Affine interval exchange maps with a wandering interval

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Abstract

For almost all interval exchange maps T_0, with combinatorics of genus $g \geq 2$, we construct affine interval exchange maps T which are semi–conjugate to T_0 and have a wandering interval.

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CONTENTS

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0. Introduction

Quasiperiodic systems play a very important role in the theory of dynamical systems and in mathematical physics.

Irrational rotations of the circle are the prototype of quasiperiodic dynamics. The suspension of these rotations produces linear flows on the two-dimensional torus. When analyzing the recurrence of rotations or the suspended flows, the modular group $\mathrm{GL}\,(2,\mathbb{Z})$ is of fundamental importance, providing the renormalization scheme associated to the continuous fraction of the rotation number.

Poincaré proved that any orientation-preserving homeomorphism of the circle with no periodic orbit is semi-conjugate to an irrational rotation. Later Denjoy constructed examples of C^r diffeomorphisms with irrational rotation number and a wandering interval if $r < 2$. He also proved that any C^2 diffeomorphism with no periodic orbit is conjugate to an irrational rotation. Actually, this result is also true for piecewise-affine homeomorphisms [He].

A natural generalization of the linear flows on the two-dimensional torus is obtained by considering linear flows on compact surfaces of higher genus, called translation surfaces. By a Poincaré section their dynamics can be reduced to (standard) interval exchange maps (i.e.m.), which generalize rotations of the circle.

Let $\mathcal{A}$ be an alphabet with $d \geq 2$ elements. A (standard) i.e.m. T on an interval I (of finite length) is determined by two partitions (I^t_a), (I^b_a), of I with I^t_a, I^b_a of the same length, the restriction of T to I^t_a being a translation with image I^b_a . Thus T is orientation-preserving and preserves Lebesgue measure. By relaxing the requirement on the lengths and only asking that the restriction of T to I^t_a is an orientation-preserving homeomorphism onto I^b_a one obtains the definition of a generalized i.e.m. A special class of generalized i.e.m. , namely affine i.e.m. are considered in this paper: we require that the restriction of T to I^t_a is affine (and orientation-preserving). When $d = 2$, by identifying the endpoints of I standard i.e.m. correspond to rotations of the circle and generalized i.e.m. to homeomorphisms of the circle.

The ordering of the subintervals in the two partitions of I constitute the combinatorial data for the i.e.m. T. One says that a standard i.e.m. has no connection if every orbit can be extended indefinitely in the future or in the past (or both) without going through the endpoints of the subintervals; Keane [Ke] has shown that such an i.e.m. is minimal. When $d = 2$, this corresponds exactly to irrational rotations.

Following Rauzy [Ra] and Veech [V1], one analyzes the dynamics of a standard i.e.m. T with no connection by considering the first return maps $T^{(n)}$ of T on a decreasing sequence of intervals $I^{(n)}$, with the same left endpoint than I. These maps are again standard i.e.m. on the same alphabet $\mathcal{A}$ but the combinatorial data may be different. The set of all possible combinatorial data accessible from the initial one by this process constitute a Rauzy class. To each Rauzy class is associated a Rauzy diagram (whose vertices are the elements in the Rauzy class and arrows are the possible transitions). The sequence of combinatorial data for the $T^{(n)}$ is an infinite path in this diagram which can be viewed as a "rotation number".

By suspending an i.e.m. through Veech zippered rectangle construction [V2], one obtains a linear flow on a translation surface. The genus g of the surface only depends on the Rauzy class.

For a generalized i.e.m. T with no connection one can still define the $T(n)$ and obtain an infinite path in a Rauzy diagram. When this path is also associated with a standard i.e.m. T_0 with no connection (one then says that T is irrational), T is semi–conjugate to T_0 .

When $d = 2$, or more generally $g = 1$, such a semi–conjugacy for an affine i.e.m is always a conjugacy as recalled above.

Levitt [L] found an example of an affine irrational i.e.m. in higher genus which has a wandering interval. The corresponding standard i.e.m is not unique in his case; this only happens in the non-uniquely ergodic case which has measure zero in parameter space [Ma], [V2].

Later Camelier and Gutierrez [CG] exhibited an example of affine irrational i.e.m. with a wandering interval such that the corresponding standard i.e.m. is uniquely ergodic. The infinite path in the Rauzy diagram in their case is periodic. The same example was studied more deeply by Cobo [Co]. In particular, he put in evidence on this example the importance of the Oseledets decomposition of the extended Zorich cocycle (see Section 3.1 below).

Very recently, Bressaud, Hubert and Maass [BHM] generalized the Camelier-Gutierrez example to a large class of periodic paths in Rauzy diagrams with $g > 1$. In the periodic case, the Zorich cocycle is just a matrix in $\mathrm{SL}\,(\mathbb{Z}, d)$ with positive coefficients. The vector of the logarithms of the slopes (for the affine i.e.m.) must lie in the Perron-Frobenius hyperplane for this matrix; however, it can have a non-zero component with respect to the next biggest eigenvalue (which is assumed to be real and conjugate to the largest one), and such a choice lead to the required examples.

Our main result is of a similar nature, but instead of starting with periodic paths (a countable set of possibilities) , we consider a set of "rotation numbers" of full measure.

Let us fix combinatorial data, such that the associated surface has genus $g > 1$. By a deep result of Avila-Viana [AV], the extended Zorich cocycle has g simple positive Lyapunov exponents $\theta_1 > \theta_2 > \ldots > \theta_g$. Let $E_0 = \mathbb{R}^{\mathcal{A}} \supset E_1 \supset E_2 \supset \ldots \supset E_g$ (with $\dim E_i = d - i$) be the corresponding filtration (defined for almost all parameter values); a necessary and sufficient condition for a vector in

$\mathbb{R}^{\mathcal{A}}$ to have for coordinates the logarithms of the slopes of an affine i.e.m. with this rotation number is that it belongs to the hyperplane E_1.

Theorem. *For almost all standard i.e.m. T_0 with the given combinatorial data, the following holds: the coordinates of any vector in $E_1 \setminus E_2$ can be realized as the logarithms of the slopes of an affine i.e.m. semi–conjugate to T_0 with a wandering interval.*

We will now summarize the contents of our paper. In the first section we introduce interval exchange maps and we develop the continued fraction algorithms. Accelerating the Rauzy–Veech map by grouping together arrows with the same type in the Rauzy diagram leads to the Zorich continued fraction algorithm (described in 1.2.4) which has the advantage of having a finite mass a.c.i.m.. The notations and the presentation of the Rauzy–Veech–Zorich algorithms follow closely the expository paper [Y1] (see also [Y2]).

Section 2 is devoted to the study of the deformations of affine interval echange maps. First we describe the compact convex set $\mathrm{Aff}^{(1)}(\underline{\gamma}, w)$ of affine i.e.m. of the unit interval whose slope vector w and orbit $\underline{\gamma}$ under the Rauzy–Veech algorithm are prescribed. Following an analogy with the theory of holomorphic motions in complex dynamics, we them define affine motions. This allows us to characterize the tangent space to $\mathrm{Aff}^{(1)}(\underline{\gamma}, w)$.

In Section 3 deals with the construction of affine interval exchange maps with a wandering interval.

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1. The continued fraction algorithm for interval exchange maps

1.1 Interval exchange maps

An interval exchange map (i.e.m.) is determined by combinatorial data on one side, length data on the other side.

