gamma^- = \gamma^- * \gamma^+ = 1$. We denote by $\Gamma_\pi(\mathcal{D})$ the subset of reduced paths starting at π and by $\pi_1(\widetilde{\mathcal{D}}, \pi)$ the group of reduced loops at π.
For each arrow γ of $\mathcal{D}$, we have defined above an isomorphism j_γ between marked Teichmüller spaces. There is a unique way to extend functorially this definition to $\Gamma(\mathcal{D})$: for each $\gamma \in \Gamma(\mathcal{D})$ starting at π and ending at π', we have an isomorphism

$$j_\gamma : \widetilde{Q}(M_\pi, \Sigma_\pi, \kappa_\pi) \longrightarrow \widetilde{Q}(M_{\pi'}, \Sigma_{\pi'}, \kappa_{\pi'}),$$

and $j_{\gamma_1 * \gamma_2} = j_{\gamma_2} \circ j_{\gamma_1}$ whenever $\gamma_1 * \gamma_2$ is defined.

In particular, when $\gamma \in \pi_1(\widetilde{\mathcal{D}}, \pi)$, j_γ is an automorphism of $\widetilde{Q}(M_\pi, \Sigma_\pi, \kappa_\pi)$. We obtain in this way a group homomorphism

$$\gamma \longmapsto j_\gamma \ ,$$

$$\pi_1(\widetilde{\mathcal{D}}, \pi) \longrightarrow \mathrm{Mod}^+(M_\pi, \Sigma_\pi) \ .$$

We now define

$$\mathcal{U}_\pi = \bigcup_{\Gamma_\pi(\mathcal{D})} j_\gamma^{-1}(i_{\pi'}(\Delta_{\pi'} \times \Theta_{\pi'})) \ ,$$

where π' is the endpoint of $\gamma \in \Gamma_\pi(\mathcal{D})$. It follows immediately from the observation at the end of subsection 9.1 that $\mathcal{U}_\pi$ is an open connected subset of $\widetilde{Q}(M_\pi, \Sigma_\pi, \kappa_\pi)$. We will denote by $\mathcal{C}_\pi$ the component of $\widetilde{Q}(M_\pi, \Sigma_\pi, \kappa_\pi)$ which contains $\mathcal{U}_\pi$.

9.3 The following result shows that, when considering some component $\mathcal{C}$ of a (marked) Teichmüller space $\widetilde{Q}(M, \Sigma, \kappa)$, there is no loss of generality if we assume that $(M, \Sigma, \kappa) = (M_\pi, \Sigma_\pi, \kappa_\pi)$ (for some appropriate combinatorial data $(\mathcal{A}, \pi)$) and $\mathcal{C} = \mathcal{C}_\pi$.

Proposition *Let (M, Σ, κ) be combinatorial data for a translation surface, let $\mathcal{C}$ be a connected component of the marked Teichmüller space $\widetilde{Q}(M, \Sigma, \kappa)$, and let $\mathcal{U}$ be the open subset of $\mathcal{C}$ formed by the translation surface structures in $\mathcal{C}$ that can be obtained through the zippered rectangle construction.*

1. *The set $\mathcal{C} - \mathcal{U}$ has real codimension $\geqslant 2$ in $\mathcal{C}$.*
2. *There exist combinatorial data $(\mathcal{A}, \pi)$ and a homeomorphism $g : (M_\pi, \Sigma_\pi, \kappa_\pi) \to (M, \Sigma, \kappa)$ such that the corresponding isomorphism g_* of marked Teichmüller spaces satisfy*
$$g_*(\mathcal{U}_\pi) = \mathcal{U}.$$
3. *Assume that $(\mathcal{A}', \pi')$ are combinatorial data and $g' : (M_{\pi'}, \Sigma_{\pi'}, \kappa_{\pi'}) \to (M, \Sigma, \kappa)$ is an homeomorphism such that*
$$g'_*(\mathcal{U}_{\pi'}) \subset \mathcal{C}.$$
Then, the Rauzy diagrams $\mathcal{D}$, $\mathcal{D}'$ spanned by π, π' are isomorphic. Moreover, assuming that $\mathcal{D} = \mathcal{D}'$, $\pi = \pi'$, the element of $\mathrm{Mod}^+(M_\pi, \Sigma_\pi)$ determined by $g^{-1} \circ g'$ belongs to the image of the group homomorphism
$$\pi_1(\mathcal{D}, \pi) \longrightarrow \mathrm{Mod}^+(M_\pi, \Sigma_\pi)$$
defined in the last subsection.

Remark It is quite possible that this homomorphism is always onto. This has been checked for $g = 1$, with any number of marked points, by Wang Zhiren.

Proof: Part 1. of the proposition is a consequence of the proposition in subsection 5.6 and the corollary in subsection 5.5: if a translation surface structure on (M, Σ, κ) has no vertical connection or no horizontal connection, it can be represented with the appropriate marking as a suspension through the zippered rectangle construction. Having both horizontal and vertical connections is indeed a codimension 2 property: this can already be seen on each orbit of the $SL(2, \mathbb{R})$ action (for instance).

By definition of $\mathcal{U}$, this open set is the union, over all combinatorial data $(\mathcal{A}, \pi)$, and all homeomorphisms $g : (M_\pi, \Sigma_\pi, \kappa_\pi) \to (M, \Sigma, \kappa)$ such that $g_*(i_\pi(\Delta_\pi \times \Theta_\pi)) \subset \mathcal{C}$, of the sets $g_*(i_\pi(\Delta_\pi \times \Theta_\pi))$. As its complement in $\mathcal{C}$ has codimension 2, the open set $\mathcal{U}$ is **connected**.

Claim *If $(\mathcal{A}, \pi, g)$, $(\mathcal{A}', \pi', g')$ satisfy*
$$g_*(i_\pi(\Delta_\pi \times \Theta_\pi)) \bigcap g'_*(i_{\pi'}(\Delta_{\pi'} \times \Theta_{\pi'})) \neq \emptyset,$$
then the Rauzy diagrams $\mathcal{D}$, $\mathcal{D}'$ spanned by π, π' are isomorphic and (assuming $\mathcal{A} = \mathcal{A}'$, $\mathcal{D} = \mathcal{D}'$) either $g_*^{-1} \circ g'_*$ or $g'^{-1}_* \circ g_*$ is equal to j_γ for a finite **oriented** *path γ in $\mathcal{D}$.*

Proof of claim: By hypothesis, there are two isomorphic translation surface structures ζ, ζ' on (M, Σ, κ) such that:

- ζ is obtained by the zippered rectangle construction from an i.e.m T acting on an interval I with combinatorial data $(\mathcal{A}, \pi)$, length data λ, suspension data τ;

- ζ' is obtained by the zippered rectangle construction from an i.e.m T' acting on an interval I' with combinatorial data $(\mathcal{A}', \pi')$, length data λ', suspension data τ'.

Let $G : (M, \Sigma, \kappa, \zeta) \to (M, \Sigma, \kappa, \zeta')$ be an isomorphism. It sends the marked outgoing horizontal separatrix for ζ isometrically onto the marked outgoing separatrix for ζ'.
If $|I| = |I'|$, we can already conclude that $T = T'$ and $\tau = \tau'$.
Assume for instance that $|I| > |I'|$. Then T' must be the first return map of T on the interval of length $|I'|$ with the same left endpoint than I. That T' is obtained from T by a finite number of steps of the Rauzy-Veech algorithm now follows from corollary 1 in subsection 7.7 (applying if necessary the same small rotation to both ζ and ζ', we may assume that ζ has no vertical connection) and the last exercise in subsection 7.2. □

End of proof of proposition: A first consequence of the claim is that the combinatorial data $(\mathcal{A}, \pi)$ such that $g_*(i_\pi(\Delta_\pi \times \Theta_\pi)) \subset \mathcal{C}$ all belong to the same Rauzy class (up to isomorphism): otherwise, the set $\mathcal{U}$ would not be connected. Once we know that, both the second and the third part of the proposition are immediate consequences of the claim. □

9.4 Rauzy diagrams and moduli spaces

Let $\mathcal{A}$, $\mathcal{R}$, $\mathcal{D}$ as above. We fix a vertex π^* of $\mathcal{D}$ and denote simply $(M_{\pi^*}, \Sigma_{\pi^*}, \kappa_{\pi^*})$, $\mathcal{U}_{\pi^*}$, $\mathcal{C}_{\pi^*}$ by (M, Σ, κ), $\mathcal{U}$, $\mathcal{C}$.
It follows from the third part of the proposition that the stabilizer of $\mathcal{C}$ (for the action of $\mathrm{Mod}^+(M, \Sigma)$ on $\widetilde{Q}(M, \Sigma, \kappa)$) is the subgroup image of $\pi_1(\mathcal{D}, \pi^*)$, which will be denoted by $\mathrm{Mod}_0(M, \Sigma)$.

We now define what amounts to a fundamental domain for the action of $\mathrm{Mod}_0(M, \Sigma)$ on $\mathcal{C}$.

For each vertex π of $\mathcal{D}$, define

$$\Delta^0_\pi = \{\lambda \in \Delta;\ 1 \leqslant \sum_\alpha \lambda_\alpha \leqslant 1 + \min(\lambda_{\alpha_t}, \lambda_{\alpha_b})\ \}$$

where $\pi_t(\alpha_t) = \pi_b(\alpha_b) = d$.

Consider then the disjoint union, over elements of $\mathcal{R}$, of the $\Delta^0_\pi \times \Theta_\pi$ and perform the following identifications on the boundaries of these sets.

The part of the boundary of $\Delta^0_\pi \times \Theta_\pi$ where $\sum_\alpha \lambda_\alpha = 1$ is called the *lower* boundary of $\Delta^0_\pi \times \Theta_\pi$; it is divided into a *top half* where $\sum_\alpha \tau_\alpha < 0$ and a *bottom half* where $\sum_\alpha \tau_\alpha > 0$.

The part of the boundary of $\Delta^0_\pi \times \Theta_\pi$ where $\sum_\alpha \lambda_\alpha = 1 + \min(\lambda_{\alpha_t}, \lambda_{\alpha_b})$ is called the *upper* boundary of $\Delta^0_\pi \times \Theta_\pi$; it is divided into a *top half* where $\lambda_{\alpha_t} > \lambda_{\alpha_b}$ and a *bottom half* where $\lambda_{\alpha_t} < \lambda_{\alpha_b}$.

For each arrow $\gamma : \pi \to \pi'$ in $\mathcal{D}$, of top type, we identify the top half of the upper boundary of $\Delta^0_\pi \times \Theta_\pi$ with the top half of the lower boundary of $\Delta^0_{\pi'} \times \Theta_{\pi'}$ through $(\lambda, \tau) \mapsto (\lambda B_\gamma^{-1}, \tau B_\gamma^{-1})$;

when γ is of bottom type, we identify similarly bottom halves.

We denote by $\mathcal{M}(\mathcal{D})$ the space obtained from $\bigsqcup_\pi \Delta^0_\pi \times \Theta_\pi$ by these identifications.
From its definition in subsection 9.2, it is clear that the set $\mathcal{U}$ is invariant under $\mathrm{Mod}_0(M,\Sigma)$. The same is true for the smaller set

$$\mathcal{V} := \bigcup_{\gamma \in \Gamma_{\pi^*}(\mathcal{D})} j_\gamma^{-1}(i_\pi(\Delta^0_\pi \times \Theta_\pi)) ,$$

where π is the endpoint of a path $\gamma \in \Gamma_{\pi^*}(\mathcal{D})$.

Proposition *There exists a unique continuous map*

$$p : \mathcal{V} \longrightarrow \mathcal{M}(\mathcal{D})$$

such that for every $\gamma \in \Gamma_{\pi^*}(\mathcal{D})$ (with endpoint π), the composition $p \circ j_\gamma^{-1} \circ i_\pi$ is the canonical map from $\Delta^0_\pi \times \Theta_\pi$ to $\mathcal{M}(\mathcal{D})$. Moreover, p is a covering map which identifies $\mathcal{M}(\mathcal{D})$ with the quotient of $\mathcal{V}$ by the action of $\mathrm{Mod}_0(M,\Sigma)$. *The set $\mathcal{U} - \mathcal{V}$ has codimension* 1.

Proof: Let γ be a path in $\Gamma_{\pi^*}(\mathcal{D})$ with endpoint π, and let γ_0 be an arrow from π to some vertex π'. The intersection

$$j_\gamma^{-1}(i_\pi(\Delta^0_\pi \times \Theta_\pi)) \bigcap j_{\gamma * \gamma_0}^{-1}(i_{\pi'}(\Delta^0_{\pi'} \times \Theta_{\pi'}))$$

is non empty; if γ_0 is for instance of top type, it is equal to the image $j_\gamma^{-1} \circ i_\pi$ of the top half of the upper boundary of $\Delta^0_\pi \times \Theta_\pi$ and also to the image by $j_{\gamma * \gamma_0}^{-1} \circ i_{\pi'}$ of the top half of the lower boundary of $\Delta^0_{\pi'} \times \Theta_{\pi'}$, the identification between these halves being exactly as in $\mathcal{M}(\mathcal{D})$. Moreover, it follows from the claim in the proof of proposition 9.3 that this is the only case where a non empty intersection occurs. As a consequence, a map p with the property required in the statement of the proposition exists, is continuous, and is uniquely defined by this property.

From the property defining p, two points in $\mathcal{V}$ have the same image under p iff they belong to the same $\mathrm{Mod}_0(M,\Sigma)$ orbit. This implies that p is a covering map.

Finally, let $[\zeta] = j_\gamma^{-1} \circ i_\pi(\lambda,\tau)$ be a point of $\mathcal{U}$ (with $\gamma \in \Gamma_{\pi^*}(\mathcal{D})$, π the endpoint of γ, $\lambda \in \Delta_\pi$, $\tau \in \Theta_\pi$). If $\lambda \in \Delta^0_\pi$, then $[\zeta]$ belongs to $\mathcal{V}$. Otherwise, $\sum_\alpha \lambda_\alpha$ is either too large or too small. If it is too large, we apply one step of the Rauzy-Veech algorithm unless $\lambda_{\alpha_t} = \lambda_{\alpha_b}$. If it is too small, we apply one step backwards unless $\sum_\alpha \tau_\alpha = 0$. Iterating this process, we will end up in $\mathcal{V}$ unless we run into one of the codimension one conditions that stops the algorithm (forwards or backwards). This proves that $\mathcal{U} - \mathcal{V}$ has codimension 1. □

9.5 Canonical volumes. Zorich acceleration

The last proposition allows us to identify $\mathcal{M}(\mathcal{D})$ with a subset of the marked moduli space $\widetilde{\mathcal{M}}(M,\Sigma,\kappa)$ whose complement has codimension 1. In particular this subset has full measure for the canonical volume of the moduli space. Observe that, in view of its relation with the period map, the canonical volume in $\mathcal{M}(\mathcal{D})$ is nothing else than the standard Lebesgue measure $d\lambda\, d\tau$ restricted to each $\Delta^0_\pi \times \Theta_\pi$.

The model $\mathcal{M}(\mathcal{D})$ for (part of) the moduli space provides us also with a natural transversal section for the Teichmüller flow, namely the union over the vertices π of $\mathcal{D}$ of the lower boundaries of the $\Delta^0_\pi \times \Theta_\pi$. Indeed, in each $\Delta^0_\pi \times \Theta_\pi$, the Teichmüller flow reads as

$$(\lambda, \tau) \mapsto (e^t\, \lambda, e^{-t}\, \tau)$$

and flows from the lower boundary of $\Delta^0_\pi \times \Theta_\pi$ to its upper boundary, being then glued as prescribed by the Rauzy-Veech algorithm to the lower boundary of some $\Delta^0_{\pi'} \times \Theta_{\pi'}$.