Let $\mathcal{A}$ be an alphabet with $d \geq 2$ elements which serve as indices for the intervals. The combinatorial data is a pair $\pi = (\pi_t, \pi_b)$ of bijections from $\mathcal{A}$ onto $\{1, \ldots, d\}$ which indicates in which order the intervals are met in the domain and in the range of the i.e.m. . We always assume that the combinatorial data are *irreducible*: for $1 \leq k < d$, we have

$$\pi_t^{-1}(\{1, \ldots, k\}) \neq \pi_b^{-1}(\{1, \ldots, k\}) \ .$$

The length data are the lengths $(\lambda_\alpha)_{\alpha \in \mathcal{A}}$ of the subintervals. Let $T = T_{\pi,\lambda}$ be the i.e.m. determined by these data; it is acting on $I = (0, \lambda^*)$, with

$$\lambda^* = \sum_{\alpha \in \mathcal{A}} \lambda_\alpha \ .$$

The subintervals in the domain are

$$I_\alpha^t = \left(\sum_{\pi_t \beta < \pi_t \alpha} \lambda_\beta, \sum_{\pi_t \beta \leq \pi_t \alpha} \lambda_\beta \right)$$

and those in the range are

$$I_\alpha^b = \left(\sum_{\pi_b \beta < \pi_b \alpha} \lambda_\beta, \sum_{\pi_b \beta \leq \pi_b \alpha} \lambda_\beta \right) \ .$$

We also write I_α for I_α^t. The translation vector $(\delta_\alpha)_{\alpha \in \mathcal{A}}$ is given by

$$\delta_\alpha = \sum_\beta \Omega_{\alpha\beta} \lambda_\beta$$

where the antisymmetric matrix $\Omega = \Omega(\pi)$ is defined by

$$\Omega_{\alpha\beta} = \begin{cases} +1 & \text{if } \pi_t \beta > \pi_t \alpha \ , \ \pi_b \beta < \pi_b \alpha, \\ -1 & \text{if } \pi_t \beta < \pi_t \alpha \ , \ \pi_b \beta > \pi_b \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

We denote the rank of Ω by $2g$; in fact g is the genus of the translation surfaces obtained from T by suspension. One has thus

$$T(x) = x + \delta_\alpha \ \text{ for } x \in I^t_\alpha \,,$$
$$T(I^t_\alpha) = I^b_\alpha \ \text{ for } \alpha \in \mathcal{A} \,.$$

We denote by $u^t_1 < \ldots < u^t_{d-1}$ the points of $I \backslash \cup_{\alpha\in\mathcal{A}} I^t_\alpha$, which we call *singularities of* T. Similarly, the points $u^b_1 < \ldots < u^b_{d-1}$ of $I \backslash \cup_{\alpha\in\mathcal{A}} I^b_\alpha$ are called the *singularities of* T^{-1}. A *connection* is a triple (u^t_i, u^b_j, m), where m is a nonnegative integer, such that

$$T^m(u^b_j) = u^t_i \,.$$

Keane has proved [Ke] that an i.e.m. with no connection is minimal, and also that an i.e.m. has no connection if the length data are independent over $\mathbb{Q}$.

1.2 The elementary step of the Rauzy–Veech algorithm

Let $T = T_{\pi,\lambda}$ be an i.e.m. . Denote by α_t, α_b the elements of $\mathcal{A}$ such that

$$\pi_t(\alpha_t) = \pi_b(\alpha_b) = d \,.$$

When $u^t_{d-1} \neq u^b_{d-1}$ (which must happen if T has no connection), we consider the first return map $\hat{T}$ on $\hat{I} = (0, \mathrm{Max}\,(u^t_{d-1}, u^b_{d-1}))$.

When $u^t_{d-1} < u^b_{d-1}$, we have

$$\hat{T}(y) = \begin{cases} T^2(y) & \text{if } y \in I^t_{\alpha_b}, \\ T(y) & \text{otherwise.} \end{cases}$$

Thus $\hat{T}$ is an i.e.m. with the same alphabet $\mathcal{A}$, length data $\hat{\lambda}$, combinatorial data $\hat{\pi}$ with

$$\hat{\lambda}_{\alpha_t} = \lambda_{\alpha_t} - \lambda_{\alpha_b} \,,$$
$$\hat{\lambda}_\alpha = \lambda_\alpha \quad, \alpha \neq \alpha_t \,,$$
$$\hat{\pi}_t = \pi_t \,,$$
$$\hat{\pi}_b(\alpha) = \begin{cases} \pi_b(\alpha) & \text{if } \pi_b(\alpha) \leq \pi_b(\alpha_t), \\ \pi_b(\alpha) + 1 & \text{if } \pi_b(\alpha_t) < \pi_b(\alpha) < d, \\ \pi_b(\alpha_t) + 1 & \text{if } \pi_b(\alpha) = d. \end{cases}$$

When $u^b_{d-1} < u^t_{d-1}$, we have

$$\hat{T}^{-1}(y) = \begin{cases} T^{-2}(y) & \text{if } y \in I^b_{\alpha_t}, \\ T^{-1}(y) & \text{otherwise.} \end{cases}$$

In this case, the length and combinatorial data for $\hat{T}$ are:

$$\begin{aligned}
\hat{\lambda}_{\alpha_b} &= \lambda_{\alpha_b} - \lambda_{\alpha_t} \ , \\
\hat{\lambda}_{\alpha} &= \lambda_{\alpha} \quad , \alpha \neq \alpha_b \ , \\
\hat{\pi}_b &= \pi_b \ , \\
\hat{\pi}_t(\alpha) &= \begin{cases} \pi_t(\alpha) & \text{if } \pi_t(\alpha) \leq \pi_t(\alpha_b), \\ \pi_t(\alpha)+1 & \text{if } \pi_t(\alpha_b) < \pi_t(\alpha) < d, \\ \pi_t(\alpha_b)+1 & \text{if } \pi_t(\alpha) = d. \end{cases}
\end{aligned}$$

We say that $\hat{T}$ is deduced from T by an elementary step of the Rauzy–Veech algorithm. We also define the Rauzy operation $\hat{\pi} = R_t(\pi)$ (respectively $\hat{\pi} = R_b(\pi)$) for the change of combinatorial data when $u^t_{d-1} < u^b_{d-1}$ (respectively $u^b_{d-1} < u^t_{d-1}$).

1.3 Rauzy diagrams

A *Rauzy class* on an alphabet $\mathcal{A}$ is a nonempty set of irreducible combinatorial data which is invariant under R_t, R_b and minimal with respect to this property. A *Rauzy diagram* is a graph whose vertices are the elements of a Rauzy class and whose arrows connect a vertex π to its images $R_t(\pi)$ and $R_b(\pi)$. Each vertex is therefore the origin of two arrows. As R_t, R_b are invertible, each vertex is also the endpoint of two arrows. It is a fact that the rank of the matrix $\Omega(\pi)$ is the same for all π in a given Rauzy class.

An arrow connecting π to $R_t(\pi)$ (respectively $R_b(\pi)$) is said to be of *top type* (resp. *bottom type*). The *winner* of an arrow of top (resp. bottom) type starting at $\pi = (\pi_t, \pi_b)$ with $\pi_t(\alpha_t) = \pi_b(\alpha_b) = d$ is the letter α_t (resp. α_b) while the *loser* is α_b (resp. α_t).

To an arrow γ of a Rauzy diagram $\mathcal{D}$ starting at π of top (resp. bottom) type, is associated the matrix $B_\gamma \in \mathrm{SL}\,(\mathbb{Z}^{\mathcal{A}})$ defined by

$$B_\gamma = \mathbb{I} + E_{\alpha_b \alpha_t}$$

(resp. $B_\gamma = \mathbb{I} + E_{\alpha_t \alpha_b}$), where $E_{\alpha\beta}$ is the elementary matrix whose only nonzero coefficient is 1 in position $\alpha\beta$. For a path γ in $\mathcal{D}$ made of the successive arrows $\gamma_1 \ldots \gamma_l$ we associate the product $B_\gamma = B_{\gamma_l} \ldots B_{\gamma_1}$. It belongs to $\mathrm{SL}\,(\mathbb{Z}^{\mathcal{A}})$ and has nonnegative coefficients.