When computing volumes, we have to normalize the area $A = \tau\, \Omega_\pi\, {}^t\lambda$. Let $\mathcal{M}^{(1)}(\mathcal{D})$ be the subset of $\mathcal{M}(\mathcal{D})$ defined by $\{A = 1\}$. We can identify the set $S(\mathcal{D})$ of subsection 9.1 with the transverse section to the Teichmüller flow in $\mathcal{M}^{(1)}(\mathcal{D})$

$$\{(\pi, \lambda, \tau) \in \bigsqcup_\pi \{\pi\} \times \Delta \times \Theta_\pi;\ \sum_\alpha \lambda_\alpha \ = \ 1,\ \ \tau\, \Omega_\pi\, {}^t\lambda \ = \ 1\}.$$

With this identification, the return map of the Teichmüller flow on $S(\mathcal{D})$ is precisely given by the Rauzy-Veech dynamics V defined in subsection 9.1. The return time is equal to

$$\log \frac{||\lambda||_1}{||\lambda\, B_\gamma^{-1}||_1},$$

where $||\ .\ ||_1$ is the ℓ^1-norm.

Observe that the return time is bounded from above, but not bounded away from 0. The unfortunate consequence, as we see below, is that the measure of $S(\mathcal{D})$ is infinite; this already happens in the elementary case $d = 2$.

In order to get nicer dynamical properties, Zorich [Zo2] considered instead a smaller transversal section $S^*(\mathcal{D}) \subset S(\mathcal{D})$ which still gives an easily understood return map but has finite measure. For an arrow $\gamma : \pi \to \pi'$ of top type (resp. of bottom type), let $S^*_\gamma(\mathcal{D})$ be the set of $(\pi, \lambda, \tau) \in S_\gamma(\mathcal{D})$ such that $\sum \tau_\alpha > 0$ (resp. $\sum \tau_\alpha < 0$). Let $S^*(\mathcal{D})$ be the union of the $S^*_\gamma(\mathcal{D})$ over all arrows of $\mathcal{D}$.
The return map of the Teichmüller flow to $S^*(\mathcal{D})$, which is also the return map of V to $S^*(\mathcal{D})$, will be denoted by V^*. It is obtained as follows: one iterates V as long as the type of the corresponding arrow does not change. It is easy to check that it is the same than to ask that the winner does not change.

This property of V^* shows the return time for V to $S^*(\mathcal{D})$ does not depends on the τ-coordinate. One can therefore define a map $V^*_+ : \Delta(\mathcal{D}) \to \Delta(\mathcal{D})$ such that V^* is fibered over V^*_+.

Example For $d = 2$, we have

$$\begin{aligned}
S(\mathcal{D}) &= \{(\lambda_A, \lambda_B, \tau_A, \tau_B);\ \lambda_A > 0,\ \lambda_B > 0,\ \tau_A > 0, \tau_B < 0,\ \lambda_A + \lambda_B = 1, \lambda_A\tau_B - \lambda_B\tau_A = 1\},\\
S_t(\mathcal{D}) &= \{(\lambda_A, \lambda_B, \tau_A, \tau_B) \in S(\mathcal{D});\ \lambda_A > \lambda_B\},\\
S_b(\mathcal{D}) &= \{(\lambda_A, \lambda_B, \tau_A, \tau_B) \in S(\mathcal{D});\ \lambda_A < \lambda_B\}\ ,\\
S^t(\mathcal{D}) &= \{(\lambda_A, \lambda_B, \tau_A, \tau_B) \in S(\mathcal{D});\ \tau_A + \tau_B < 0\},\\
S^b(\mathcal{D}) &= \{(\lambda_A, \lambda_B, \tau_A, \tau_B) \in S(\mathcal{D});\ \tau_A + \tau_B > 0\},\\
S_t^*(\mathcal{D}) &= \{(\lambda_A, \lambda_B, \tau_A, \tau_B) \in S(\mathcal{D});\ \lambda_A > \lambda_B,\ \tau_A + \tau_B > 0\}\ ,\\
S_b^*(\mathcal{D}) &= \{(\lambda_A, \lambda_B, \tau_A, \tau_B) \in S(\mathcal{D});\ \lambda_A < \lambda_B,\ \tau_A + \tau_B > 0\}\ .
\end{aligned}$$

For $(\lambda_A, \lambda_B, \tau_A, \tau_B) \in S_t(\mathcal{D})$, we have

$$V(\lambda_A, \lambda_B, \tau_A, \tau_B) = (\lambda_A\lambda_B^{-1}, 1 - \lambda_A\lambda_B^{-1}, \lambda_B\tau_A, \lambda_B(\tau_B - \tau_A)).$$

For $(\lambda_A, \lambda_B, \tau_A, \tau_B) \in S_t^*(\mathcal{D})$, we have

$$V^*(\lambda_A, \lambda_B, \tau_A, \tau_B) = (\lambda_A\Lambda^{-1}, 1 - \lambda_A\Lambda^{-1}, \Lambda\tau_A, \Lambda(\tau_B - n\tau_A)),$$

where $\Lambda = \lambda_B - (n-1)\lambda_A,\ n\lambda_A < \lambda_B < (n+1)\lambda_A,\ n \geqslant 1.$

On the λ-coordinate, V_+^* is essentially given by the Gauss map.

9.6 Volume estimates: the key combinatorial lemmas

We will present three volume estimates: two for the measures of $S(\mathcal{D})$ and $S^*(\mathcal{D})$ and one for the measure of $\mathcal{M}^{(1)}(\mathcal{D})$, i.e the integral over $S(\mathcal{D})$ of the return time for the Teichmüller flow.

Before doing that, we consider the case $d = 2$ as an example of what happens in general. We first integrate over the τ variables. For a point (λ_A, λ_B) with $\lambda_B > \lambda_A > 0,\ \lambda_A + \lambda_B = 1$,

- the integral over $\{\tau_A > 0, \tau_B < 0, \lambda_B\tau_A - \lambda_A\tau_B = 1\}$ gives $\lambda_A^{-1}\lambda_B^{-1}$;
- the integral over $\{\tau_A + \tau_B > 0, \tau_B < 0, \lambda_B\tau_A - \lambda_A\tau_B = 1\}$ gives $(\lambda_A + \lambda_B)^{-1}\lambda_B^{-1} = \lambda_B^{-1}$.

Formulas for $\lambda_A > \lambda_B > 0,\ \lambda_A + \lambda_B = 1$ are symmetric.

For the measure of $S(\mathcal{D})$, we have therefore to integrate

$$\int_0^{\frac{1}{2}} \frac{d\,\lambda}{\lambda(1-\lambda)}$$

with the pole at 0 making the integral divergent.

For the measure of $S^*(\mathcal{D})$, we have to integrate

$$\int_0^{\frac{1}{2}} \frac{d\,\lambda}{1-\lambda}$$

on a domain away from the pole; the integral is equal to $\log 2$.

For the measure of $\mathcal{M}^{(1)}(\mathcal{D})$, the return time is $-\log(1-\lambda)$; the zero at 0 cancels the pole and we obtain

$$-\int_0^{\frac{1}{2}} \frac{d\,\lambda}{\lambda(1-\lambda)} \log(1-\lambda) \;=\; \frac{\pi^2}{12}.$$

The measure of $\mathcal{M}^{(1)}(\mathcal{D})$ is twice this.

We come back to the general case. Again, we want first to perform the integration over the τ variables. These variables run over the convex cone Θ_π but are restricted by the area condition. Define as usual α_t, α_b by $\pi_t(\alpha_t) = \pi_b(\alpha_b) = d$. Set

$$\begin{aligned} h^t_\alpha &= \sum_{\pi_t(\beta)\leqslant\pi_t(\alpha)} \tau_\beta \ , \\ h^b_\alpha &= -\sum_{\pi_b(\beta)\leqslant\pi_b(\alpha)} \tau_\beta \ , \\ \check{h}^t_\alpha &= \sum_{\pi_t(\beta)<\pi_t(\alpha)} \tau_\beta \ , \\ \check{h}^b_\alpha &= -\sum_{\pi_b(\beta)<\pi_b(\alpha)} \tau_\beta \ . \end{aligned}$$

With $h = -\Omega\, {}^t\tau$ as in subsection 4.3, we have

$$h_\alpha = h^t_\alpha + h^b_\alpha = \check{h}^t_\alpha + \check{h}^b_\alpha,$$

for all $\alpha \in \mathcal{A}$ and $h^t_{\alpha_t} + h^b_{\alpha_b} = 0$. The suspension conditions are

$$h^t_\alpha > 0 \quad \text{for } \ \alpha \neq \alpha_t, \qquad\qquad h^b_\alpha < 0 \quad \text{for } \ \alpha \neq \alpha_b.$$

Consider for instance the top half of $\Delta = \Delta_\pi$ where $\lambda_{\alpha_t} > \lambda_{\alpha_b}$ (the other case is symmetric); we write

$$\begin{aligned} \widehat{\lambda}_\alpha &= \lambda_\alpha && \text{for } \ \alpha \neq \alpha_t, \\ \widehat{\lambda}_{\alpha_t} &= \lambda_{\alpha_t} - \lambda_{\alpha_b} \ , && \\ \widehat{h}_\alpha &= h_\alpha && \text{for } \ \alpha \neq \alpha_b, \\ \widehat{h}_{\alpha_b} &= h_{\alpha_b} + h_{\alpha_t}. && \end{aligned}$$

The area is given by

$$A = \sum_\alpha \lambda_\alpha h_\alpha = \sum_\alpha \widehat{\lambda}_\alpha \widehat{h}_\alpha.$$

We decompose Θ_π into a finite family $\mathcal{G}(\pi)$ of simplicial disjoint cones. Let Γ be a cone in this family, and let $\tau^{(1)}, \cdots, \tau^{(d)}$ be a base of $\mathbb{R}^{\mathcal{A}}$ of volume 1 such that

$$\Gamma \;=\; \{ \ \sum_1^d t_i \tau_i \ ; \ t_i > 0 \ \ \} \ .$$

Writing $h^{(i)} = -\Omega_\pi \,{}^t\tau^{(i)}$ for $1 \leqslant i \leqslant d$, the area condition becomes

$$\sum_1^d t_i \,(\sum_\alpha \widehat{\lambda}_\alpha \,\widehat{h}_\alpha^{(i)}\,) \;=\; 1,$$

and therefore the integral over Γ gives

$$\frac{1}{(d-1)!}\;[\prod_1^d (\sum_\alpha \widehat{\lambda}_\alpha \,\widehat{h}_\alpha^{(i)}\,)\,]^{-1}.$$

To get the measure of $S(\mathcal{D})$, we should then integrate this quantity over the top half of Δ_π (normalized by $\sum_\alpha \lambda_\alpha = 1$), sum over $\Gamma \in \mathcal{G}(\pi)$, sum over π and finally add the symmetric contribution of the bottom halves.

Only the first step presents a finiteness problem. To deal with it, given a proper subset $\mathcal{B}$ of $\mathcal{A}$, we introduce the subspace $E_\mathcal{B}$ of $\mathbb{R}^\mathcal{A}$ generated by the τ in the closure of Θ_π such that $\widehat{h}_\alpha = 0$ for all $\alpha \in \mathcal{B}$ (with again $h = -\Omega \,{}^t\tau$).

Lemma 1 *We have* $\mathrm{codim} E_\mathcal{B} \geqslant \#\mathcal{B}$*, and even* $\mathrm{codim} E_\mathcal{B} > \#\mathcal{B}$ *when* $\alpha_b \in \mathcal{B}$*.*

Proof: We will find sufficiently many independent linear forms vanishing on $E_\mathcal{B}$.

Assume first that $\alpha_b \notin \mathcal{B}$. Let τ be a vector in the closure of Θ_π, such that $\widehat{h}_\alpha = 0$ for all $\alpha \in \mathcal{B}$. For $\alpha \in \mathcal{B}$, we have

$$\widehat{h}_\alpha = h_\alpha = h_\alpha^t + h_\alpha^b = \check{h}_\alpha^t + \check{h}_\alpha^b,$$

with $h_\alpha^b \geqslant 0$, $h_\alpha^t \geqslant 0$ (if $\alpha \neq \alpha_t$), $\check{h}_\alpha^t \geqslant 0$, $\check{h}_\alpha^b \geqslant 0$.

We have therefore $h_\alpha^t = 0$ for $\alpha \in \mathcal{B}$, $\alpha \neq \alpha_t$, and also $\check{h}_\alpha^t = 0$ for $\alpha \in \mathcal{B}$, $\pi_t(\alpha) > 1$. This gives at least $\#\mathcal{B}$ independent linear forms vanishing on such vectors τ, and thus also on $E_\mathcal{B}$ (the independence of the forms come from the triangular form of the h_α^t, $\check{h}_\alpha^t$).

Assume now that $\alpha_b \in \mathcal{B}$. Let τ be a vector in the closure of Θ_π, such that $\widehat{h}_\alpha = 0$ for all $\alpha \in \mathcal{B}$. The relation $\widehat{h}_{\alpha_b} = 0$ implies $h_{\alpha_t} = h_{\alpha_b} = 0$, hence

$$h_{\alpha_t}^t + h_{\alpha_t}^b = h_{\alpha_b}^b + h_{\alpha_b}^t = 0.$$

As we have $h_{\alpha_t}^b \geqslant 0$, $h_{\alpha_b}^t \geqslant 0$, $h_{\alpha_t}^t + h_{\alpha_b}^b = 0$, we conclude that

$$h_{\alpha_t}^t = h_{\alpha_t}^b = h_{\alpha_b}^b = h_{\alpha_b}^t = 0.$$

We have therefore

- $h_\alpha^b = \check{h}_\alpha^b = 0$ for all $\alpha \in \mathcal{B}$;
- $h_\alpha^t = \check{h}_\alpha^t = 0$ for all $\alpha \in \mathcal{B}$.

The first set of relations gives at least $\#\mathcal{B}+1$ independent linear forms vanishing on $E_{\mathcal{B}}$ unless $\pi_b(\mathcal{B}) = \{1, \cdots, \#\mathcal{B}\}$. The same is true for the second set of relations unless $\pi_t(\mathcal{B}) = \{1, \cdots, \#\mathcal{B}\}$. By irreducibility, the two exceptional cases are mutually exclusive and the proof of the lemma is complete. □

When we deal with $S^*(\mathcal{D})$, we should replace Θ_π by

$$\Theta^t_\pi = \{\tau \in \Theta_\pi;\ \sum_\alpha \tau_\alpha \ > \ 0\ \}$$

when we deal with the top half of Δ_π. We proceed in the same way, decomposing Θ_π into a finite family of simplicial cones Γ^*. We now define $E^*_{\mathcal{B}}$ as the subspace of $\mathbb{R}^{\mathcal{A}}$ generated by the vectors τ in the closure of Θ^t_π, such that $\widehat{h}_\alpha = 0$ for all $\alpha \in \mathcal{B}$.

Lemma 2 *We have* $\operatorname{codim} E^*_{\mathcal{B}} > \#\mathcal{B}$ *for all proper subsets* $\mathcal{B}$ *of* $\mathcal{A}$.

Proof: Obviously we have $E^*_{\mathcal{B}} \subset E_{\mathcal{B}}$, therefore the case where $\alpha_b \in \mathcal{B}$ is given by Lemma 1. We therefore assume that $\alpha_b \notin \mathcal{B}$.
Let τ be a vector in the closure of Θ^t_π, such that $\widehat{h}_\alpha = 0$ for all $\alpha \in \mathcal{B}$. For $\alpha \in \mathcal{B}$, we have

$$0 = \widehat{h}_\alpha = h_\alpha = h^t_\alpha + h^b_\alpha = \check{h}^t_\alpha + \check{h}^b_\alpha,$$

with $\check{h}^t_\alpha \geqslant 0$, $\check{h}^b_\alpha \geqslant 0$, $h^b_\alpha \geqslant 0$ (because $\alpha \neq \alpha_b$), $h^t_\alpha \geqslant 0$ (even for $\alpha = \alpha_t$). We therefore have

- $h^b_\alpha = \check{h}^b_\alpha = 0$ for all $\alpha \in \mathcal{B}$,
- $h^t_\alpha = \check{h}^t_\alpha = 0$ for all $\alpha \in \mathcal{B}$,

and conclude as in Lemma 1. □

9.7 Finiteness of volume for $\mathcal{M}^{(1)}(\mathcal{D})$ and $S^*(\mathcal{D})$.

The combinatorial facts proven in the last subsection will be combined with the following simple analytic lemma. Let

$$\Delta^{(1)} = \{\lambda \in \mathbb{R}^{\mathcal{A}};\ \lambda_\alpha > 0,\ \sum_\alpha \lambda_\alpha = 1\ \}.$$

For $\mathcal{B} \subset \mathcal{A}$, define also

$$\Delta^{(1)}_{\mathcal{B}} = \{\lambda \in \Delta^{(1)};\ \lambda_\alpha = 0 \ \text{ for } \alpha \notin \mathcal{B}\}.$$

Consider linear forms $L_1, \cdots, L_p, M_1, \cdots, M_q$ on $\mathbb{R}^{\mathcal{A}}$ which are positive on $\Delta^{(1)}$, and the rational map

$$R := \frac{L_1 \cdots L_p}{M_1 \cdots M_q}.$$

For $\mathcal{B} \subset \mathcal{A}$, let

$$\begin{aligned} m_+(\mathcal{B}) &= \#\{i;\ \ L_i(\lambda) = 0 \text{ for all } \lambda \in \Delta^{(1)}_{\mathcal{B}}\ \},\\ m_-(\mathcal{B}) &= \#\{j;\ M_j(\lambda) = 0 \text{ for all } \lambda \in \Delta^{(1)}_{\mathcal{B}}\ \},\\ m(\mathcal{B}) &= m_+(\mathcal{B}) - m_-(\mathcal{B}). \end{aligned}$$

Lemma *Assume that $d + m(\mathcal{B}) > \#\mathcal{B}$ holds for all proper subsets of $\mathcal{A}$. Then R is integrable on $\Delta^{(1)}$.*

Remark The converse is also true but will not be used.

Proof : We decompose $\Delta^{(1)}$ as follows: let

$$\mathcal{N} := \{n \in \mathbb{N}^{\mathcal{A}};\ \min_{\alpha} n_\alpha = 0\}.$$

For $n \in \mathcal{N}$, let $\Delta^{(1)}(n)$ be the set of $\lambda \in \Delta^{(1)}$ such that $\lambda_\alpha \geqslant \frac{1}{2d}$ if $n_\alpha = 0$ and

$$\frac{1}{2d} 2^{1-n_\alpha} > \lambda_\alpha \geqslant \frac{1}{2d} 2^{-n_\alpha}$$

if $n_\alpha > 0$. We have indeed

$$\Delta^{(1)} = \bigsqcup_{\mathcal{N}} \Delta^{(1)}(n)$$

and also

$$C^{-1}\, 2^{-\sum n_\alpha}\ \leqslant\ \mathrm{vol}\, \Delta^{(1)}(n)\ \leqslant C\, 2^{-\sum n_\alpha}.$$

Fix $n \in \mathcal{N}$. Let $0 = n^0 < n^1 < \cdots$ be the distinct values, in increasing order, taken by the n_α, and let

$$\mathcal{B}_i := \{\alpha \in \mathcal{A};\ n_\alpha\ \geqslant\ n^i\ \}.$$

Let L be a linear form on $\mathbb{R}^{\mathcal{A}}$, positive on $\Delta^{(1)}$. There is a maximal subset $\mathcal{B}(L) \subset \mathcal{A}$ such that $L(\lambda) = 0$ for all $\lambda \in \Delta^{(1)}_{\mathcal{B}(L)}$. We have then, for $n \in \mathcal{N}$, $\lambda \in \Delta^{(1)}(n)$

$$C_L^{-1} 2^{-m} \leqslant L(\lambda) \leqslant C_L 2^{-m}, \qquad \text{with } m = \min_{\mathcal{A}-\mathcal{B}(L)} n_\alpha.$$

The definition of m shows that $m \geqslant n^i$ iff $\mathcal{A} - \mathcal{B}(L) \subset \mathcal{B}_i$ and $m = n^i$ iff $\mathcal{B}_i^c \subset \mathcal{B}(L)$ but $\mathcal{B}_{i+1}^c \not\subset \mathcal{B}(L)$. From this, we see that for $n \in \mathcal{N}$, $\lambda \in \Delta^{(1)}(n)$, we have

$$C_R^{-1} 2^{-N} \leqslant R(\lambda) \leqslant C_R 2^{-N},$$

with

$$N = \sum_{i \geqslant 0} n^i\ (m(\mathcal{B}_i^c) - m(\mathcal{B}_{i+1}^c)) = \sum_{i>0} (n^i - n^{i-1})\ m(\mathcal{B}_i^c).$$

Using the hypothesis of the lemma, we have, for $i > 0$

$$m(\mathcal{B}_i^c) \geqslant \#\mathcal{B}_i^c - d + 1 = 1 - \#\mathcal{B}_i\ ,$$

and therefore

$$N\ \geqslant\ \sum_{i>0} (n^i - n^{i-1}) - \sum_{i \geqslant 0} n^i (\#\mathcal{B}_i - \#\mathcal{B}_{i+1}) = \max_{\alpha} n_\alpha - \sum_{\alpha} n_\alpha\ .$$

We conclude that the integral of R on $\Delta^{(1)}(n)$ is at most of the order of $2^{-\max_\alpha n_\alpha}$. Summing over $\mathcal{N}$ gives the required result. $\square$

We can now prove the finiteness of the measures of $\mathcal{M}^{(1)}(\mathcal{D})$ and $S^*(\mathcal{D})$. As explained in subsection 9.6, the total masses of these measures are expressed as finite sums of certain integrals over top or bottom halves of the $\Delta_\pi^{(1)}$. We will consider the case of top halves, the other case being symmetric. Observe that the top half of $\Delta_\pi^{(1)}$ is characterized by the inequalities $\widehat{\lambda}_\alpha > 0, \forall \alpha \in \mathcal{A}$. We will therefore in both cases apply the lemma above **in the $\widehat{\lambda}$ variables**. We don't have $\sum \widehat{\lambda}_\alpha = 1$, but observe that $\sum \lambda_\alpha = 1$ implies $\frac{1}{2} \leqslant \sum \widehat{\lambda}_\alpha \leqslant 1$, which is good enough.

- We start with $\mathcal{M}^{(1)}(\mathcal{D})$. The return time of the Teichmüller flow to $S(\mathcal{D})$ is equal to $-\log \sum \widehat{\lambda}_\alpha = -\log(1-\lambda_{\alpha_b})$ on the top half of $\Delta_\pi^{(1)}$.

  According to subsection 9.6, we have to integrate

  $$\frac{-\log(1-\lambda_{\alpha_b})}{(d-1)!} \; [\prod_1^d (\, \Sigma_\alpha \, \widehat{\lambda}_\alpha \, \widehat{h}_\alpha^{(i)} \,) \,]^{-1}$$

  over the top half of $\Delta_\pi^{(1)}$. The vectors $h^{(i)} = -\Omega_\pi \, {}^t\tau^{(i)}$ are obtained here from vectors $\tau^{(i)}$ generating a simplicial cone $\Gamma \subset \Theta_\pi$.

  We apply the lemma above with $p = 1$, $q = d$. We take $L(\lambda) = \lambda_{\alpha_b} = \widehat{\lambda}_{\alpha_b}$, a linear form of the same order than the return time $-\log(1-\lambda_{\alpha_b})$. The linear forms M_i are the $\Sigma_\alpha \, \widehat{\lambda}_\alpha \, \widehat{h}_\alpha^{(i)}$.

  We check the hypothesis of the lemma. Let $\mathcal{B} \subset \mathcal{A}$ be a proper subset. First, we have $m_+(\mathcal{B}) = 0$ if $\alpha_b \in \mathcal{B}$, $m_+(\mathcal{B}) = 1$ if $\alpha_b \notin \mathcal{B}$. Next we have

  $$\Sigma_\alpha \, \widehat{\lambda}_\alpha \, \widehat{h}_\alpha^{(i)} = 0, \quad \text{for all } \widehat{\lambda} \in \Delta_{\mathcal{B}}^{(1)}$$

  iff $\widehat{h}_\alpha^{(i)} = 0$ for all $\alpha \in \mathcal{B}$. By definition of $E_{\mathcal{B}}$, this happens iff $\tau^{(i)} \in E_{\mathcal{B}}$. As the $\tau^{(i)}$ are independent, Lemma 1 in the last subsection gives $m_-(\mathcal{B}) \leqslant d - \#\mathcal{B}$ if $\alpha_b \notin \mathcal{B}$, $m_-(\mathcal{B}) < d - \#\mathcal{B}$ if $\alpha_b \in \mathcal{B}$. The hypothesis of the lemma above is thus satisfied, and its conclusion gives the finiteness of the measure of $\mathcal{M}^{(1)}(\mathcal{D})$.

- We now deal with $S^*(\mathcal{D})$. According to subsection 9.6, we have to integrate

  $$\frac{1}{(d-1)!} \; [\prod_1^d (\, \Sigma_\alpha \, \widehat{\lambda}_\alpha \, \widehat{h}_\alpha^{(i)} \,) \,]^{-1}$$

  over the top half of $\Delta_\pi^{(1)}$. The vectors $h^{(i)} = -\Omega_\pi \, {}^t\tau^{(i)}$ are obtained here from vectors $\tau^{(i)}$ generating a simplicial cone $\Gamma^* \subset \Theta_\pi^t$.

  We will apply the lemma above with $p = 0$, $q = d$. The linear forms M_i are the $\Sigma_\alpha \, \widehat{\lambda}_\alpha \, \widehat{h}_\alpha^{(i)}$.

  We check the hypothesis of the lemma . For a proper subset $\mathcal{B} \subset \mathcal{A}$, we have

  $$\Sigma_\alpha \, \widehat{\lambda}_\alpha \, \widehat{h}_\alpha^{(i)} = 0, \quad \text{for all } \widehat{\lambda} \in \Delta_{\mathcal{B}}^{(1)}$$

iff $\widehat{h}^{(i)}_\alpha = 0$ for all $\alpha \in \mathcal{B}$. By definition of $E^*_{\mathcal{B}}$, this happens iff $\tau^{(i)} \in E^*_{\mathcal{B}}$. As the $\tau^{(i)}$ are independent, Lemma 2 in the last subsection guarantees that there are less than $d - \#\mathcal{B}$ such indices i. The hypothesis of the lemma above is thus satisfied, and its conclusion gives the finiteness of the measure of $S^*(\mathcal{D})$.

We have thus proved a first statement in the theorems of Masur and Veech presented in subsection 6.11, the finiteness of the volume of the moduli space of translation surfaces. Except in the simplest cases, it seems difficult to get the exact value of this volume through this method. Exact formulas for the volumes of the moduli spaces have been obtained by Eskin and Okounkov [EOk] using a different approach.

We end this section with the following statement, which is an easy consequence of the lemma above.

Proposition *The canonical measure on $S^*(\mathcal{D})$ satisfies, for all $\varepsilon > 0$*

$$m(\{(\pi, \lambda, \tau) \in S^*(\mathcal{D});\ min_\alpha \lambda_\alpha < \varepsilon\}) \leqslant C\varepsilon(\log \varepsilon)^{d-2},$$

where the constant C depends only on d.

Proof: In the context of the proof of the lemma, it is sufficient to observe that the number of $n \in \mathcal{N}$ such that $\max_\alpha n_\alpha = N$ is of the order of N^{d-2}. □

10 Ergodicity and unique ergodicity

In this section, we complete the proofs of the theorems of Masur and Veech presented in subsection 6.11.

10.1 Hilbert metric

Let C be an open set in the projective space $\mathbb{P}^N$ which is the image of an open convex cone in $\mathbb{R}^{N+1}$ whose closure intersects some hyperplane only at the origin.

Given two distinct points $x, y \in C$, the intersection of the line through x, y with C is a segment (a, b). The crossratio of the points a, b, x, y gives rise to a distance on C called the Hilbert metric on C:

$$d_C(x, y) := |\log \frac{x-a}{y-a} \frac{x-b}{y-b}|.$$

Exercise Check the triangle inequality.

The following properties are easily verified.

- Let X be a subset of C; then the closure $\overline{X}$ of X in $\mathbb{P}^N$ is contained in C iff X has finite diameter for d_C.

- If $\varphi : \mathbb{P}^N \to \mathbb{P}^N$ is a projective isomorphism, then, for all $x, y \in C$

$$d_{\varphi(C)}(\varphi(x), \varphi(y)) = d_C(x, y).$$

- If $C' \subset C$ is a smaller set satisfying the same hypothesis than C, then, for all $x, y \in C'$

$$d_C(x, y) \leqslant d_{C'}(x, y).$$

- If C' is a set satisfying the same hypothesis than C and $\overline{C}' \subset C$, there exists $k \in (0, 1)$ such that, for all $x, y \in C'$

$$d_C(x, y) \leqslant k\ d_{C'}(x, y).$$

Thus, if $\varphi : \mathbb{P}^N \to \mathbb{P}^N$ is a projective isomorphism satisfying $\overline{\varphi(C)} \subset C$, there exists $k \in (0, 1)$ such that, for all $x, y \in C'$ we have

$$d_C(\varphi(x), \varphi(y)) \leqslant k\ d_C(x, y).$$

10.2 Almost sure unique ergodicity

We prove that, for every combinatorial data $(\mathcal{A}, \pi)$, and almost every length vector $\lambda \in \mathbb{R}^{\mathcal{A}}$, the corresponding i.e.m is uniquely ergodic.

The set of i.e.m having a connection has codimension 1. Therefore, almost surely the Rauzy-Veech algorithm does not stop and associates to the i.e.m T an infinite path γ_T starting at π in the Rauzy diagram $\mathcal{D}$ constructed from $(\mathcal{A}, \pi)$. According to subsection 8.1, we have to prove that the closed convex cone $\mathcal{C}(\gamma_T)$ determined by γ_T is almost surely a ray.

By Poincaré recurrence of the Teichmüller flow and subsection 7.7, for almost every length vector λ, there exists an initial segment γ_s of γ_T which occurs infinitely many times in γ_T and such that all coefficients of the matrix B_{γ_s} are positive. We write γ_T as a concatenation

$$\gamma_T = \gamma_s * \gamma_1 * \gamma_s * \gamma_2 * \cdots.$$

Let C be the open set in $\mathbb{P}(\mathbb{R}^{\mathcal{A}})$ image of the positive cone in $\mathbb{R}^{\mathcal{A}}$. From the last property in the last subsection, there exists $k \in (0, 1)$ such that B_{γ_s} decreases the Hilbert metric d_C at least by a factor k, while the B_{γ_i}, $i = 1, 2, \cdots$ do not increase d_C. The first image CB_{γ_s} has closure contained in C hence has finite diameter K for d_C. We then have

$$\operatorname{diam}(CB_{\gamma_s * \cdots \gamma_i}) \leqslant K k^{i-1}.$$

It follows that the image in $\mathbb{P}(\mathbb{R}^{\mathcal{A}})$ of $\mathcal{C}(\gamma_T)$ is a point. The result is proved.