A path γ in $\mathcal{D}$ is *complete* if each letter in $\mathcal{A}$ is the winner of at least one arrow in γ; it is *k–complete* if γ is the concatenation of k complete paths. An infinite path is ∞*–complete* if it is the concatenation of infinitely many complete paths. By [MMY, Section 1.2.4], if a path γ is $(2d-3)$–complete, then all coefficients of B_γ are strictly positive.

1.4 The Rauzy-Veech and Zorich algorithms

Let $T^{(0)} = T_{(\lambda^{(0)},\pi^{(0)})}$ be an i.e.m. with no connection. We denote by $\mathcal{A}$ the alphabet for $\pi^{(0)}$ and by $\mathcal{D}$ the Rauzy diagram on $\mathcal{A}$ having $\pi^{(0)}$ as a vertex. The i.e.m. $T^{(1)} = T_{(\lambda^{(1)},\pi^{(1)})}$ deduced from $T^{(0)}$ by the elementary step of the Rauzy–Veech algorithm has also no connection. It is therefore possible to iterate this elementary step indefinitely and get a sequence $T^{(n)} = T_{(\lambda^{(n)},\pi^{(n)})}$ of i.e.m. acting on a decreasing sequence $I^{(n)}$ of intervals and a sequence $\gamma(n, n+1)$ of arrows in $\mathcal{D}$ from $\pi^{(n)}$ to $\pi^{(n+1)}$. For $m < n$, we also write $\gamma(m, n)$ for the path from $\pi^{(m)}$ to $\pi^{(n)}$ composed of the $\gamma(l, l+1)$, $m \leq l < n$. One has

$$\lambda^{(m)} =^t B_{\gamma(m,n)}\lambda^{(n)} ,$$
$$\delta^{(n)} = B_{\gamma(m,n)}\delta^{(m)} .$$

Conversely, if it is possible to iterate indefinitely the Rauzy–Veech elementary step starting from $T^{(0)}$, then $T^{(0)}$ has no connection.

Let $\underline{\gamma}$ be the infinite path starting at $\pi^{(0)}$ obtained by concatenation of the $\gamma(n, n+1)$; then $\underline{\gamma}$ is ∞–complete. Conversely, if an infinite path $\underline{\gamma}$ is ∞–complete, it is associated by the Rauzy–Veech algorithm to some $T = T_{\lambda,\pi}$ with no connection. This T is unique up to rescaling if and only if it is uniquely ergodic; this last property is true for almost all λ ([Ma], [V2]).

Following Zorich [Z1] it is often convenient to group together in a single Zorich step successive elementary steps of the Rauzy–Veech algorithm whose corresponding arrows have the same type (or equivalently the same winner); we therefore introduce a sequence $0 = n_0 < n_1 < \ldots$ such that for each k all arrows in $\gamma(n_k, n_{k+1})$ have the same type and this type is alternatively top and bottom. For $n \geq 0$, the integer k such that $n_k \leq n < n_{k+1}$ is called the *Zorich time* and denoted by $Z(n)$.

1.5 Dynamics of the continued fraction algorithms

Let $\mathcal{R}$ be a Rauzy class on an alphabet $\mathcal{A}$. The elementary step of the Rauzy–Veech algorithm,

$$(\pi, \lambda) \mapsto (\hat{\pi}, \hat{\lambda}) ,$$

considered up to rescaling, defines a map from $\mathcal{R} \times \mathbb{P}((\mathbb{R}^+)^{\mathcal{A}})$ to itself, denoted by Q_{RV}. There exists a unique absolutely continuous measure invariant under these dynamics ([V2]); it is conservative and ergodic but has infinite total mass, which does not allow all ergodic–theoretic machinery to apply. Replacing a Rauzy–Veech elementary step by a Zorich step gives a new map Q_{Z} on $\mathcal{R} \times \mathbb{P}((\mathbb{R}^+)^{\mathcal{A}})$. This map has now a *finite* absolutely continuous invariant measure, which is ergodic ([Z1]).

It is also useful to consider the natural extensions of the maps Q_{RV} and Q_{Z}, defined through the suspension data which serve to construct translation surfaces from i.e.m. . For $\pi \in \mathcal{R}$, let Θ_π be the convex open cone in $\mathbb{R}^{\mathcal{A}}$ defined by the inequalities

$$\sum_{\pi_t \alpha \leq k} \tau_\alpha > 0 \ , \qquad \sum_{\pi_b \alpha \leq k} \tau_\alpha < 0 \ , \quad 1 \leq k < d \ .$$

Define also

$$\Theta_\pi^t = \{\tau \in \Theta_\pi \ , \sum_\alpha \tau_\alpha < 0\} \ ,$$
$$\Theta_\pi^b = \{\tau \in \Theta_\pi \ , \sum_\alpha \tau_\alpha > 0\} \ .$$

Let $\gamma : \pi \rightarrow \hat{\pi}$ be an arrow in the Rauzy diagram $\mathcal{D}$ associated to $\mathcal{R}$. Then ${}^t B_\gamma^{-1}$ sends Θ_π isomorphically onto $\Theta_{\hat{\pi}}^t$ (resp. $\Theta_{\hat{\pi}}^b$) when γ is of top type (resp. bottom type). The natural extension $\hat{Q}_{\mathrm{RV}}$ is then defined on $\sqcup_{\pi \in \mathcal{R}} \{\pi\} \times \mathbb{P}((\mathbb{R}^+)^{\mathcal{A}}) \times \mathbb{P}(\Theta_\pi)$ by

$$(\pi, \lambda, \tau) \ \mapsto (\hat{\pi}, {}^t B_\gamma^{-1} \lambda, {}^t B_\gamma^{-1} \tau)$$

where γ is the arrow starting at π, associated to the map Q_{RV} at (π, λ). The map $\hat{Q}_{\mathrm{RV}}$ has again a unique absolutely continuous invariant measure; it is ergodic, conservative but infinite. One defines similarly a natural extension $\hat{Q}_{\mathrm{Z}}$ for Q_{Z}; it has a unique absolutely continuous invariant measure, which is finite and ergodic.

1.6 The continued fraction algorithm for generalized and affine i.e.m.

Let $\mathcal{A}$ be an alphabet and $\pi = (\pi_t, \pi_b)$ be irreducible combinatorial data over $\mathcal{A}$. Let $I = (0, \lambda^*)$ be an interval and let

$$0 = u_0^t < u_1^t < \ldots < u_d^t = \lambda^* \ ,$$
$$0 = u_0^b < u_1^b < \ldots < u_d^b = \lambda^* \ ,$$

two sets of points in $\overline{I}$. Define

$$I_\alpha^t = \left(u_{\pi_t(\alpha)-1}^t, u_{\pi_t(\alpha)}^t\right) \ ,$$
$$I_\alpha^b = \left(u_{\pi_b(\alpha)-1}^b, u_{\pi_b(\alpha)}^b\right) \ .$$

A *generalized i.e.m.* with combinatorial data π is a map on I whose restriction to each I_α^t is a non decreasing homeomorphism onto I_α^b (for some choice of the u_i^t, u_j^b). When these restrictions are affine, we say that T is an *affine* i.e.m. .