10.3 Ergodicity of the Teichmüller flow

We will prove in this subsection that the Teichmüller flow on $\mathcal{M}^{(1)}(\mathcal{D})$ and its return maps V on $S(\mathcal{D})$ and V^* on $S^*(\mathcal{D})$ are ergodic. In view of the relation between these three dynamical systems, the three statements are equivalent. We will prove that V^* is ergodic.

From the ergodicity of V and V^*, it follows that the maps V_+ and V_+^* on $\Delta(\mathcal{D})$ are also ergodic.

By Birkhoff's ergodic theorem, for every continuous function φ on $S^*(\mathcal{D})$, there exists an almost everywhere defined fuction $\overline{\varphi}$ such that, for almost every $(\pi, \lambda, \tau) \in S^*(\mathcal{D})$, one has

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{0}^{n-1} \varphi((V^*)^m(\pi, \lambda, \tau)) = \overline{\varphi}(\pi, \lambda, \tau),$$

and also

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{0}^{n-1} \varphi((V^*)^{-m}(\pi, \lambda, \tau)) = \overline{\varphi}(\pi, \lambda, \tau).$$

To prove ergodicity, it is sufficient to show that $\overline{\varphi}$ is almost everywhere constant, for any continuous function φ.

Starting from almost every (π, λ, τ), one can iterate the Rauzy-Veech algorithm both forward and backward. This leads to a biinfinite path $\gamma = \gamma^+ * \gamma^-$ in the Rauzy diagram $\mathcal{D}$, where γ^+ depends only on (π, λ) and γ^- depends only on (π, τ).

By Poincaré recurrence, for almost every (π, τ), there is a finite path γ_e at the end of γ^- such that all coefficients of B_{γ_e} are positive and which appears infinitely many times in γ^-. Let again C be the open set in $\mathbb{P}(\mathbb{R}^{\mathcal{A}})$ image of the positive cone in $\mathbb{R}^{\mathcal{A}}$, d_C the associated Hilbert metric. Let λ, $\lambda' \in \Delta_\pi$; for $m \geqslant 0$, let $\lambda_{-m}, \lambda'_{-m}$ be the respective λ-components of $(V^*)^{-m}(\pi, \lambda, \tau), (V^*)^{-m}(\pi, \lambda', \tau)$. By the same argument that in the last subsection, we have

$$\lim_{m \to +\infty} d_C(\lambda_{-m}, \lambda'_{-m}) = 0.$$

This implies that, for almost every (π, τ), $\overline{\varphi}(\pi, \lambda, \tau)$ does not depend on λ.

We claim that the same argument works exchanging λ and τ, future and past. For almost every (π, λ), we want to find a finite path γ_s at the beginning of γ^+ which appears infinitely many times in γ^+ (this is guaranteed by Poincaré recurrence) and satisfies

$$\overline{\Theta}_\pi B_{\gamma_s}^{-1} \subset \Theta_{\pi'} \cup \{0\}$$

where π' is the endpoint of γ_s. Then, using the Hilbert metrics relative to the open sets images in $\mathbf{P}(\mathbb{R}^{\mathcal{A}})$ of the Θ_π, we conclude in the same way as above that , for almost every (π, λ, τ), $\overline{\varphi}(\pi, \lambda, \tau)$ does not depend on τ. Thus, almost surely, $\overline{\varphi}(\pi, \lambda, \tau)$ does not depend on λ and τ. But $\overline{\varphi}(\pi, \lambda, \tau)$ is also V^*-invariant, therefore it must be almost everywhere constant.

It remains to prove that, almost surely, some initial path γ_s of γ^+ satisfies $\overline{\Theta}_\pi B_{\gamma_s}^{-1} \subset \Theta_{\pi'} \cup \{0\}$. This is a consequence of the following result.

Lemma *If a finite path $\underline{\gamma}$ in $\mathcal{D}$, from a vertex π to a vertex π', is the concatenation of $3d-4$ complete paths, then we have*

$$\overline{\Theta}_\pi B_{\underline{\gamma}}^{-1} \subset \Theta_{\pi'} \cup \{0\}.$$

Proof: For combinatorial data π and $\tau \in \mathbb{R}^{\mathcal{A}}$, we write as before

$$h^t_\alpha = \sum_{\pi_t(\beta) \leqslant \pi_t(\alpha)} \tau_\beta, \qquad h^b_\alpha = - \sum_{\pi_b(\beta) \leqslant \pi_b(\alpha)} \tau_\beta, \qquad h_\alpha = h^t_\alpha + h^b_\alpha.$$

We write $\gamma_1, \gamma_2, \cdots, \gamma_m$ for the successive arrows of $\underline{\gamma}$.
Starting from $\pi =: \pi^0$ with a nonzero vector $\tau^0 \in \mathbb{R}^{\mathcal{A}}$ satisfying

(2) $$h^{0,t}_\alpha \geqslant 0 \quad \text{for } \pi^0_t(\alpha) < d, \qquad h^{0,b}_\alpha \geqslant 0 \quad \text{for } \pi^0_b(\alpha) < d,$$

we have to show that

(3) $$h^{m,t}_\alpha > 0 \quad \text{for } \pi^m_t(\alpha) < d, \qquad h^{m,b}_\alpha > 0 \quad \text{for } \pi^m_b(\alpha) < d,$$

where π_j is the endpoint of γ_j and $h^{j,t}$, $h^{j,b}$ are calculated from $\tau^j := \tau^{j-1} B_{\gamma_j}^{-1}$.

The heigth vectors h^j are column vectors related by

$$h^j = B_{\gamma_j} h^{j-1}$$

and their entries are nonnegative. Let $m' < m$ is the smallest integer such that the initial part $\gamma_1 * \cdots * \gamma_{m'}$ of $\underline{\gamma}$ is the concatenation of $2d-3$ complete paths. By proposition 2 in subsection 7.7, we have

(4) $$h^j_\alpha > 0, \qquad \forall\, \alpha \in \mathcal{A}, \forall\, j \geqslant m'.$$

If γ_j is of top type, one has $\pi^j_t = \pi^{j-1}_t$ and

(5) $$h^{j,t}_\alpha = h^{j-1,t}_\alpha, \qquad \text{if } \pi^j_t(\alpha) < d,$$
(6) $$h^{j,b}_\alpha = h^{j-1,b}_\alpha, \qquad \text{if } \pi^{j-1}_t(\alpha) < d, \text{ and } \pi^{j-1}_b(\alpha) < d,$$
(7) $$h^{j,b}_{\alpha_b} = h^{j-1,b}_{\alpha_t}, \qquad \text{with } \pi^{j-1}_t(\alpha_t) = \pi^{j-1}_b(\alpha_b) = d,$$
(8) $$h^{j,b}_{\alpha_t} = h^{j-1,b}_{\alpha_*} + h^{j-1}_{\alpha_t}, \qquad \text{with } \pi^{j-1}_b(\alpha_*) = d-1.$$

Let $\ell^t(j)$ (resp. $\ell^b(j)$) be the largest integer ℓ such that $h^{j,t}_\alpha > 0$ for $\pi^j_t(\alpha) < \ell$ (resp. $h^{j,b}_\alpha > 0$ for $\pi^j_b(\alpha) < \ell$). We want to show that $\ell^t(m) = \ell^b(m) = d$. This implies the required conclusion.

We always have (trivially) $\ell^t(j) \geqslant 1$, $\ell^b(j) \geqslant 1$. Assume for instance that γ_j is of top type as above. Then relation (5) and $\pi^j_t = \pi^{j-1}_t$ imply that $\ell^t(j) \geqslant \ell^t(j-1)$. If $\pi^j_b(\alpha_t) = \pi^{j-1}_b(\alpha_t) > \ell^b(j-1)$, we have $\ell^b(j) \geqslant \ell^b(j-1)$ from (6). On the other hand, if $\pi^j_b(\alpha_t) \leqslant \ell^b(j-1)$ and

$j > m'$, it follows from relations (4),(6),(7),(8) that $\ell^b(j) > \ell^b(j-1)$. We first conclude that ℓ^t, ℓ^b are non-decreasing functions of $j \geqslant m'$.

Let $m' < m_0 < m_1 \leqslant m$ be such that $\gamma_{m_0} * \cdots * \gamma_{m_1-1}$ is complete. Observe that there is a letter ${}_b\alpha$ such that $\pi({}_b\alpha) = 1$ for all vertices π of $\mathcal{D}$. Let $m_0 \leqslant j < m_1$ such that ${}_b\alpha$ is the winner of γ_j. Then γ_j is of top type so, in the notations above, we have ${}_b\alpha = \alpha_t$, $1 = \pi_b^j(\alpha_t) \leqslant \ell^b(j-1)$ and $\ell^b(j) > \ell^b(j-1)$. As we can find $d-1$ disjoint such complete subpaths between m' and m, this shows that $\ell^b(m) = d$. The proof that $\ell^t(m) = d$ is symmetric. □

The proof of ergodicity is now complete. We recall the full statement.

Theorem *The maps V (on $S(\mathcal{D})$), V^* (on $S^*(\mathcal{D})$),V_+ and V_+^* (on $\Delta(\mathcal{D})$) are ergodic. The restriction of the Teichmüller flow to any component of the marked moduli space $\widetilde{\mathcal{M}}^{(1)}(M, \Sigma, \kappa)$ is ergodic. The action of $SL(2,\mathbb{R})$ on any such component is therefore also ergodic.*

11 Lyapunov exponents

The remaining sections are planned as introductions to further reading. The results are presented mostly without proofs. In this section, we introduce the Kontsevich-Zorich cocycle [Kon] and present the results of Forni [For2] and Avila-Viana [AvVi1].

11.1 Oseledets multiplicative ergodic theorem

Let $(X, \mathcal{B}, \mu)$ be a probability space, and let $T : X \to X$ be a measure-preserving **ergodic** transformation. Let also

$$A : X \longrightarrow GL(d, \mathbb{R})$$

be a measurable function. We assume that both $\log ||A||$ and $\log ||A^{-1}||$ are integrable. These data allow to define a linear cocycle

$$X \times \mathbb{R}^d \longrightarrow X \times \mathbb{R}^d$$

$$(x, v) \longmapsto (Tx, A(x)v).$$

Iterating this map leads to consider, for $n \geqslant 0$, the matrices

$$A^{(n)}(x) := A(T^{n-1}x) \cdots A(x).$$

When T is invertible, one can also consider, for $n < 0$

$$A^{(n)}(x) := (A^{(-n)}(T^n x))^{-1} = (A(T^n x)^{-1} \cdots (A(T^{-1}x))^{-1}.$$

To state Oseledets multiplicative theorem, we distinguish the case where T is invertible, which allows a stronger conclusion, from the general case.

Theorem (Oseledets [Os])

1. **The invertible case** *There exist numbers $\lambda_1 > \cdots > \lambda_r$ (the* **Lyapunov exponents***) and, at almost every point $x \in X$, a decomposition*

$$\mathbb{R}^d = F_1(x) \oplus \cdots \oplus F_r(x)$$

depending measurably on x, which is invariant under the action of the cocycle

$$A(x)F_i(x) = F_i(Tx)$$

and such that, for $1 \leqslant i \leqslant r$, $v \in F_i(x)$, $v \neq 0$, one has

$$\lim_{n \to \pm\infty} \frac{1}{n} \log ||A^{(n)}(x)v|| \ = \ \lambda_i.$$

2. **The general case** *There exist numbers $\lambda_1 > \cdots > \lambda_r$ and, at almost every point $x \in X$, a filtration*

$$\mathbb{R}^d = E_0(x) \supset E_1(x) \supset \cdots \supset E_r(x) = \{0\}$$

depending measurably on x, which is invariant under the action of the cocycle

$$A(x)E_i(x) = E_i(Tx)$$

and such that, for $v \in E_{i-1}(x) - E_i(x)$, one has

$$\lim_{n \to +\infty} \frac{1}{n} \log ||A^{(n)}(x)v|| \ = \ \lambda_i.$$

Remarks

1. In the invertible case, one obtains the second statement from the first by setting

$$E_i(x) = \oplus_{i+1}^r F_j(x).$$

2. When A is independent of x, the Lyapunov exponents are the logarithms of the moduli of the eigenvalues of A and the F_i are the sums of the corresponding generalized eigenspaces.

3. The statements above require obvious modifications for continuous time, i.e for flows and semiflows.

11.2 The Kontsevich-Zorich cocycle (discrete version)

Let $\mathcal{R}$ be a Rauzy class, $\mathcal{D}$ the associated Rauzy diagram.

We have defined in subsection 9.1 the map V_+ on the space $\Delta(\mathcal{D})$ which is the dynamics in parameter space defined by the Rauzy-Veech algorithm. There is a partition mod.0

$$\Delta(\mathcal{D}) = \bigcup_{\gamma} \{\pi\} \times \mathbb{P}(\Delta_\gamma)$$

over arrows $\gamma : \pi \to \pi'$ of $\mathcal{D}$, such that on $\{\pi\} \times \mathbb{P}(\Delta_\gamma)$, V_+ is given by

$$V_+(\pi,\lambda) = (\pi', \lambda\, B_\gamma^{-1})\ .$$

The **(extended) Kontsevich-Zorich cocycle** is the linear cocycle $V_{+,KZ} : \Delta(\mathcal{D}) \times \mathbb{R}^{\mathcal{A}} \to \Delta(\mathcal{D}) \times \mathbb{R}^{\mathcal{A}}$ over V_+ defined on $\{\pi\} \times \mathbb{P}(\Delta_\gamma) \times \mathbb{R}^{\mathcal{A}}$ by

$$V_{+,KZ}(\pi,\lambda,w) = (V_+(\pi,\lambda), B_\gamma\, w).$$

Over the accelerated Zorich dynamics V_+^* on $\Delta(\mathcal{D})$, we similarly define

$$V_{+,KZ}^*(\pi,\lambda,w) = (V_+^*(\pi,\lambda), B_{\underline{\gamma}}\, w),$$

where $\underline{\gamma}$ is the path in $\mathcal{D}$ (formed of arrows of the same type, having the same winner) associated to a single iteration of V^* at the point (π,λ) under consideration.

The extended Kontsevich-Zorich cocycle has a natural interpretation in terms of Birkhoff sums. Let T be an i.e.m with combinatorial data π, length data λ, acting on an interval I. Assume that T has no connection. Let T_n (with combinatorial data $\pi^{(n)}$, length data $\lambda^{(n)}$, acting on an interval $I^{(n)} \subset I$) be the i.e.m obtained from T after n steps of the Rauzy-Veech algorithm.

For any function φ on I, one can associate a new function $S^{(n)}\varphi$ on $I^{(n)}$ by

$$S^{(n)}\varphi(x) = \sum_{0 \leqslant i < r(x)} \varphi(T^i(x)),$$

where $r(x)$ is the return time in $I^{(n)}$ of $x \in I^{(n)}$.

Let $w \in \mathbb{R}^{\mathcal{A}}$. Consider w as the function on I which takes on I_α^t the constant value w_α. Then it is easy to see that the function $S^{(n)}w$ is constant on each interval $I_\alpha^{(n),t} \subset I^{(n)}$ and thus can also be considered as a vector in $\mathbb{R}^{\mathcal{A}}$. It follows from the properties of the matrices $B_{\gamma(m,n)}$ mentioned at the end of section 7.5 that one has

$$V_{+,KZ}^n(\pi,\lambda,w) = (\pi^{(n)}, \lambda^{(n)}, S^{(n)}w).$$

As was mentioned in subsection 7.6, for any arrow $\gamma : \pi \to \pi'$, the image of Im Ω_π under B_γ is equal to Im $\Omega_{\pi'}$. One obtains the **restricted** Kontsevich-Zorich cocycle by allowing only, in the definition of $V_{+,KZ}$ or $V_{+,KZ}^*$, the vector w to vary in Im Ω_π.