Connections for generalized i.e.m. are again defined by some relation $T^m(u_j^b) = u_i^t$, with $m \geq 0$, $0 < i, j < d$. When T has no connection, one has in particular

$u^t_{d-1} \neq u^b_{d-1}$. One then defines $\hat{I} = (0, \mathrm{Max}\,(u^t_{d-1}, u^b_{d-1}))$ and $\hat{T}$ as the first return map of T in $\hat{I}$. Then $\hat{T}$ is again a generalized i.e.m. (affine if T was affine), the combinatorial data being $R_t(\pi)$ if $u^t_{d-1} < u^b_{d-1}$, $R_b(\pi)$ if $u^b_{d-1} < u^t_{d-1}$. Also, $\hat{T}$ has no connection, hence we can iterate the processus.

A difference with the case of standard i.e.m. is that the infinite path $\underline{\gamma}$ in the Rauzy diagram $\mathcal{D}$ having π as a vertex is not always ∞–complete.

When this path $\underline{\gamma}$ is ∞–complete, there exists also a standard i.e.m. T_0 associated to $\underline{\gamma}$, and any two such T_0 are topologically conjugate. Let I_0 be the interval on which acts T_0. Then there exists a unique semiconjugacy from T to T_0, i.e. a continuous non–decreasing surjective map h from I onto I_0 such that $h \circ T = T_0 \circ h$.

2. Deformations of affine interval exchange maps

Let $\mathcal{D}$ be a Rauzy diagram on the alphabet $\mathcal{A}$ and let $\underline{\gamma}$ be an ∞–complete path in $\mathcal{D}$ issued from (π_t, π_b).

An affine i.e.m. with combinatorial data π is uniquely defined by the lengths $|I^t_\alpha|$ and $|I^b_\alpha|$ subjected to the only constant $\sum_\alpha |I^t_\alpha| = \sum_\alpha |I^b_\alpha|$.

Let $w \in \mathbb{R}^{\mathcal{A}}$. We will describe the set $\mathrm{Aff}\,(\underline{\gamma}, w)$ of the affine interval exchange maps whose orbit under the Rauzy–Veech algorithm is given by $\underline{\gamma}$ and with slope vector $\exp w$:

$$|I^b_\alpha| = \exp w_\alpha |I^t_\alpha| \ , \ \forall \alpha \in \mathcal{A} \ . \tag{1}$$

We denote by $\mathrm{Aff}^{(1)}\,(\underline{\gamma}, w)$ the set of affine i.e.m. in $\mathrm{Aff}\,(\underline{\gamma}, w)$ whose domain is $[0,1]$.

When $w = 0$ it is known ([Ka], [V1]) that the set of length vectors λ corresponding to a fixed Rauzy–Veech expansion $\underline{\gamma}$ is a simplicial cone of dimension $\leq g$ (where g is the genus of the surface associated to the diagram $\mathcal{D}$). In the remaining part of Section 2 we assume that $w \neq 0$.

2.1 The set $\mathrm{Aff}^{(1)}\,(\underline{\gamma}, w)$.

We will first determine a necessary and sufficient condition for $\mathrm{Aff}\,(\underline{\gamma}, w) \neq \emptyset$.

Lemma 1 *Let α_t, α_b the elements of $\mathcal{A}$ such that $\pi_t(\alpha_t) = \pi_b(\alpha_b) = d$. There exists an affine interval exchange map of slope $\exp w$ verifying $|I^t_{\alpha_t}| > |I^b_{\alpha_b}|$ if and only if the intersection*

$$\{\sum \lambda_\alpha w_\alpha = 0\} \cap \{\lambda_\alpha > 0 \, , \lambda_{\alpha_t} > \lambda_{\alpha_b}\}$$

is not empty.

Proof. There exists an affine i.e.m. of slope $\exp w$ verifying $|I^t_{\alpha_t}| > |I^b_{\alpha_b}|$ if and only if the hyperplane $\{\sum_\alpha |I^t_\alpha|(\exp w_\alpha - 1) = 0\}$ intersects the cone $\{|I^t_\alpha| > 0\,,\, |I^t_{\alpha_t}| > \exp w_{\alpha_b} |I^t_{\alpha_b}|\}$.

Let $a \neq 0$ in $\mathbb{R}^{\mathcal{A}}$. The hyperplane $\{\sum_\alpha a_\alpha x_\alpha = 0\}$ does not intersect the positive cone $x_\alpha > 0$ if and only if either all $a_\alpha \geq 0$ or all $a_\alpha \leq 0$.

Set first $x_\alpha = |I^t_\alpha|$ for $\alpha \neq \alpha_t$, $x_{\alpha_t} = |I^t_{\alpha_t}| - \exp(w_{\alpha_b})|I^t_{\alpha_b}|$, $a_\alpha = \exp w_\alpha - 1$ for $\alpha \neq \alpha_b$, $a_{\alpha_b} = \exp(w_{\alpha_t}) - \exp(-w_{\alpha_b})$. We have $\sum_\alpha a_\alpha x_\alpha = \sum_\alpha |I^t_\alpha|(\exp w_\alpha - 1)$. Therefore the hyperplane $\{\sum_\alpha |I^t_\alpha|(\exp w_\alpha - 1) = 0\}$ does not intersect the cone $\{|I^t_\alpha| > 0\,,\, |I^t_{\alpha_t}| > \exp w_{\alpha_b} |I^t_{\alpha_b}|\}$ iff

- either $\exp w_\alpha - 1 \geq 0$ for $\alpha \neq \alpha_b$ and $\exp w_{\alpha_t} - \exp(-w_{\alpha_b}) \geq 0$,
- or $\exp w_\alpha - 1 \leq 0$ for $\alpha \neq \alpha_b$ and $\exp w_{\alpha_t} - \exp(-w_{\alpha_b}) \leq 0$.

This is in turn respectively equivalent to

- $w_\alpha \geq 0$ for $\alpha \neq \alpha_b$ and $w_{\alpha_t} + w_{\alpha_b} \geq 0$,
- $w_\alpha \leq 0$ for $\alpha \neq \alpha_b$ and $w_{\alpha_t} + w_{\alpha_b} \leq 0$.

Take now $x_\alpha = \lambda_\alpha$ for $\alpha \neq \alpha_t$, $x_{\alpha_t} = \lambda_{\alpha_t} - \lambda_{\alpha_b}$; $a_\alpha = w_\alpha$ for $\alpha \neq \alpha_b$, $a_{\alpha_b} = w_{\alpha_t} + w_{\alpha_b}$. We have $\sum_\alpha a_\alpha x_\alpha = \sum_\alpha \lambda_\alpha w_\alpha$. Therefore the hyperplane $\{\sum_\alpha \lambda_\alpha w_\alpha = 0\}$ does not intersect $\lambda_\alpha > 0\,,\, \lambda_{\alpha_t} > \lambda_{\alpha_b}\}$ if and only if

- either $w_\alpha \geq 0$ for $\alpha \neq \alpha_b$ and $w_{\alpha_t} + w_{\alpha_b} \geq 0$,
- or $w_\alpha \leq 0$ for $\alpha \neq \alpha_b$ and $w_{\alpha_t} + w_{\alpha_b} \leq 0$.