When necessary, the Kontsevich-Zorich cocycle (in its extended or restricted version) can also be viewed as a linear cocycle over V or V^*. This is important when one wants to use the Oseledets theorem for invertible maps.

11.3 The Kontsevich-Zorich cocycle (continuous version)

The continuous version of the Kontsevich-Zorich cocycle is defined over the Teichmüller flow $(\mathcal{T}_t)_{t\in\mathbb{R}}$ (on the moduli space $\mathcal{M}(M,\Sigma,\kappa)$, or the marked moduli space $\widetilde{\mathcal{M}}(M,\Sigma,\kappa)$) in the following way.

Consider for instance the case of the marked moduli space. Recall that we denote by $\widetilde{\mathcal{Q}}(M,\Sigma,\kappa)$ the associated marked Teichmüller space. On the product $\widetilde{\mathcal{Q}}(M,\Sigma,\kappa)\times H^1(M-\Sigma,\mathbb{R})$, we define a linear cocycle over the Teichmüller flow on $\widetilde{\mathcal{Q}}(M,\Sigma,\kappa)$ by

$$\mathcal{T}_t^{KZ}(\zeta,\theta) = (\mathcal{T}_t(\zeta),\theta).$$

The modular group $\mathrm{Mod}(M,\Sigma)$ acts in a non trivial canonical way on both factors of the product $\widetilde{\mathcal{Q}}(M,\Sigma,\kappa)\times H^1(M-\Sigma,\mathbb{R})$. The quotient is a vector bundle over the marked moduli space $\widetilde{\mathcal{M}}(M,\Sigma,\kappa)$, equipped with a flow fibered over the Teichmüller flow: this flow is the continuous version of the extended Kontsevich-Zorich cocycle. One gets the restricted version by restricting the fiber to the subspace $H^1(M,\mathbb{R})\subset H^1(M-\Sigma,\mathbb{R})$.

Let us explicit the relation between the discrete and continuous version of the KZ-cocycle.

Let (π,λ,τ) be an element of $S(\mathcal{D})$, viewed both as (cf. subsection 9.1) the domain of the natural extension of the Rauzy-Veech dynamics and as (cf. subsection 9.5) a transverse section to the Teichmüller flow in $\mathcal{M}^{(1)}(\mathcal{D})$. Let $w\in\mathbb{R}^{\mathcal{A}}$. Let (M,Σ,κ,ζ) be the translation surface obtained from (π,λ,τ) by the zippered rectangle construction. As seen in subsection 4.5, this construction provides us with a canonical basis $(\zeta_\alpha)_{\alpha\in}$ of the homology group $H_1(M,\Sigma,\mathbb{Z})$. We associate to w the homology class $\zeta_w=\sum_\alpha w_\alpha\zeta_\alpha\in H_1(M,\Sigma,\mathbb{R})$, which can also be viewed as a cohomology class in $H^1(M-\Sigma,\mathbb{R})$ from the duality provided by the intersection form.

We assume that (M,Σ,κ,ζ) has no vertical connection. From (π,λ,τ) viewed as a point in $\mathcal{M}^{(1)}(\mathcal{D})\subset\widetilde{\mathcal{M}}(M,\Sigma,\kappa)$, we flow with the Teichmüller flow during a time t to a point $(\pi',\lambda',\tau')\in\mathcal{M}^{(1)}(\mathcal{D})$. The continuous Teichmüller trajectory corresponds to a path $\underline{\gamma}$ from π to π' in $\mathcal{D}$. As seen in subsection 7.4, the translation surface (M,Σ,κ,ζ) is canonically isomorphic to the translation surface constructed from the data $(\pi',e^{-t}\lambda',e^t\tau')$. This isomorphism and the combinatorial data π' provides another basis $(\zeta'_\alpha)_{\alpha\in}$ for $H_1(M,\Sigma,\mathbb{Z})$ (or $H^1(M-\Sigma,\mathbb{Z})$). We express ζ_w as $\zeta_w=\sum_\alpha w'_\alpha\zeta'_\alpha$. Then, we have

$$w' = B_{\underline{\gamma}}w.$$

The two versions of the KZ-cocycle are thus seen to be equivalent.

11.4 Lyapunov spectrum of the Kontsevich-Zorich cocycle

We start with some simple observations which follow from subsections 7.6, 9.7 and 10.3.

It follows from the proposition in subsection 7.6 that one can choose, for each vertex π of $\mathcal{D}$, a basis for the quotient space $\mathbb{R}^{\mathcal{A}}/\mathrm{Im}\,\Omega_\pi$, in such a way that, for every arrow $\gamma:\pi\to\pi'$, the homomorphism from $\mathbb{R}^{\mathcal{A}}/\mathrm{Im}\,\Omega_\pi$ to $\mathbb{R}^{\mathcal{A}}/\mathrm{Im}\,\Omega_{\pi'}$ induced by B_γ corresponds to the identity matrix in the selected bases.

As a consequence, vectors in these quotient spaces stay bounded under the action of the KZ-cocycle. It follows that 0 is the unique Lyapunov exponent associated with this part of the KZ-cocycle. The multiplicity of this exponent is $s-1=d-2g$.

By the Masur-Veech theorem stated in subsection 6.11 and proved in subsections 9.7 and 10.3, the canonical measures on $\mathcal{M}(M,\Sigma,\kappa)$ and $\widetilde{\mathcal{M}}(M,\Sigma,\kappa)$ have finite total masses, and the Teichmüller flow is ergodic with respect to these invariant measures. As seen in Subsection 9.7 and first proved by Zorich, the canonical invariant measure on $\widetilde{\mathcal{M}}(M,\Sigma,\kappa)$ induces on $S^*(\mathcal{D})$ a finite measure which is equivalent to Lebesgue measure and invariant under V^*. This measure can be projected to $\Delta(\mathcal{D})$ to obtain a finite measure, equivalent to Lebesgue measure, which is invariant under V_+^*.

We can thus apply the Oseledets theorem to the restricted KZ-cocycle, either in the continuous version over the Teichmüller flow or in the discrete version over V^* or V_+^*.

However, one has first to check the integrability condition of subsection 10.1. We do that for the discrete version of the cocycle. From the definition of the Zorich acceleration V_+^* of the Rauzy-Veech dynamics, the norm of the matrix $B_{\underline{\gamma}}$ defining the KZ-cocycle at a point (π,λ) is bounded by

$$||B_{\underline{\gamma}}|| \leqslant C\, \frac{\sum_\alpha \lambda_\alpha}{\min_\alpha \lambda_\alpha}.$$

The same estimate holds for the inverse of this matrix. But the proposition at the end of subsection 9.7 states that the majorant in the inequality above is larger than A on a set of measure at most $A^{-1}(\log A)^{d-2}$, which easily implies the required integrability.

Observe that the same computation shows that the return time for the Teichmüller flow on $S^*(\mathcal{D})$ is integrable. By Birkhoff's ergodic theorem, the mean value θ_1^* over $S^*(\mathcal{D})$ of this return time has the following property: for almost any point in $\zeta \in \widetilde{\mathcal{M}}(M,\Sigma,\kappa)$, we have

$$\lim_{T\to+\infty} \frac{1}{T}\,\#\{t\in[0,T];\ \mathcal{T}_t(\zeta)\in S^*(\mathcal{D})\} = \frac{1}{\theta_1^*}.$$

As a consequence, the Lyapunov exponents for the discrete KZ-cocycle over V^* or V_+^* are proportional by a factor θ_1^* to those of the continuous KZ-cocycle over $\mathcal{T}$.

Exercise: Show that the largest Lyapunov exponent of the continuous KZ-cocycle over $\mathcal{T}$ is equal to 1, and that the largest Lyapunov exponent of the discrete KZ-cocycle over V^* or V_+^* is equal to θ_1^*.

Exercise: Use the ergodicity of V_+^* to show that the largest Lyapunov exponent of the KZ-cocycle is simple.

Let $\gamma : \pi \to \pi'$ be an arrow of $\mathcal{D}$. We have also seen in subsection 7.6 that, when we equip $\operatorname{Im}\Omega_\pi$ and $\operatorname{Im}\Omega_{\pi'}$ with the symplectic structures defined by Ω_π, $\Omega_{\pi'}$ respectively, the restriction of B_γ to $\operatorname{Im}\Omega_\pi$ is symplectic. This implies that the Lyapunov spectrum (i.e the Lyapunov exponents, counted with multiplicities) of the restricted KZ-cocycle is symmetric with respect to 0: counted with multiplicities the Lyapunov exponents of the continuous restricted KZ-cocycle have the form

$$1=\theta_1 > \theta_2 \geqslant \dots \theta_g \geqslant \theta_{g+1} = -\theta_g \geqslant \dots \geqslant \theta_{2g-1} = -\theta_2 > \theta_{2g} = -1,$$

the Lyapunov exponents for the discrete restricted KZ-cocycle over V^* or V^*_+ being the $\theta^*_i := \theta^*_1\, \theta_i$.

Kontsevich and Zorich conjectured that all Lyapunov exponents of the restricted KZ-cocycle are simple. In particular, this stipulates that $\theta_g > \theta_{g+1} = -\theta_g$, hence that the restricted KZ-cocycle is *hyperbolic* in the sense that it does not have 0 as Lyapunov exponent. Forni then proved the hyperbolicity of the restricted KZ-cocycle before Avila and Viana proved the full conjecture of Kontsevich and Zorich.

Theorem (Forni [For2],[Kri]) *The restricted Kontsevich-Zorich cocycle is hyperbolic.*

The (Lyapunov) hyperbolicity of the KZ-cocycle holds w.r.t the invariant measure equivalent to Lebesgue measure, but not to any invariant measure.

Exercise : In the Rauzy diagram with $g = 2, d = 4$, find a **complete** loop γ such that B_γ has two eigenvalues of modulus 1.

Observe that when $g = 2$, Forni's theorem already implies that the Lyapunov spectrum of the KZ-cocycle is simple. For higher genus, we have

Theorem (Avila-Viana [AvVi1],[AvVi2]) *The Lyapunov spectrum of the restricted Kontsevich-Zorich cocycle is simple.*

The proofs of both theorems (Avila-Viana's approach is quite different from Forni's) are beyond the scope of these notes.

The Lyapunov exponents of the restricted discrete KZ-cocycle over V^* and V^*_+ are the same. The conclusions of the Oseledets theorem are however slightly different.

- For almost every $(\pi, \lambda, \tau) \in S^*(\mathcal{D})$, there exists a direct sum decomposition into 1-dimensional subspaces
$$\operatorname{Im} \Omega_\pi = \oplus_1^{2g} F_i(\pi, \lambda, \tau),$$
such that, for $w \in F_i(\pi, \lambda, \tau), w \neq 0$, we have, writing $(V^*_{KZ})^n(\pi, \lambda, \tau, w) = ((V^*)^n(\pi, \lambda, \tau), w_n)$
$$\lim_{n\to\pm\infty} \frac{1}{n} \log \frac{||w_n||}{||w||} = \theta^*_i.$$

- For almost every $(\pi, \lambda) \in \Delta(\mathcal{D})$, there exists a filtration
$$\operatorname{Im} \Omega_\pi = E_0(\pi, \lambda) \supset E_1(\pi, \lambda) \supset \ldots \supset E_{2g}(\pi, \lambda) = \{0\},$$
with $\operatorname{codim} E_i(\pi, \lambda) = i$, such that, for $w \in E_{i-1}(\pi, \lambda) - E_i(\pi, \lambda)$, writing $(V^*_{+,KZ})^n(\pi, \lambda, w) = ((V^*_+)^n(\pi, \lambda), w_n)$, we have
$$\lim_{n\to+\infty} \frac{1}{n} \log \frac{||w_n||}{||w||} = \theta^*_i.$$

For almost every $(\pi, \lambda, \tau) \in S^*(\mathcal{D})$, and every $0 \leqslant i < 2g$, the direct sum $\oplus_{i+1}^{2g} F_i(\pi, \lambda, \tau)$ is independent of τ and equal to $E_i(\pi, \lambda)$. Symmetrically, for almost every $(\pi, \lambda, \tau) \in S^*(\mathcal{D})$, and every $0 < i \leqslant 2g$, the direct sum $\oplus_1^i F_i(\pi, \lambda, \tau)$ is independent of λ.

When one considers (assuming $s > 1$) the **extended** KZ-cocycle over V^* or V_+^*, one obtains moreover

- In the invertible case, a subspace $F_*(\pi, \lambda, \tau)$, which complements $\operatorname{Im} \Omega_\pi$ and has dimension $s - 1$, associated to the exponent 0;
- In the non invertible case, the subspaces associated to the positive exponents are the

$$E_i^*(\pi, \lambda) := E_i(\pi, \lambda) \oplus F_*(\pi, \lambda, \tau), \qquad\qquad \forall\, 0 \leqslant i \leqslant g,$$

  which satisfy $\operatorname{codim} E_i^*(\pi, \lambda) = i$. The subspace associated to the exponent 0 is

$$E^*(\pi, \lambda) := F_*(\pi, \lambda, \tau) \oplus E_{g+1}(\pi, \lambda),$$

  and those associated with the negative exponents θ_i^*, $g < i \leqslant 2g$ are the $E_i(\pi, \lambda)$.

11.5 Lyapunov exponents of the Teichmüller flow

Recall that $S(\mathcal{D})$ was identified in subsection 9.5 with the transverse section to the Teichmüller flow in $\mathcal{M}^{(1)}(\mathcal{D})$

$$\{(\pi, \lambda, \tau) \in \bigsqcup_\pi \{\pi\} \times \Delta \times \Theta_\pi;\ \sum_\alpha \lambda_\alpha \ =\ 1,\ \ \tau\, \Omega_\pi\, {}^t\lambda \ =\ 1\},$$

the return map being given by the Rauzy-Veech invertible dynamics V. Thus, a number of iterations of V, associated to a path $\underline{\gamma} : \pi \longrightarrow \pi'$ in $\mathcal{D}$, correspond to the Teichmüller time

$$\log \frac{||\lambda||_1}{||\,\lambda\, B_{\underline{\gamma}}^{-1}||_1} = -\log ||\,\lambda\, B_{\underline{\gamma}}^{-1}||_1$$

and to the return map

$$(\pi, \lambda, \tau) \mapsto (\pi', \frac{\lambda\, B_{\underline{\gamma}}^{-1}}{||\,\lambda\, B_{\underline{\gamma}}^{-1}||_1}, ||\,\lambda\, B_{\underline{\gamma}}^{-1}||_1\, \tau\, B_{\underline{\gamma}}^{-1}).$$

From subsection 7.6, we know that the action of $B_{\underline{\gamma}}^{-1}$ on row vectors in $\operatorname{Ker} \Omega_\pi$ is neutral and the action on the quotient $\mathbb{R}^{\mathcal{A}} / \operatorname{Ker} \Omega_\pi \simeq \operatorname{Im} \Omega_\pi$ is given by $B_{\underline{\gamma}}$. From this we deduce immediately the Lyapunov exponents of the Teichmüller flow on $\mathcal{M}^{(1)}(\mathcal{D})$ (with respect to the canonical invariant measure)

- There are, counted with multiplicities, $d-1=(2g-1)+(s-1)$ positive Lyapunov exponents which are the simple exponents

$$2 = 1+\theta_1 > 1+\theta_2 > \ldots > 1+\theta_{2g-1}$$

  and, when $s>1$, the exponent 1 (between $1+\theta_g$ and $1+\theta_{g+1}$) with multiplicity $s-1$.