We have shown that the negations of both statements considered in the Lemma are equivalent to the same set of inequalities. Hence the proof of the Lemma is complete. □

If an affine interval exchange map verifies (1) and $|I^t_{\alpha_t}| > |I^b_{\alpha_b}|$, one can apply a step of the Rauzy–Veech algorithm. The new affine i.e.m. $\hat{T}$ is the return map of T on $\cup_{\alpha \neq \alpha_b} I^b_\alpha$ and its slope vector $\exp \hat{w}$ is given by

$$\begin{aligned} \hat{w}_\alpha &= w_\alpha \ , \ \text{if}\, \alpha \neq \alpha_b \,, \\ \hat{w}_{\alpha_b} &= w_{\alpha_b} + w_{\alpha_t} \,. \end{aligned}$$

The corresponding lengths are

$$\begin{aligned} |\hat{I}^t_\alpha| &= |I^t_\alpha| \ , \ \text{if}\, \alpha \neq \alpha_t \,, \\ |\hat{I}^t_{\alpha_t}| &= |I^t_{\alpha_t}| - \exp(w_{\alpha_b})|I^t_{\alpha_b}| \,. \end{aligned}$$

It is easy to check that the maps $\hat{T}$ obtained in this way (as T varies) are determined by the only constraint

$$(1') \qquad |\hat{I}^b_\alpha| = \exp \hat{w}_\alpha |\hat{I}^t_\alpha| \,.$$

Similarly, the top Rauzy–Veech operation maps the set

$$\{\sum \lambda_\alpha w_\alpha = 0\,, \lambda_\alpha > 0\,,\, \lambda_{\alpha_t} > \lambda_{\alpha_b}\}$$

onto the set

$$\{\sum \hat{\lambda}_\alpha \hat{w}_\alpha = 0\,, \hat{\lambda}_\alpha > 0\}\ ,$$

where $\hat{\lambda}$ is connected to λ by the formulas of Section 1.2.

Lemma 1 and the subsequent discussion have a symmetric reformulation for the bottom Rauzy–Veech operation ($|I^t_{\alpha_t}| < |I^b_{\alpha_b}|$, $\lambda_{\alpha_t} < \lambda_{\alpha_b}$).

By applying several times the top or bottom versions of Lemma 1 and the subsequent discussion one obtains

Lemma 2. *Let $\underline{\gamma}^*$ be a finite initial segment of $\underline{\gamma}$. There exists an affine interval exchange map satisfying (1) whose orbit under the Rauzy–Veech algorithm begins with $\underline{\gamma}^*$ if and only if the set $\{\sum \lambda_\alpha w_\alpha = 0\,, \lambda_\alpha > 0\}$ contains a* standard *i.e.m. whose expansion under the the Rauzy–Veech algorithm begins with $\underline{\gamma}^*$.*

We now give a necessary and sufficient condition for Aff $(\underline{\gamma}, w)$ to be non empty.

Proposition *The set Aff$(\underline{\gamma}, w)$ is not empty if and only if the hyperplane $\{\sum \lambda_\alpha w_\alpha = 0\}$ contains a standard interval exchange map whose Rauzy–Veech expansion is equal to $\underline{\gamma}$. In this case, the set Aff$^{(1)}$ $(\underline{\gamma}, w)$, parametrized by the $|I^t_\alpha|$, is convex and compact.*

Proof. For γ an arrow of $\mathcal{D}$, we define a matrix $B_\gamma[w] \in \mathrm{SL}\,(\mathbb{R}^{\mathcal{A}})$ with nonnegative coefficients in the following way. Let $\pi = (\pi_t, \pi_b)$ be the origin of γ, $\alpha_t, \alpha_b \in \mathcal{A}$ such that $\pi_t(\alpha_t) = \pi_b(\alpha_b) = d$. If γ is of top type, set

$$B_\gamma[w] = \mathbb{I} + \exp w_{\alpha_b} E_{\alpha_b \alpha_t}\ .$$

If γ is of bottom type, set

$$B_\gamma[w] = \mathbb{I} + (\exp w_{\alpha_b} - 1) E_{\alpha_t \alpha_t} + \exp(-w_{\alpha_b}) E_{\alpha_t \alpha_b}\ .$$

Observe that $B_\gamma[0]$ is the matrix B_γ introduced in 1.3. The positive coefficients for $B_\gamma[w]$ and B_γ appear at the same positions. If T is an affine i.e.m. with combinatorial data π, slope $\exp w$ and $\hat{T}$ is deduced from T by the Rauzy–Veech operation associated to γ, the respective lengths $|I^t_\alpha|$, $|\hat{I}^t_\alpha|$ are related by

$$|I^t| =^t B_\gamma[w] |\hat{I}^t|\,,$$

in view of the formulas in the discussion following Lemma 1. If $\gamma = \gamma_1 \dots \gamma_l$ is a path in $\mathcal{D}$, we define

$$B_\gamma[w] = B_{\gamma_l}[w_{l-1}] \dots B_{\gamma_1}[w_0]\,,$$

with $w_0 = w$, $w_j = B_{\gamma_1 \dots \gamma_j}[w]$ for $j > 0$. If $\hat{T}$ is deduced from T by a sequence of Rauzy–Veech operations corresponding to γ, we still have

$$|I^t| =^t B_\gamma[w] |\hat{I}^t|\,.$$

Observe also that the positive coefficients of $B_\gamma[w]$ and $B_\gamma[0] = B_\gamma$ appear again at the same positions. Let now be $\underline{\gamma}$ be an ∞–complete path in $\mathcal{D}$. Let $\mathrm{Aff}\,(\underline{\gamma}(0,n), w)$ be the set of lengths (I^t_α) for affine i.e.m. T whose Rauzy–Veech expansion starts with the initial segment $\underline{\gamma}(0,n)$ of $\underline{\gamma}$. We have

$$\begin{aligned}
\mathrm{Aff}\,(\underline{\gamma}(0,n), w) &= {}^t\, B_{\underline{\gamma}(0,n)}[w]((\mathbb{R}^+)^{\mathcal{A}})\,,\\
\mathrm{Aff}\,(\underline{\gamma}(0,n), 0) &= {}^t\, B_{\underline{\gamma}(0,n)}((\mathbb{R}^+)^{\mathcal{A}})\,,\\
\mathrm{Aff}\,(\underline{\gamma}, w) &= \cap_{n\geq 0} \mathrm{Aff}\,(\underline{\gamma}(0,n), w)\,,\\
\mathrm{Aff}\,(\underline{\gamma}, 0) &= \cap_{n\geq 0} \mathrm{Aff}\,(\underline{\gamma}(0,n), 0)\,.
\end{aligned}$$

Let $n > m$ be such that $\gamma(m,n)$ is $(2d-3)$–complete. Then, as recalled in Section 1.3, all coefficients of $B_{\gamma(m,n)}$ are positive. Therefore, the same is true for $B_{\gamma(m,n)}[B_{\gamma(0,m)}w]$. We therefore have

$$\overline{\mathrm{Aff}\,(\underline{\gamma}(0,n), 0)} = {}^t\, B_{\gamma(0,n)}(\overline{(\mathbb{R}^+)^{\mathcal{A}}}) = {}^t\, B_{\gamma(0,m)}(\overline{{}^tB_{\gamma(m,n)}((\mathbb{R}^+)^{\mathcal{A}})} \subset \{0\} \cup \mathrm{Aff}\,(\gamma(0,m), 0)\,,$$

and similarly

$$\overline{\mathrm{Aff}\,(\underline{\gamma}(0,n), w)} \subset \{0\} \cup \mathrm{Aff}\,(\gamma(0,m), w)\;.$$

It follows that

$$\{0\} \cup \mathrm{Aff}\,(\underline{\gamma}, 0) = \cap_{n\geq 0} \overline{\mathrm{Aff}\,(\underline{\gamma}(0,n), 0)}\,, \tag{2}$$

$$\{0\} \cup \mathrm{Aff}\,(\underline{\gamma}, w) = \cap_{n\geq 0} \overline{\mathrm{Aff}\,(\underline{\gamma}(0,n), w)}\,. \tag{3}$$

We conclude that $\mathrm{Aff}\,(\underline{\gamma}, w)$ is nonempty if and only if $\mathrm{Aff}\,(\underline{\gamma}(0,n), w)$ is nonempty for all $n \geq 0$; by Lemma 2 this happens if and only if $\mathrm{Aff}\,(\underline{\gamma}(0,n), 0)$ intersects the hyperplane $\{\sum_\alpha \lambda_\alpha w_\alpha = 0\}$ for all $n \geq 0$; in view of the formula (2) above, this last condition holds if and only if the hyperplane $\{\sum_\alpha \lambda_\alpha w_\alpha = 0\}$ meets $\mathrm{Aff}\,(\underline{\gamma}, 0)$. This proves the first statement in the proposition.

The second statement follows from formula (3) and the fact that

$$\overline{\mathrm{Aff}\,(\underline{\gamma}(0,n), w)} = {}^t\, B_{\gamma(0,n)}[w](\overline{(\mathbb{R}^+)^{\mathcal{A}}})$$

is a closed convex cone for $n \geq 0$. □

When there exists a unique (up to rescaling) standard i.e.m. whose expansion under the Rauzy–Veech algorithm is $\underline{\gamma}$ the condition stated in the Proposition above means that the vector w belongs to the hyperplane

$$\{\sum \lambda_\alpha w_\alpha = 0\}\;.$$

In general, as already mentioned, $\mathrm{Aff}\,(\underline{\gamma}, 0)$ is a simplicial cone of dimension $r \le g$. Let us denote by $\lambda^{(1)}, \dots, \lambda^{(r)}$ the normalized extremal vectors of this simplicial cone. The necessary and sufficient condition which guarantees that $\mathrm{Aff}\,(\underline{\gamma}, w)$ is not empty is that the numbers

$$\sum_{\alpha \in \mathcal{A}} \lambda_\alpha^{(j)} w_\alpha \ , \quad j = 1, \dots, r$$

are neither all strictly positive, nor all strictly negative.

Remark. For fixed combinatorial data, normalized affine i.e.m. form a manifold of dimension $(2d-2)$, and the standard i.e.m. have dimension $(d-1)$. As almost all i.e.m. are uniquely ergodic, one can think that $(d-1)$ is also the "dimension" od the set of paths $\underline{\gamma}$. When $\underline{\gamma}$ corresponds to the uniquely ergodic standard i.e.m. , the constraint $\sum_\alpha \lambda_\alpha w_\alpha = 0$ defines a $(d-1)$ dimensional space. Therefore one can expect that for most $(\underline{\gamma}, w)$ the set $\mathrm{Aff}^{(1)}\,(\underline{\gamma}, w)$ is of dimension $(2d-2)-(d-1)-(d-1) = 0$. As $\mathrm{Aff}^{(1)}\,(\underline{\gamma}, w)$ is convex and compact this would mean that $\mathrm{Aff}^{(1)}\,(\underline{\gamma}, w)$ is reduced to a point. The problem with this heuristic argument is that the map which associates to an affine i.e.m. T its "rotation number" $\underline{\gamma}$ is not smooth.

2.2 Affine motions.

Let $w \neq 0$ and $T^* \in \mathrm{Aff}^{(1)}\,(\underline{\gamma}, 0)$ such that

$$\sum_\alpha \lambda_\alpha^* w_\alpha = 0 \ .$$

We choose an affine i.e.m T_0 in the intrinsic interior of the nonempty compact convex set $\mathrm{Aff}^{(1)}\,(\underline{\gamma}, w)$. There exists a unique semiconjugacy H of T_0 towards T^*.

We denote by u_i^b, u_i^t $(1 \le i \le d-1)$ the singularities of T_0^{-1} and T_0 respectively.

Let $(T_s)_{s \in (-1,+1)}$ be an open segment passing through T_0 and contained in $\mathrm{Aff}^{(1)}\,(\underline{\gamma}, w)$, with an affine parametrization. Let $u_i^t(s)$ and $u_i^b(s)$ denote the singularities of T_s and T_s^{-1} respectively. Since the parametrization is affine we can write

$$u_i^t(s) = u_i^t + s\nu(u_i^t)\ , \ s \in (-1,+1)\ ,$$
$$u_i^b(s) = u_i^b + s\nu(u_i^b)\ , \ s \in (-1,+1)\ ,$$

with certain numbers $\nu(u_i^t), \nu(u_i^b)$ (note that the u_i^t are all distinct from the u_i^b since T_0 has no connection). Since all the maps T_s are semi–conjugate to T^*, we can also write

$$u_{i,n}^t(s) = u_{i,n}^t + s\nu(u_{i,n}^t)\ , \ n \le 0\ ,$$
$$u_{i,n}^b(s) = u_{i,n}^b + s\nu(u_{i,n}^b)\ , \ n \ge 0\ ,$$

where we have set

$$\begin{aligned} u^t_{i,n}(s) &= T^n_s(u^t_i(s))\,,\; n \le 0 \\ u^t_{i,n} &= T^n_0(u^t_i)\,,\; n \le 0 \\ u^b_{i,n}(s) &= T^n_s(u^b_i(s))\,,\; n \ge 0 \\ u^b_{i,n} &= T^n_0(u^b_i)\,,\; n \ge 0 \end{aligned}$$

Let

$$Z = \{u^t_{i,n}, u^b_{j,m}\,,\; n \le 0, m \ge 0\,, 1 \le i,j \le d-1\} \cup \{0,1\}\;\;,$$

and let $\nu(0) = \nu(1) = 0$.

In analogy with the notion of holomorphic motions [MSS], we will say that one has an *affine motion* for the set Z parametrized by the interval $(-1,+1)$: for each $s \in (-1,+1)$ the map h_s

$$\begin{aligned} Z &\hookrightarrow [0,1] \\ u^t_{i,n} &\mapsto u^t_{i,n}(s) \\ u^b_{j,m} &\mapsto u^b_{j,m}(s) \end{aligned}$$

is *injective* and the dependence w.r.t. s is affine. The application ν (or rather its derivative) plays the role of a "Beltrami form" .

Proposition 1. *The map $\nu : Z \to \mathbb{R}$ is 1–Lipschitz.*

Proof. Indeed if this not true there exists x_0, x_1 with $|\nu(x_0) - \nu(x_1)| > |x_0 - x_1|$. Then the maps $s \to x_0 + \nu(x_0)s$, $s \to x_1 + \nu(x_1)s$ are equal at the point $s^* = -(x_1 - x_0)/(\nu(x_1) - \nu(x_0)) \in (-1,+1)$ which contradicts the injectivity of h_{s^*}. □

Extending by continuity ν to $\overline{Z}$ we obtain an affine motion of $\overline{Z}$. If $\overline{Z} \ne [0,1]$, i.e. if T_0 has a wandering interval, one can extend the affine motion to the whole interval $[0,1]$ by linear interpolation, i.e. one extends ν to $[0,1]$ in such a way that ν is affine on each component of $[0,1] \setminus \overline{Z}$. This extension of ν to $[0,1]$ is still 1–Lipschitz.

This leads to a one–parameter family $(h_s)_{s\in(-1,+1)}$ of homeomorphisms of $[0,1]$. By construction, h_s is a conjugacy between T_0 and T_s:

$$\begin{aligned} T_s(h_s(x)) &= h_s(T_0(x))\,,\; x \ne u^t_i\,, \\ T_s^{-1}(h_s(x)) &= h_s(T_0^{-1}(x))\,,\; x \ne u^b_j\;. \end{aligned}$$

Let χ_α denote the (constant) value of $\frac{\partial}{\partial s} T_s|_{s=0}$ on $I^t_\alpha(T_0)$. If we derive the above relations w.r.t. s we obtain

$$\nu(T_0(x)) = \nu(x)\exp w_\alpha + \chi_\alpha\,,\; x \in I^t_\alpha(T_0)\;.$$

Since ν is 1–Lipschitz, its derivative (in the sense of distributions) is a function in $L^\infty([0,1])$. It verifies

$$D\nu(T_0(x)) = D\nu(x)\,,$$
$$\|D\nu\|_{L^\infty} \le 1\ .$$

Moreover, since one has extended ν by linear interpolation to all wandering intervals of T_0, $D\nu$ is *constant* on any wandering interval. Finally, as $\nu(0) = \nu(1) = 0$, $D\nu$ has zero mean.