- There are symmetrically $d-1=(2g-1)+(s-1)$ negative Lyapunov exponents which are the simple exponents

$$-1+\theta_2 > \ldots > -1+\theta_{2g-1} > -1+\theta_{2g} = -2$$

  and, when $s>1$, the exponent -1 with multiplicity $s-1$.

- Finally, the exponent $0 = 1+\theta_{2g} = -1+\theta_1$ was killed by the normalization conditions on λ and τ, but is still present with multiplicity 1 in the direction of the flow.

- When considering the flow in $\mathcal{M}(\mathcal{D})$, the exponent 0 has multiplicity 2 because the foliation by the levels of the area map A is invariant.

- The strong local stable manifold of a point $(\pi,\lambda_0,\tau_0) \in \mathcal{M}(\mathcal{D})$ has equation $\{\lambda=\lambda_0,\ (\tau-\tau_0)\ \Omega_\pi\ {}^t\lambda_0 = 0\}$. Similarly, the strong local unstable manifold has equation $\{\tau=\tau_0,\ \tau_0\ \Omega_\pi\ {}^t(\lambda-\lambda_0)=0\}$.

11.6 Deviation of ergodic averages

Let T be an i.e.m with combinatorial data $(\mathcal{A},\pi)$ and domain $\sqcup I^t_\alpha$. Given a point x_0, a letter $\alpha\in\mathcal{A}$ and an integer k, denote the number of visits to I^t_α of the orbit of x_0 up to time k by

$$\chi_\alpha(k) := \#\{i\in[0,k)\,;\ T^i(x_0)\in I^t_\alpha\}\ .$$

How do these numbers behave as k goes to $+\infty$? This was one of the questions that led Kontsevich and Zorich to introduce their cocycle.

A first answer is provided by Birkhoff's theorem: by the theorem of Masur and Veech, for almost all length data λ, T is ergodic w.r.t Lebesgue measure. Therefore, for such a T, one has , for all $\alpha\in\mathcal{A}$ and almost all x_0

$$\lim_{k\to+\infty}\frac{1}{k}\chi_\alpha(k) = |I^t_\alpha| = \lambda_\alpha\ .$$

A slightly better answer is obtained by using that, by the same theorem of Masur and Veech, almost all T are actually uniquely ergodic. Indeed, if f is a uniquely ergodic minimal homeomorphism of a compact metric space X and φ is a continuous function on X, the convergence of the Birkhoff sums of φ holds for **any** initial value x_0 and is uniform in x_0.

Here, T is not a homeomorphism and the characteristic function of I^t_α is not continuous but this is not a problem in reason of the following trick.

Split any point in the forward orbit of the singularities of T^{-1} and the backward orbit of the singularities of T into its left and right limit. One obtains, equipped with the order topology, a compact metric space $\widehat{I}$. The i.e.m T induces on $\widehat{I}$ a homeomorphism $\widehat{T}$ which is easily seen to be uniquely ergodic when T is. Also, the interval I^t_α corresponds to a clopen set in $\widehat{I}$ so its characteristic function is continuous.

A much more precise answer on the speed of convergence of the $\frac{1}{k}\chi_\alpha(k)$ is obtained using the KZ-cocycle.

Assume that T has no connection. Let $(I^{(n)})_{n\geqslant 0}$ be the intervals of induction for the Rauzy-Veech algorithm, $(T_n)_{n\geqslant 0}$ the corresponding i.e.m, $w \in \mathbb{R}^{\mathcal{A}}$. Viewing w as the function on $\sqcup I^t_\alpha$ with constant value w_α on I^t_α, the Birkhoff sums of w are given by

$$S_k w(x_0) = \sum_\alpha w_\alpha \chi_\alpha(k) \ .$$

On the other hand, we have seen in subsection 11.2 that the KZ-cocycle is directly related to the Birkhoff sums $S^{(n)}w$ of w corresponding to the return to $I^{(n)}$.

In order to relate $S_k w(x_0)$ to the $S^{(n)}w$, we introduce the point x^* of the orbit $\{T^j(x_0);\ 0 \leqslant j \leqslant k\}$ which is closest to the left endpoint of I. We consider separately in $S_k w(x_0)$ the part of the sum which is before x^* and the part which is after x^*. Thus, we have just to consider Birkhoff sums $S_j w(x^*)$ (with $j \in \mathbb{Z}$).

Consider such a sum $S_j w(x^*)$, with for instance $j \geqslant 0$ (the case $j \leqslant 0$ is similar). In the orbit $\{T^\ell(x^*);\ 0 \leqslant \ell \leqslant j\}$, there exists a unique subsequence $(x^*_s)_{0\leqslant s\leqslant r} = (T^{j_s}(x^*))_{0\leqslant s\leqslant r}$, and a sequence $(n_s)_{0\leqslant s\leqslant r}$ with the following properties:

- $0 = j_0 < j_1 < \ldots < j_r = j$;
- $0 \leqslant n_r \leqslant \ldots \leqslant n_0$;
- the point x^*_s belongs to $I^{(n_s)}$ for $0 \leqslant s \leqslant r$;
- the point x^*_s does not belong to $I^{(n_s+1)}$ for $1 \leqslant s \leqslant r$;
- the point $T^\ell(x^*)$ does not belong to $I^{(n_s)}$ for $1 \leqslant s \leqslant r$, $j_{s-1} < \ell < j_s$;

This means that the sum $\sum_{j_{s-1}}^{j_s-1} w(T^\ell(x^*))$ corresponds to a first return in $I^{(n_s)}$. Writing α_s for the letter such that $x^*_{s-1} \in I^{t,(n_s)}_{\alpha_s}$, we have

$$S_j w(x^*) = \sum_1^r (S^{(n_s)}w)_{\alpha_s} \ .$$

As the return time of T_n in $I^{(n+1)}$ is 1 or 2, we have actually $n_0 > n_1 > \ldots > n_r$. On the other hand, assume that the data (π, λ) of T are typical for Oseledets theorem applied to the KZ-cocycle; when $w \in E_i^*(\pi, \lambda)$ for some $0 \leqslant i < g$ (resp. $w \in E_g^*(\pi, \lambda)$), one has

$$\limsup \frac{\log ||S^{(n)} w||}{\log ||S^{(n)} 1||} = \theta_{i+1},$$

$$(\text{resp.} \limsup \frac{\log ||S^{(n)} w||}{\log ||S^{(n)} 1||} = 0. \,)$$

From this, one obtains the following result

Theorem [Zo3] *For almost every i.e.m. $T = T_{\pi,\lambda}$, and all $x \in I$, one has*

$$\limsup \frac{\log |S_k w(x)|}{\log k} \leqslant \theta_{i+1}$$

if $w \in E_i^*(\pi, \lambda)$ for some $0 \leqslant i < g$ and

$$\limsup \frac{\log |S_k w(x)|}{\log k} = 0$$

if $w \in E_g^*(\pi, \lambda)$.

There is a similar interpretation of the KZ-cocycle in terms of the way that the orbits of the vertical flow of a typical translation surface wind around the surface: see [Zo1],[Zo4].

12 The cohomological equation

We present in this section the main result of [MmMsY]. Let $f : X \to X$ be a map. The *cohomological equation* associated to this dynamical system is

$$\psi \circ f - \psi = \varphi,$$

where φ is a given function on X (usually assumed to have some degree of smoothness), and ψ is an unknown function on X (generally required to have another degree of smoothness).

12.1 Irrational numbers of Roth type

Definition An irrational number α is of *Roth type* if, for every $\varepsilon > 0$, there exists $C = C_\varepsilon > 0$ such that, for every rational $\frac{p}{q}$, one has

$$|\alpha - \frac{p}{q}| \geqslant \frac{C}{q^{2+\varepsilon}}.$$

The reason for the terminology is the celebrated result

Theorem (Roth) *Every irrational algebraic number is of Roth type.*

Let $\alpha = [a_0; a_1, \ldots]$ be the continuous fraction decomposition of the irrational number α, and let $(\frac{p_n}{q_n})$ be the associated convergents of α. Then α is of Roth type iff $q_{n+1} = \mathcal{O}(q_n^{1+\varepsilon})$ for all $\varepsilon > 0$; this can be reformulated as $a_{n+1} = \mathcal{O}(q_n^{\varepsilon})$ for all $\varepsilon > 0$.

The set of irrational numbers of Roth type has full Lebesgue measure: indeed, for every $q \geqslant 1,\ C > 0$, the set of $\alpha \in (0,1)$ such that

$$|\alpha - \frac{p}{q}| < \frac{C}{q^{2+\varepsilon}}$$

for some $p \in \mathbb{Z}$ has measure $\leqslant 2Cq^{-1-\varepsilon}$ and the series $\sum_{q \geqslant 1} q^{-1-\varepsilon}$ is convergent.

Standard methods of harmonic analysis allow to prove the following fundamental result, where R_α denotes the rotation $x \mapsto x + \alpha$ on $\mathbb{T}$.

Theorem *Let α be an irrational number of Roth type and let r, s be nonnegative real numbers with $r - s > 1$. For every function $\varphi \in C^r(\mathbb{T})$ of mean value 0, there exists a unique function $\psi \in C^s(\mathbb{T})$ of mean value 0 such that*

$$\psi \circ R_\alpha - \psi = \varphi.$$

12.2 Interval exchange maps of Roth type

Let T be an interval exchange map, $(\mathcal{A}, \pi)$ its combinatorial data; denote by $\mathcal{R}$ the Rauzy class of π and by $\mathcal{D}$ the associated Rauzy diagram.

We assume that T has no connection. The Rauzy-Veech algorithm applied to T produces an infinite path γ in $\mathcal{D}$ starting from π. From Proposition 1 in subsection 7.7, the path γ is ∞-complete. We can therefore write in a unique way γ as a concatenation

$$\gamma = \gamma_1 * \gamma_2 * \ldots * \gamma_n * \ldots$$

where each γ_i is complete but no strict initial subpath of γ_i is complete. We write $\gamma(n)$ for the initial part

$$\gamma(n) = \gamma_1 * \gamma_2 * \ldots * \gamma_n$$

of γ.

We say that T is an i.e.m *of Roth type* if it satisfies the three conditions (a), (b), (c) below.

(a) *For every $\varepsilon > 0$, there exists $C = C_\varepsilon$ such that, for all $n > 0$, one has*

$$||B_{\gamma_n}|| \leqslant C\, ||B_{\gamma(n-1)}||^{\varepsilon}.$$

Exercise Let $x \mapsto x+\alpha$ be an irrational rotation on $\mathbb{T}$ and let T be the associated i.e.m with two intervals. Show that α is of Roth type iff T satisfies condition (a).

Let $\lambda \in \mathbb{R}^{\mathcal{A}}$ be the length data of T and let $E_1 = \{\sum_\alpha \lambda_\alpha w_\alpha = 0\}$; this hyperplane of $\mathbb{R}^{\mathcal{A}}$ should be viewed as the space of functions w, constant on each I^t_α, of mean value 0.

(b) *There exists $\theta > 0$, $C > 0$, such that, for all $n > 0$, one has*

$$||B_{\gamma(n)}|_{E_1}|| \leqslant C\,||B_{\gamma(n)}||^{1-\theta}.$$

Exercise Show that condition (b) is always satisfied when $d = 2$.

Exercise Show that condition (b) implies that T is uniquely ergodic.

Exercise Assume that T satisfies the following reinforcement of condition (a): there exists $C > 0$ such that $||B_{\gamma_n}|| < C$ for all $n > 0$. Show that this imply that T satisfies condition (b).

Exercise Show that the condition of the last exercise is satisfied iff the orbit of (π, λ) under V is relatively compact in $\Delta(\mathcal{D})$.

In order to state part (c) of the definition of Roth type i.e.m, we define, for $\ell \geqslant k$

$$\gamma(k,\ell) = \gamma_{k+1} * \ldots * \gamma_\ell$$

and introduce, for $k \geqslant 0$

$$E^s(k) := \{w \in \mathbb{R}^{\mathcal{A}}\,;\ \limsup_{\ell\to+\infty} \frac{\log||B_{\gamma(k,\ell)}w||}{\log||B_{\gamma(k,\ell)}||} < 0\}.$$

Observe that $E^s(k)$ is a vector subspace of $\mathbb{R}^{\mathcal{A}}$ which is sent by $B_{\gamma(k,\ell)}$ onto $E^s(\ell)$. Denote by $B^\flat_{k,\ell}$ the restriction of $B_{\gamma(k,\ell)}$ to $E^s(k)$ and by $B^\sharp_{k,\ell}$ the map from $\mathbb{R}^{\mathcal{A}}/E^s(k)$ to $\mathbb{R}^{\mathcal{A}}/E^s(\ell)$ induced by $B_{\gamma(k,\ell)}$.

(c) *For every $\varepsilon > 0$, there exists $C = C_\varepsilon$ such that, for all $\ell \geqslant k$, we have*

$$\begin{aligned} ||B^\flat_{k,\ell}|| &\leqslant C\,||B_{\gamma(\ell)}||^\varepsilon, \\ ||(B^\sharp_{k,\ell})^{-1}|| &\leqslant C\,||B_{\gamma(\ell)}||^\varepsilon. \end{aligned}$$

Assume that μ is a probability measure which is invariant under the dynamics V generated by the Rauzy-Veech algorithm or the accelerated version V^*. Assume also that the integrability condition of Oseledets's theorem is satisfied by the Kontsevich-Zorich cocycle w.r.t μ. For instance, μ could be the canonical V^*-invariant measure absolutely continuous w.r.t Lebesgue, or could be supported by a periodic orbit of V (or more generally a compact V-invariant subset of $\Delta(\mathcal{D})$).

Then, property (c) is satisfied by μ-almost all T. The spaces $E^s(k)$ are the stable subspaces associated to the negative Lyapunov exponents (relative to μ) and the estimates in (c) follow from the conclusions of Oseledets's theorem.

Property (b) is also satisfied by μ-almost all T. Indeed, the largest Lyapunov exponent for μ is simple, with associated hyperplane equal to E_1 (the simplicity of the largest exponent for μ is proven from the positivity of the matrices B as in subsection 10.2).

Regarding property (a), no general statement w.r.t any invariant probability μ as above is known. On the other hand, with respect to the canonical V^*-invariant measure absolutely continuous w.r.t Lebesgue, (or equivalently w.r.t Lebesgue measure), almost all T satisfy property (a): this follows from a stronger statement that will be presented in section 14. We thus obtain

Proposition *For any combinatorial data $(\mathcal{A}, \pi)$, and Lebesgue almost any length vector λ, the i.e.m T constructed from these data is of Roth type.*

12.3 The cohomological equation for interval exchange maps

The first and decisive breakthrough concerning the cohomological equation for i.e.m of higher genus was obtained by Forni [For1]. He actually works with the (nonzero) constant vectorfields X on a translation surface $(M, \Sigma, \kappa, \zeta)$ for which the cohomological equation takes the form

$$X.\Psi = \Phi.$$

He defines from the flat metrics associated to the structure of translation surface a family $H^s(M)$ of Sobolev spaces and obtains the following result

Theorem (Forni[For1],[For3]) *Let $k \geqslant 0$ be an integer and r, s be real numbers satisfying $s - 3 > k > r$. For almost all constant unit vectorfields X on $(M, \Sigma, \kappa, \zeta)$, and all functions $\Phi \in H^s(M)$ satisfying $D.\Phi = 0$ for all $D \in \mathcal{I}^s_X$, there exists $\Psi \in H^r(M)$ such that $X.\Psi = \Phi$. Here, $\mathcal{I}^s_X$ is the finite-dimensional space of X-invariant distributions in $H^{-s}(M)$.*

A slight drawback of Forni's theorem is that no explicit description of the set of "good" directions for which it is possible to solve the cohomological equation is given. This is addressed by the next result.