Conversely, let us suppose that one has a function $\mu \in L^\infty([0,1])$ which verifies

- $\|\mu\|_{L^\infty} \le 1$;
- μ has zero mean;
- μ is T_0–invariant;
- μ is constant on each wandering interval.

Then one can realize a segment $(T_s)_{s\in(-1,+1)}$ in $\mathrm{Aff}^{(1)}(\underline{\gamma}, w)$: we denote by ν the primitive of μ which vanishes at 0 and 1. The function ν is 1–Lipschitz on $[0,1]$. For $s \in (-1,+1)$ one defines $h_s : [0,1] \to \mathbb{R}$ by

$$h_s(x) = x + s\nu(x)\ .$$

One has $h_s(0) = 0$, $h_s(1) = 1$; h_s is continuous since ν is continuous and it is *injective* since ν is 1–Lipschitz; thus h_s is a homeomorphism of $[0,1]$. One defines T_s by

$$T_s(y) = h_s \circ T_0 \circ h_s^{-1}(y)$$

if $y \ne h_s(u_j^t)$. T_s is a generalized i.e.m. conjugate to T_0. The T_0–invariance of μ implies that T_s is *affine* and belongs to $\mathrm{Aff}^{(1)}(\underline{\gamma}, w)$. Finally, since the $h_s(u_j^t)$, $h_s(u_j^b)$ have an affine dependence on s the parametrization we have obtained is also affine. Summarizing:

Proposition 2. *The tangent space to $Aff^{(1)}(\underline{\gamma}, w)$ at T_0 is canonically identified with the vector space of L^∞ functions on $[0,1]$ which are T_0–invariant, constant on each wandering interval of T_0 and have zero mean.*

It is easy to compute the dimension $r-1$ of this tangent space in terms of the "ergodic components" of T_0. Indeed we will have $r = r_d + r_c$ with:

- r_d is the number of orbits of (maximal) wandering intervals of T_0;
- $r_c > 0$ if and only if $\mathrm{Leb}(\overline{Z}) > 0$; if this is the case, one has a partition $\overline{Z} = Z_1 \sqcup \ldots \sqcup Z_{r_c}$ of $\overline{Z}$ into T_0–invariant sets, of positive Lebesgue measure, and ergodic (i.e. the restriction of the quasi–invariant Lebesgue measure to Z_i is ergodic).

3. Wandering intervals for affine interval exchange maps

3.1 The Zorich cocycle

Let $\mathcal{R}$ be a Rauzy class on an alphabet $\mathcal{A}$, $\mathcal{D}$ the associated Rauzy diagram. For $T = T_{\pi,\lambda}$ a standard i.e.m. acting on some interval I with combinatorial data $\pi \in \mathcal{R}$, define E_T to be the vector space of functions on I which are constant on each subinterval I^t_α. This vector space is canonically isomorphic to $\mathbb{R}^{\mathcal{A}}$. Let $\hat{T} = T_{\hat{\pi},\hat{\lambda}}$ be the i.e.m. deduced from T by one step of the Rauzy–Veech algorithm, let γ be the corresponding arrow from π to $\hat{\pi}$ in $\mathcal{D}$, let $\hat{I}$ be the interval on which $\hat{T}$ acts and $\hat{I}^t_\alpha$ the associated subintervals. For $\varphi \in E_T$, one defines a function $\hat{\varphi} \in E_{\hat{T}}$ by

$$\hat{\varphi}(x) = \sum_{i=0}^{q(x)-1} \varphi(T^i x) ,$$

where $q(x)$ is the return time of x in $\hat{I}$ (equal to 1 or 2). The matrix of the linear map $\varphi \mapsto \hat{\varphi}$ from E_T to $E_{\hat{T}}$ in the canonical bases of these spaces is B_γ.

At the projective level, the fibered map

$$(\pi, \lambda, \varphi) \mapsto (Q_{\mathrm{RV}}(\pi, \lambda), B_\gamma \varphi) ,$$
$$\mathcal{R} \times \mathbb{P}((\mathbb{R}^+)^{\mathcal{A}}) \times \mathbb{R}^{\mathcal{A}} \to \mathcal{R} \times \mathbb{P}((\mathbb{R}^+)^{\mathcal{A}}) \times \mathbb{R}^{\mathcal{A}} ,$$

is called the *extended Zorich cocycle* over the Rauzy–Veech dynamics Q_{RV}.

There is an invariant subbundle under this cocycle whose fiber over (π, λ) is

$$H(\pi) = \mathrm{Im}\, \Omega(\pi) .$$

Indeed, we have

$$B_\gamma \Omega(\pi) = \Omega(\hat{\pi})\, {}^t B_\gamma^{-1} .$$

It also follows that the restriction to the cocycle to this subbundle, called the *Zorich cocycle*, is symplectic (for the symplectic form defined by the $\Omega(\pi)$). To analyze the extended Zorich cocycle, one goes to the accelerated dynamics Q_{Z}, i.e. one reparametrizes the time in the algorithm in order to apply the Oseledets multiplicative ergodic theorem. Then, the Lyapunov exponents on the quotient $\mathbb{R}^{\mathcal{A}}/H(\pi)$ are all equal to zero. Avila–Viana ([AV], see also [Fo]) have proved that the Lyapunov exponents on $H(\pi)$ are all simple, hence by symplecticity they can be written as

$$\theta_1 > \theta_2 > \ldots > \theta_g > -\theta_g > \ldots > -\theta_1 .$$

Here $g = \frac{1}{2}\dim H(\pi)$ is the genus of the surface obtained by suspension. Associated to these exponents, we have for almost all T a filtration

$$E_T = \mathbb{R}^{\mathcal{A}} = E_0 \supset E_1 \supset \ldots \supset E_g ,$$

with $\dim E_i = d - i$. Here, we have

$$E_1 = \{\varphi \in E_T \, , \, \int_I \varphi(x)dx = 0\} \, .$$

3.2 Statement of the main result

We assume $g \geq 2$. We recall the statement of the Theorem in the introduction.

Theorem. *For all vertices π of $\mathcal{D}$, for almost all $\lambda \in (\mathbb{R}^+)^{\mathcal{A}}$, for any $w \in E_1(\pi,\lambda) \setminus E_2(\pi,\lambda)$, there exists an affine i.e.m. $T^* = T^*_{\pi,\lambda,w}$ with the following properties:*
(i) $T^* \in Aff(\underline{\gamma}, w)$;
(ii) T^* has a wandering interval.

Remarks.
1. For almost all (π,λ), $T_{\pi,\lambda}$ is uniquely ergodic; then $w \in E_1(\pi,\lambda)$ is a necessary condition for an affine i.e.m. to satisfy (i).
2. Actually the proof of the theorem shows that any affine i.e.m. in Aff $(\underline{\gamma}, w)$ has a wandering interval: see the remark at the end of Section 3.7. Moreover, in view of this remark and of the remark at the end of Section 2.1, it appears very probable that there is up to scaling only one affine i.e.m. in Aff $(\underline{\gamma}, w)$

3.3 Reduction to a statement on Birkhoff sums

3.3.1 The main step in the proof of the theorem is the following result

Proposition. *For all vertices π of $\mathcal{D}$, for almost all $\lambda \in (\mathbb{R}^+)^{\mathcal{A}}$, for all $w \in E_1(\pi,\lambda) \setminus E_2(\pi,\lambda)$, there exists x^*, not in the orbits of the singularities of $T^{\pm 1}_{\pi,\lambda}$, such that the Birkhoff sums of w at x^* satisfy, for all $\varepsilon > 0$ and a constant $C(\varepsilon) > 0$ independent of $n \in \mathbb{Z}$,*

$$S_n w(x^*) \leq C(\varepsilon) - |n|^{\theta_2/\theta_1 - \varepsilon} \, .$$

The Birkhoff sums are here defined as usual as

$$S_n w(x^*) = \begin{cases} \sum_{i=0}^{n-1} w_{\beta_i} & \text{for } n \geq 0, \\ -\sum_{i=n}^{-1} w_{\beta_i} & \text{for } n < 0, \end{cases}$$

with $T^i_{\pi,\lambda}(x^*) \in I^t_{\beta_i}$.