Let T be an interval exchange map, $(\mathcal{A}, \pi)$ its combinatorial data, $\sqcup I^t_\alpha$ the domain of T. We denote by $\mathrm{BV}^1_*(\sqcup I^t_\alpha)$ the Banach space of functions φ on $\sqcup I^t_\alpha$ with the following properties

- the restriction of φ to each I^t_α is absolutely continuous and its derivative is a function of bounded variation;
- the mean value of the derivative $D\varphi$ over $\sqcup I^t_\alpha$ is 0.

Remark The first property implies that the limits $\varphi((u_i^t)^+)$ (for $0 \leqslant i < d$) and $\varphi((u_i^t)^-)$ (for $0 < i \leqslant d$) exist, where $u_0 = u_0^t$, $u_d = u_d^t$ are the endpoints of the domain of t and $u_1^t < \ldots < u_{d-1}^t$ are the singularities of T. Then the second condition is

$$\sum_1^{d-1}(\varphi((u_i^t)^+) - \varphi((u_i^t)^-)) + \varphi(u_0^+) - \varphi(u_d^-) = 0.$$

Theorem [MmMsY] *Assume that T has no connection and is of Roth type. Then, for every function $\varphi \in \mathrm{BV}_*^1(\sqcup I_\alpha^t)$, there exists a bounded function ψ on $\sqcup I_\alpha^t$ and a function χ which is constant on each I_α^t such that*

$$\psi \circ T - \psi = \varphi - \chi \quad .$$

Remark The solution (ψ, χ) of the equation is unique if one restricts ψ, χ to smaller subspaces. More precisely, let E_T be the subspace of $\mathbb{R}^{\mathcal{A}}$ formed of the functions χ, constant on each I_α^t, which can be written as $\psi \circ T - \psi$ for some bounded function ψ; let E_T^* be a complementary subspace of E_T in $\mathbb{R}^{\mathcal{A}}$. Then, under the hypotheses of the theorem, one can find a unique pair (ψ, χ) satisfying moreover that ψ has mean value 0 and that $\chi \in E_T^*$. The quotient space $\mathbb{R}^{\mathcal{A}}/E_T$ can thus be seen as the obstruction to solve the cohomological equation for the smoothness data under consideration.

As the derivative of T is 1 on each I_α^t, differentiating the cohomological equation leads to the same equation for derivatives of φ, ψ, with only constants of integration to keep under control. A result on the cohomological equation in higher smoothness is therefore easily deduced from the basic result above.

For $r \geqslant 1$, let $\mathrm{BV}_*^r(\sqcup I_\alpha^t)$ be the space of functions φ on $\sqcup I_\alpha^t$ such that

- the restriction of φ to each I_α^t is of class C^{r-1}, $D^{r-1}\varphi$ is absolutely continuous on I_α^t and $D^r\varphi$ is a function of bounded variation;
- the mean value of the derivative $D^j\varphi$ over $\sqcup I_\alpha^t$ is 0 for every integer $0 < j \leqslant r$.

On the other hand, let I be the interval supporting the action of T. Denote for $r \geqslant 2$ by $C^{r-2+\mathrm{Lip}}(I)$ the space of functions ψ on I which are of class C^{r-2} **on all of** I and such that $D^{r-2}\psi$ is Lipschitz on I.

Finally, for $r \geqslant 1$, let $E(r)$ be the space of functions χ on $\sqcup I_\alpha^t$ such that

- the restriction of χ to each I_α^t is a polynomial of degree $< r$;
- the mean value of the derivative $D^j\chi$ over $\sqcup I_\alpha^t$ is 0 for every integer $0 < j < r$.

One has then

Theorem [MmMsY] *Assume that T has no connection and is of Roth type. Let r be an* *integer* $\geqslant 2$. *Then, for every function* $\varphi \in \mathrm{BV}_*^r(\sqcup I_\alpha^t)$, *there exists a function* $\psi \in C^{r-2+\mathrm{Lip}}(I)$ *and a function* $\chi \in E(r)$ *such that*

$$\psi \circ T - \psi = \varphi - \chi \quad .$$

12.4 Sketch of the proof

We give some indications about the steps of the proof of the theorem.

We want to use the following classical result.

Theorem (Gottschak-Hedlund) *Let f be a minimal homeomorphism of a compact metric space X, let x_0 be a point of X, and let φ be a continuous function on X. The following are equivalent:*

1. *The Birkhoff sums* $\sum_0^{n-1} \varphi \circ f^i(x_0)$ *are bounded.*
2. *There exists a continuous function ψ on X such that*

$$\psi \circ f - \psi = \varphi.$$

By splitting each point in the orbits of the singularities of T and T^{-1} into its left and right limit, one obtain a compact metric space $\widehat{I}$ on which T induces a minimal homeomorphism. Moreover, every continuous function $\widehat{\psi}$ on $\widehat{I}$ induces a bounded function on I. Therefore, in view of the theorem of Gottschalk-Hedlund, it is sufficient to find, for every $\varphi \in \mathrm{BV}_*^1(\sqcup I_\alpha^t)$, a function χ, constant on each I_α^t, such that the Birkhoff sums of $\varphi - \chi$ are bounded.

Let $\mathrm{BV}(\sqcup I_\alpha^t)$ be the Banach space of functions φ_1 of bounded variation on $\sqcup I_\alpha^t$, equipped with the norm

$$\begin{aligned} ||\varphi_1||_{BV} &:= \sup_{\sqcup I_\alpha^t} |\varphi_1(x)| \ + \ |\varphi_1|_{BV}, \\ |\varphi_1|_{BV} &:= \sum_\alpha \mathrm{Var}_{I_\alpha^t} \ \varphi_1 \ . \end{aligned}$$

Let $I^{(n)} = \sqcup I_\alpha^{t,(n)} \subset I$ be the interval of induction for the step of the Rauzy-Veech algorithm associated to the initial path $\gamma(n)$ of γ (notations of subsection 12.2). A simple but crucial observation, in the spirit of the Denjoy estimates for circle diffeomorphisms, is that, for $\varphi_1 \in \mathrm{BV}(\sqcup I_\alpha^t)$, the Birkhoff sum $S^{(n)}\varphi_1$ corresponding to returns in $I^{(n)}$ (see subsection 11.2) satisfy $S^{(n)}\varphi_1 \in \mathrm{BV}(\sqcup I_\alpha^{t,(n)})$ with

$$|S^{(n)}\varphi_1|_{BV} \leqslant |\varphi_1|_{BV} \ .$$

This estimate is the basic ingredient in the proof of the

Proposition *Assume that T has no connection and satisfy conditions (a) and (b) of subsections 12.2. For every function $\varphi_1 \in \mathrm{BV}(\sqcup I^t_\alpha)$ of mean value 0, and every $n \geqslant 0$, we have*

$$\sup_{\sqcup I_\alpha^{t,(n)}} |S^{(n)}\varphi_1(x)| \;\leqslant\; C\,||B_{\gamma(n)}||^{1-\frac{\theta}{2d}}||\varphi_1||_{BV},$$

where C depends only on the constants in condition (a) and (b).

From condition (a), the lengths $|I_\alpha^{t,(n)}|$ satisfy

$$\lim_{n\to+\infty} \frac{\log |I_\alpha^{t,(n)}|}{\log ||B_{\gamma(n)}||} = -1 \,.$$

Therefore, for every $\varphi_1 \in \mathrm{BV}(\sqcup I^t_\alpha)$ of mean value 0, and every $n \geqslant 0$, there exists a primitive $\varphi_0 \in \mathrm{BV}^1_*(\sqcup I^t_\alpha)$ of φ_1 (one constant of integration being chosen for each I^t_α) such that

$$\sup_{\sqcup I_\alpha^{t,(n)}} |S^{(n)}\varphi_0(x)| \;\leqslant\; C\,||B_{\gamma(n)}||^{-\frac{\theta}{3d}}||\varphi_1||_{BV}.$$

Using condition (c) of subsection 12.2, one can change the order of the quantifiers to make the primitive φ_0 independent of n and still satisfy

$$\sup_{\sqcup I_\alpha^{t,(n)}} |S^{(n)}\varphi_0(x)| \;\leqslant\; C\,||B_{\gamma(n)}||^{-\omega}||\varphi_1||_{BV},$$

for some $\omega > 0$. But the last estimate, together with condition (a) of 12.2, easily imply that the Birkhoff sums of φ_0 are bounded. This proves the required result: starting from any $\varphi \in \mathrm{BV}^1_*(\sqcup I^t_\alpha)$, we take $\varphi_1 := D\varphi \in \mathrm{BV}(\sqcup I^t_\alpha)$; it has mean value 0 and therefore has a primitive φ_0 such that the Birkhoff sums of φ_0 are bounded. The difference $\varphi - \varphi_0$ is constant on every I^t_α.

13 Connected components of the moduli spaces

We present in this section the classification of the connected components of the moduli space $\mathcal{M}(M,\Sigma,\kappa)$ by Kontsevich and Zorich [KonZo]. The classification of the connected components of the marked moduli space is the same: it is easy to see that the canonical covering map from $\widetilde{\mathcal{M}}(M,\Sigma,\kappa)$ to $\mathcal{M}(M,\Sigma,\kappa)$ induces a bijection at the π_0 level. Observe also that for classification purposes, we can and will assume that all ramification indices κ_i are > 1.

13.1 Hyperelliptic components

Let $d \geqslant 4$ be an integer, and let $P \in \mathbb{C}[z]$ be a polynomial of degree $d+1$ with simple roots. Adding 1 or 2 points at infinity (depending on whether d is even or odd) to the complex curve $\{w^2 = P(z)\}$, one obtains an hyperelliptic compact Riemann surface M of genus $g = [\frac{d}{2}]$. The holomorphic 1-form $\omega := \frac{dz}{w}$ has no zero at finite distance. When d is even, it has a zero of order $d-2 = 2g-2$ at the single point A_1 at infinity. When d is odd, it has a zero of the same order $g-1 = \frac{d-3}{2}$ at each of the two points A_1, A_2 at infinity.

The translation surface defined by (M, ω) has therefore the following data:

- $s = 1$, $\kappa_1 = 2g-1$ if d is even;
- $s = 2$, $\kappa_1 = \kappa_2 = g$ if d is odd.

Moreover we have $d = 2g + s - 1$ in all cases so d is the complex dimension of the corresponding moduli space.

Observe that, for $a \in \mathbb{C}^*, b \in \mathbb{C}$, the polynomials P and $a^{-2}P(az+b)$ produce isomorphic translation surfaces. One has therefore exactly d independent complex parameters to deform the translation surface through a change of polynomial P. It is not difficult to see that one gets in this way, for each integer $d \geqslant 4$, a whole connected component of the corresponding moduli space. Such connected components are called *hyperelliptic*.

Hyperelliptic components correspond to the simplest Rauzy classes. Let $\#\mathcal{A} = d$. A Rauzy class containing some combinatorial data $\pi = (\pi_t, \pi_b)$ such that $\pi_t(\alpha) + \pi_b(\alpha) = d+1$ for all $\alpha \in \mathcal{A}$ is associated to the hyperelliptic component of dimension d.

When $g = 2$, the values $d = 4$ and $d = 5$ correspond to a double zero or two simple zeros for ω respectively. It is immediate to check that the hyperelliptic Rauzy classes described above are the only ones giving these values of (g, s, κ). Therefore, the two strata of the moduli space in genus 2 are connected and hyperelliptic.

Kontsevich and Zorich discovered that the situation is quite different in genus $\geqslant 3$.

13.2 Parity of spin structure

Let $(M, \Sigma, \kappa, \zeta)$ be a translation surface such that all κ_i are **odd**. We denote as usual $\Sigma = (A_1, \ldots, A_s)$. The divisor $D = \sum \frac{(\kappa_i - 1)}{2} A_i$ defines a *spin structure* on the Riemann surface M (equipped with the complex structure defined by the structure of translation surface). The *parity* of this spin structure is the parity of the dimension of the space of meromorphic functions f on M such that $(f) + D \leqslant 0$.

The reader should consult [At], [Mil] for some fundamental facts and results about spin structures and their parity. A fundamental result is that the parity of the spin structure is invariant under deformation, and is therefore the same for all translation surfaces in a same connected component of the moduli space.

The parity of the spin structure can be computed in the following way. For a smooth loop $c : \mathbb{S}^1 \to M - \Sigma$, define the index $ind(c)$ to be the degree mod 2 of the map which associates to $t \in \mathbb{S}^1$ the angle between the tangent vector $\dot{c}(t)$ and the horizontal direction at $c(t)$. As all ramification indices κ_i are odd, the index depends only on the class of c in $H_1(M, \mathbb{Z})$. Now let a_i, b_i, $1 \leqslant i \leqslant g$ be smooth loops in $M - \Sigma$ such that their homology classes form a standard symplectic basis of $H_1(M,\mathbb{Z})$. The parity of the spin structure for the translation surface (M,Σ,κ,ζ) is then given by

$$\sum_1^g (ind(a_i)+1)(ind(b_i)+1) \quad \text{mod. } 2.$$

13.3 Classification

Kontsevich and Zorich have shown that hyperellipticity and parity of spin structure are sufficient to classify components. More precisely

Theorem [KonZo] *Let (g,s,κ) be combinatorial data (with all $\kappa_i > 1$) determining a moduli space for translation surfaces.*

1. *If at least one of the κ_i is even, the moduli space is connected, except when $s=2$, $\kappa_1=\kappa_2=g\geqslant 4$. In this case, the moduli space has two components, one hyperelliptic and the other not hyperelliptic.*

2. *If all κ_i are odd and either $s\geqslant 3$ or $s=2$ and $\kappa_1\neq\kappa_2$, then the moduli space has two connected components, one with even spin structure and the other with odd spin structure.*

3. *If either $s=1, g\geqslant 4$ or $s=2, \kappa_1=\kappa_2=g$ odd $\geqslant 5$, the moduli space has three connected components: one hyperelliptic and two non hyperelliptic distinguished by the parity of the spin structure.*

4. *If $g=3$, $s=1$ or $s=2$, $\kappa_1=\kappa_2=3$, the moduli space has two components, one hyperelliptic and the other not hyperelliptic. If $g=2$,$s=1$, the moduli space is connected.*

We just say a few words of the scheme of the proof. The confluence of the zeros of the 1-form associated to the structure of translation surface organizes the various moduli spaces as the strata of a stratification. The minimal stratum S_{min} corresponds to a single zero of maximal multiplicity $2g-2$.

Kontsevich and Zorich establish the following fact, which allows to rely any stratum to S_{min}: for any stratum S, and any connected component C of S_{min}, there exists exactly one component of S which contains C in its closure.

The determination of the connected components of the minimal stratum S_{min} is by induction on the genus g. First, using a local construction first described in [EMaZo], they show that there are at least as many components as stated in the theorem: given a translation surface with a single zero A_1 of multiplicity $2g-2$, they split A_1 into two zeros A_1', A_1'' of respective multiplicities k_1', k_1'', slit the surface along a segment joining A_1' and A_1'', and glue the two sides to the two boundary components of a cylinder. The resulting translation surface has genus $g+1$, a single zero of maximal multiplicity $2g$ and the parity of its spin structure changes when the parity of k_1' change.

That there are no more components of S_{min} that as stated in the theorem is also proved by induction. The idea is to present any generic translation surface in S_{min} as the suspension, via the zippered rectangle construction, of an i.e.m and then take off a handle by an appropriate reduction operation.

14 Exponential mixing of the Teichmüller flow

We present in this section the main results from [AvGoYo].

14.1 Exponential mixing

Let $(X, \mathcal{B}, m)$ be a probability space, and let (T^t) be a measure-preserving dynamical system. We allow here for discrete time ($t \in \mathbb{Z}$) as well as continuous time ($t \in \mathbb{R}$). We denote by $L^2_0(X)$ the Hilbert space of square-integrable functions of mean value 0, by U^t the unitary operator $\varphi \mapsto \varphi \circ T^t$ of $L^2_0(X)$. For φ, $\psi \in L^2_0(X)$, we define the **correlation coefficient** of φ, ψ by

$$c_{\varphi,\psi}(t) := < \varphi, U^t \psi > .$$

We recall that

- T^t is ergodic iff, for all φ, $\psi \in L^2_0(X)$, $c_{\varphi,\psi}(t)$ converges to 0 in the sense of Cesaro as $t \to +\infty$;
- T^t is mixing iff, for all φ, $\psi \in L^2_0(X)$, $c_{\varphi,\psi}(t)$ converges to 0 as $t \to +\infty$.

Exponential mixing requires that this convergence is exponentially fast. However, simple examples (for instance, the shift map) show that this cannot happen, even in the most chaotic dynamical systems, for **all** functions φ, $\psi \in L^2_0(X)$. One generally requires that φ, ψ belong to some Banach space E of "regular" functions on X, dense in $L^2_0(X)$. Then the correlation coefficients should satisfy

$$c_{\varphi,\psi}(t) \leqslant C||\varphi||_E \, ||\psi||_E \exp(-\delta\, t),$$

where $\delta > 0$ is independent of φ, $\psi \in E$. Observe that this indeed imply mixing.

Exponential mixing, unlike ergodicity or mixing, is **not** a spectral notion (one which depends only on the properties of the unitary operators U^t).

Theorem [AvGoYo] *The Teichmuller flow is exponentially mixing on any connected component of any marked moduli space $\widetilde{\mathcal{M}}^{(1)}(M, \Sigma, \kappa)$.*

The subspace E of "regular" functions will be explicited below; for any $1 \geqslant \beta > 0$, it can be chosen to contain all β-Hölder functions with compact support.

14.2 Exponential mixing and irreducible unitary representations of $SL(2,\mathbb{R})$

The theorem has an interesting consequence with respect to the representation of $SL(2,\mathbb{R})$ determined by the action of this group on the marked moduli spaces.

Ler $\mathcal{C}$ be a connected component of some marked moduli space $\widetilde{\mathcal{M}}^{(1)}(M,\Sigma,\kappa)$. Denote by H the Hilbert space of zero mean L^2 functions on $\mathcal{C}$. The action of $SL(2,\mathbb{R})$ induces an unitary representation of $SL(2,\mathbb{R})$ in H. As any unitary representation of $SL(2,\mathbb{R})$, it decomposes as an hilbertian sum

$$H = \int H_\xi d\mu(\xi),$$

where, for each ξ, the representation of $SL(2,\mathbb{R})$ in H_ξ is irreducible.

According to Bargmann, the nontrivial irreducible unitary representations of $SL(2,\mathbb{R})$ are divided into three families, the *discrete, principal and complementary series.* This corresponds to an orthogonal decomposition into invariant subspaces

$$H = H_{tr} \oplus H_d \oplus H_{pr} \oplus H_c \, .$$

The ergodicity of the action of $SL(2,\mathbb{R})$ (Masur-Veech) means that $H_{tr} = \{0\}$.

Write g_t for the diagonal element (e^t, e^{-t}) of $SL(2,\mathbb{R})$ corresponding to the Teichmüller flow. In general, for vectors v, v' belonging both to the discrete component H_d or the principal component H_{pr} of the representation, one has, for $t \leqslant 1$

$$< g_t(v), v' > \; \leqslant \; C \, t \, e^{-t} \, ||v|| \, ||v'|| \, .$$

On the other hand, the complementary series is parametrized by a parameter $s \in (0,1)$, with

$$\mathcal{H}_s = \{ f \, : \mathbb{S}^1 \to \mathbb{C} \, , \; ||f||^2 := \int\int \frac{f(z)\overline{f}(z')}{|z-z'|^{1-s}} \, dz \, dz' < +\infty \},$$

the representation of $SL(2,\mathbb{R})$ in $\mathcal{H}_s$ being given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} . f(z) = |\overline{\beta} z + \overline{\alpha}|^{-1-s} \, f\left(\frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}} \right) \, ,$$

with

$$\begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} .$$

We observe that the norm in $\mathcal{H}_s$ is equivalent to the norm

$$||f||' = \left(\sum (1+|n|)^{-s} |\widehat{f}(n)|^2 \right)^{\frac{1}{2}} .$$

The integral powers $e_n(z) := z^n$, $n \in \mathbb{Z}$, are eigenfunctions for the action of $SO(2,\mathbb{R})$:

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} e_n \;=\; \exp(2i\pi n\theta)\, e_n \;.$$

An easy calculation show that, for $m,n \in \mathbb{Z}$, $t \geqslant 1$

$$|< g_t\, e_m \,,\, e_n >| \quad \leqslant \quad < g_t\, e_0 \,,\, e_0 >,$$

$$C_s^{-1}\, e^{t(s-1)} \leqslant \quad < g_t\, e_0 \,,\, e_0 > \quad \leqslant C_s\, e^{t(s-1)},$$

with $C_s > 0$ depending on s but not on t.

Definition A unitary representation $\mathcal{H}$ of $SL(2,\mathbb{R})$ has an *almost invariant vector* if, given any compact subset K of $SL(2,\mathbb{R})$ and $\varepsilon > 0$, there exists a unit vector $v \in \mathcal{H}$ such that

$$||g.v - v|| < \varepsilon$$

for all $g \in K$.

A unitary representation $\mathcal{H}$ of $SL(2,\mathbb{R})$ with no almost invariant vector is said to have a *spectral gap.*

Let $H = \int H_\xi d\mu(\xi)$ be the decomposition of a unitary representation $\mathcal{H}$ of $SL(2,\mathbb{R})$ into irreducible representations. Then $\mathcal{H}$ has a spectral gap iff there exists $s_0 \in (0,1)$ such that, for almost every ξ, H_ξ is neither the trivial representation nor isomorphic to a representation in the complementary series with parameter $s \in (s_0, 1)$.

Definition Let $\mathcal{H}$ be a unitary representation of $SL(2,\mathbb{R})$. A vector $v \in \mathcal{H}$ is C^r-$SO(2,\mathbb{R})$-smooth if the function

$$\theta \to \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} .v$$

is of class C^r.

Proposition (Ratner [Rat]) *If the unitary representation $\mathcal{H}$ has a spectral gap, then it is exponential mixing for C^2-$SO(2,\mathbb{R})$-smooth vectors: there exists $\delta > 0$ and $C > 0$ such that, for any C^2-$SO(2,\mathbb{R})$-smooth $v, v' \in \mathcal{H}$, $t \geqslant 1$*

$$|< g_t.v, v' >| \;\leqslant\; C\; \exp(-\delta t)\; ||v||_2\; ||v'||_2 \;,$$

where $||v||_2$ is the sum of the norm of v and the norm of the second derivative at 0 of $\theta \to \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} .v$.

Sketch of proof: It is sufficient to consider unit vectors v, v' in the complementary component of the representation. There exists $s_0 \in (0,1)$ such that $s(\xi) \notin [s_0, 1)$ for almost every ξ, hence we have

$$| < g_t.e_m, e_n >_\xi | \leqslant C \exp(t(s_0 - 1))$$

for all $t \geqslant 1$, $m, n \in \mathbb{Z}$, and almost every ξ. Let v, v' be C^2-$SO(2,\mathbb{R})$-smooth vectors in the complementary component of $\mathcal{H}$. Write

$$v = \int v(\xi) d\mu(\xi) = \int \sum v_m(\xi) e_m d\mu(\xi),$$

and similarly for v'. Then $v(\xi)$ is C^2-$SO(2,\mathbb{R})$-smooth for almost all ξ. From the remark on the norm in $\mathcal{H}_s$ above, this gives, for all $m \in \mathbb{Z}$

$$|v_m(\xi)| \leqslant C ||v(\xi)||_2 (1 + |m|)^{\frac{s(\xi)}{2} - 2}.$$

We conclude that

$$\begin{aligned}
| < g_t.v,\, v' > | &\leqslant \int | < g_t.v(\xi),\, v'(\xi) > | d\mu(\xi) \\
&\leqslant \int | \sum_m \sum_n v_m(\xi) \overline{v}'_n(\xi) < g_t.e_m, e_n >_\xi | d\mu(\xi) \\
&\leqslant C \ \exp(t(s_0 - 1)) \int ||v(\xi)||_2 ||v'(\xi)||_2 d\mu(\xi) \\
&\leqslant C \ \exp(t(s_0 - 1)) \ ||v||_2 ||v'||_2. \qquad \square
\end{aligned}$$

Remark: The absence of a trivial component, i.e the ergodicity of the action of $SL(2,\mathbb{R})$, already imply that the action of the diagonal subgroup is mixing: for vectors of the form $v = \int_{0<s(\xi)<s_0} \sum_{|m|\leqslant M} v_m(\xi) e_m d\mu(\xi), v' = \int_{0<s(\xi)<s_0} \sum_{|m|\leqslant M} v'_m(\xi) e_m d\mu(\xi)$, for some $M > 0$, $s_0 \in (0,1)$, we have that $| < g_t.v, v' > |$ converges to 0 by the calculation above. These vectors are dense in the complementary component of $\mathcal{H}$, and the mixing property follows.

Conversely

Proposition *Assume that there exists $\delta > 0$ and a dense subset E of vectors v in the space of $SO(2,\mathbb{R})$-invariant vectors in $\mathcal{H}$ for which the correlation coefficients $< g_t.v, v >$ are $\mathcal{O}(\exp(-\delta t))$. Then $\mathcal{H}$ has a spectral gap.*

Proof: We may assume that $0 < \delta < 1$. Assume by contradiction that $\mathcal{H}$ has no spectral gap. The complementary component v_c of any $SO(2,\mathbb{R})$-invariant vector takes the form $v_c = \int v_0(\xi) e_0 d\mu(\xi)$, with $v_0 \in L^2(\mu)$. As E is dense in the space of $SO(2,\mathbb{R})$-invariant vectors in $\mathcal{H}$, we can find $v \in E$ such that

$$\mu\{\xi,\ s(\xi) > 1 - \delta \text{ and } v_0(\xi) \neq 0\} > 0\ .$$

Then we have

$$\begin{aligned} < g_t.v_c, v_c > &= \int |\phi(\xi)|^2 < g_t.e_0, e_0 > d\mu(\xi) \\ &\geqslant \int |\phi(\xi)|^2 C_{s(\xi)}^{-1} \exp(t(s(\xi)-1)) d\mu(\xi); \end{aligned}$$

here $s(\xi)$ is the parameter in the complementary series associated to H_ξ. Thus $< g_t.v_c, v_c >$ is not $\mathcal{O}(\exp(-\delta t))$. But v does not have a discrete component, and the principal component v_p satisfies $< g_t.v_p, v_p >= \mathcal{O}(t\exp(-t))$. This contradicts the property of E. $\square$

Coming back to the setting of the theorem in subsection 14.1, let $\mathcal{C}$ be a component of some marked moduli space $\widetilde{\mathcal{M}}^{(1)}(M, \Sigma, \kappa)$. The space of compactly supported mean zero smooth $SO(2,\mathbb{R})$-invariant functions on $\mathcal{C}$ is dense in the subspace of $SO(2,\mathbb{R})$-invariant functions in $L_0^2(\mathcal{C})$. Therefore the representation of $SL(2,\mathbb{R})$ in $L_0^2(\mathcal{C})$ has a spectral gap.

14.3 Diophantine estimates

Exponential mixing is a classical property of uniformly hyperbolic transformations preserving a smooth volume form.

Exercise : Let $A \in SL(d,\mathbb{R})$ be a hyperbolic matrix. The induced diffeomorphism of $\mathbb{T}^d$ preserves Lebesgue measure. Prove that , if φ, ψ are Hölder functions on $\mathbb{T}^d$ with zero mean-value, the correlation coefficient $c_{\varphi,\psi}(n) := \int_{\mathbb{T}^d} \varphi \;\; \psi \circ A^n$ satisfy

$$|c_{\varphi,\psi}(n)| \leqslant C||\varphi|| \; ||\psi|| \; \exp(-\delta n),$$

where δ depends only on A and the Hölder exponent of φ, ψ.

With respect to this very basic case, the Teichmüller flow presents three difficulties:

- the time is continuous rather than discrete;
- hyperbolicity is non uniform;
- distortion for large time is not controlled as simply than in the uniformly hyperbolic setting on a compact manifold.

As the constant time suspension of an Anosov diffeomorphism is obviously not mixing, the first difficulty is quite serious. The ideas which allow to deal with it were first introduced by Dolgopyat ([Do]) and later developped by Baladi-Vallée ([BaVa]).

The other two difficulties are related to a lack of compactness of the moduli spaces of translation surfaces. To get uniform hyperbolicity and bounded distortion, one is led to introduce the return map of the Teichmüller flow to a suitably small transversal section (smaller than the ones considered in Section 9). The problem is then to control the return time to this transversal section. This is done through diophantine estimates which we will now present.

Let $\mathcal{R}$ be a Rauzy class on an alphabet $\mathcal{A}$, let $\mathcal{D}$ be the associated Rauzy diagram. The estimates depend on a parameter $q \in \mathbb{R}_+^{\mathcal{A}}$. For such q, we define a probability measure P_q on $\mathbb{P}(\mathbb{R}_+^{\mathcal{A}})$ by

$$P_q(A) := \frac{\text{Leb}(\mathbb{R}_+ A \bigcap \Lambda_q)}{\text{Leb}(\Lambda_q)},$$

where $\Lambda_q = \{\lambda \in \mathbb{R}_+^{\mathcal{A}};\ < \lambda, q > \ < 1\ \}$. Define also, for $q \in \mathbb{R}_+^{\mathcal{A}}$, $M(q) := \max_{\alpha \in \mathcal{A}} q_\alpha$, $m(q) := \min_{\alpha \in \mathcal{A}} q_\alpha$. For a finite path γ in $\mathcal{D}$, starting from a vertex π, we denote by Δ_γ the set of $\lambda \in \Delta_\pi$ whose Rauzy-Veech path starts with γ.

Let now $0 \leqslant m \leqslant M$ be integers, $q \in \mathbb{R}_+^{\mathcal{A}}$, $\pi \in \mathcal{R}$. Define $\Gamma_0 = \Gamma_0(m, M, q, \pi)$ to be the set of finite paths $\gamma \in \mathcal{D}$ starting from π such that

$$M(B_\gamma q) > 2^M\ M(q), \qquad\qquad m(B_\gamma q) < 2^{M-m}\ M(q).$$

Theorem [AvGoYo] *There exist constants θ, C depending only on $\#\mathcal{A}$ such that*

$$P_q(\bigcup_{\gamma \in \Gamma_0} \mathbb{P}(\Delta_\gamma)) \leqslant C(m+1)^\theta 2^{-m}.$$

A closely connected estimate is the following. Let M be an integer and $q \in \mathbb{R}_+^{\mathcal{A}}$, $\pi \in \mathcal{R}$. Define $\Gamma_1 = \Gamma_1(M, q, \pi)$ to be the set of finite paths $\gamma \in \mathcal{D}$ starting from π such that γ is not complete and $M(B_\gamma q) > 2^M\ M(q)$.

Theorem [AvGoYo] *There exist constants θ, C depending only on $\#\mathcal{A}$ such that*

$$P_q(\bigcup_{\gamma \in \Gamma_1} \mathbb{P}(\Delta_\gamma)) \leqslant C(M+1)^\theta 2^{-M}.$$

Exercise Use these estimates to show that almost all i.e.m are of Roth type.

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