3.3.2 The theorem follows from the proposition by the usual Denjoy construction. Let π, λ, w, x^* be as in the Proposition and $I^{(0)}$ be the interval of definition of $T_{\pi,\lambda}$. Define, for $n \in \mathbb{Z}$

$$l_n = \exp\{S_n w(x^*)\} .$$

From the Proposition it follows that

$$L = \sum_{n\in\mathbb{Z}} l_n < +\infty .$$

For $x \in I^{(0)}$ set

$$l^-(x) = \sum_{T^n_{\pi,\lambda}(x^*)<x} l_n ,$$

$$l^+(x) = \sum_{T^n_{\pi,\lambda}(x^*)\leq x} l_n ,$$

and let $h : [0, L] \to I^{(0)}$ be the continuous non decreasing map such that

$$h^{-1}(x) = [l^-(x), l^+(x)] .$$

One then defines the affine i.e.m. T^* on $[0, L]$ by

- $T^*(l^\pm(x)) = l^\pm(T_{\pi,\lambda}(x))$,
- when $l^-(x) < l^+(x)$, T^* is affine from the interval $[l^-(x), l^+(x)]$ onto the interval $[l^-(T_{\pi,\lambda}(x)),l^+(T_{\pi,\lambda}(x))]$.

Then, the fact that T^* is an affine i.e.m. with the required slopes follow from the definition of the l_i. The semi–conjugacy to $T_{\pi,\lambda}$ is built in the construction (using also that $T_{\pi,\lambda}$ is minimal). Finally, the interval $h^{-1}(x^*)$ is wandering.

3.4 Limit shapes for Birkhoff sums

3.4.1 In order to prove the Proposition in 3.3.1, we construct some functions closely related to the Zorich cocycle. Such functions have also been considered in a different setting in [BHM]. Instead of acting on (π, λ) we consider the natural extension of the Rauzy–Veech dynamics (and the Zorich acceleration) acting on (π, λ, τ), where $\tau \in \mathbb{R}^{\mathcal{A}}$ is a suspension datum satisfying the usual conditions (for $1 \leq k \leq d$)

$$\sum_{\pi_t\alpha<k} \tau_\alpha > 0 , \quad \sum_{\pi_b\alpha<k} \tau_\alpha < 0 .$$

Instead of a filtration

$$E_0 = \mathbb{R}^{\mathcal{A}} \supset E_1(\pi, \lambda) \supset E_2(\pi, \lambda) \supset \ldots$$

as above, we get from Oseledets theorem 1–dimensional subspaces $F_i(\pi, \lambda, \tau)$ associated to the Lyapunov exponent θ_i, generated by a vector in $E_{i-1}(\pi, \lambda) \setminus E_i(\pi, \lambda)$. Moreover the sums $\bigoplus_{j=1}^{i} F_j(\pi, \lambda, \tau)$ *depend only on* (π, τ). (This is the subspace of vectors decreasing in the past under the Zorich cocycle at a rate at least $-\theta_i$).

In particular F_1 depends only on (π, τ), not on λ; because the matrices B of the Zorich cocycle only have non negative entries (and positive entries after appropriate iteration), the subspace $F_1(\pi, \tau)$ is contained in the positive cone $(\mathbb{R}^+)^{\mathcal{A}}$; we write $q(\pi, \lambda)$ for a *positive* vector generating $F_1(\pi, \tau)$, normalized by

$$\sum_{\alpha} q_\alpha^2(\pi, \tau) = 1\,.$$

Next, we consider the 2–dimensional subspace $F_1 \oplus F_2$, depending only on (π, τ): we choose a vector $v(\pi, \tau)$ satisfying

$$\sum_{\alpha} v_\alpha^2(\pi, \tau) = 1\,,$$

$$\sum_{\alpha} v_\alpha(\pi, \tau) q_\alpha(\pi, \tau) = 0\,.$$

There are two choices for v, differing by a sign, both of them being relevant in the following; we fix such a choice.

From q and v, it is easy to find a generator w for $F_2(\pi, \lambda, \tau)$. Indeed we have

$$F_2(\pi, \lambda, \tau) \subset E_1(\pi, \lambda)\,,$$

with

$$E_1(\pi, \lambda) = \{w\,, \sum_{\alpha} \lambda_\alpha w_\alpha = 0\}\,.$$

Therefore, we will take

$$w(\pi, \lambda, \tau) = v(\pi, \tau) - t(\pi, \lambda, \tau) q(\pi, \tau)$$

with

$$t(\pi, \lambda, \tau) = \frac{\langle \lambda, v \rangle}{\langle \lambda, q \rangle}\,.$$

Proposition. *For almost all* (π, λ, τ) *and all* $(n_\alpha) \in \mathbb{N}^{\mathcal{A}}$*, not all equal to* 0*, we have*

$$\sum_{\alpha} n_\alpha w_\alpha(\pi, \lambda, \tau) \neq 0\,.$$

Proof. Indeed, fixing (n_α), we have

$$\sum_\alpha n_\alpha w_\alpha = 0 \Leftrightarrow t = \frac{\langle n, v\rangle}{\langle n, q\rangle}$$

where $\langle n, q\rangle > 0$ as $n_\alpha \geq 0$, $q_\alpha > 0$. In view of the formula for t, for fixed (π, λ) this happens with measure 0 w.r.t. λ. The conclusion follows by Fubini's theorem. □

3.4.2 The functions $V_\alpha(\pi, \lambda)$. Let (π, τ) be a typical point (for backward time Rauzy–Veech–Zorich dynamics). Let $(\pi^{(-n)}, \tau^{(-n)})$ be its backwards orbit for the Rauzy–Veech dynamics. Let $q^{(-n)}(\pi, \tau)$, $v^{(-n)}(\pi, \tau)$ be the images of $q(\pi, \tau)$, $v(\pi, \tau)$ under the Zorich cocycle. From the invariance of F_1 and $F_1 \oplus F_2$ w.r.t. the Zorich cocycle we can write

$$\begin{aligned} q^{(-n)}(\pi, \tau) &= \Theta_1^{(-n)} q(\pi^{(-n)}, \tau^{(-n)}) \, , \\ v^{(-n)}(\pi, \tau) &= \Theta_2^{(-n)} v(\pi^{(-n)}, \tau^{(-n)}) + \Theta^{(-n)} q(\pi^{(-n)}, \tau^{(-n)}) \, , \end{aligned}$$

where $\Theta_1^{(-n)}$, $\Theta_2^{(-n)}$ and $\Theta^{(-n)}$ are real numbers depending on π, τ, n, $\Theta_1^{(-n)} > 0$. We will always make a coherent choice for the vectors $v(\pi^{(-n)}, \tau^{(-n)})$ along an orbit in order to have $\Theta_2^{(-n)} > 0$. The coefficient $\Theta_1^{(-n)}$ is exponentially small (in Zorich reparametrized time) at rate θ_1, $|\Theta_2^{(-n)}|$ is exponentially small at rate θ_2, and $|\Theta^{(-n)}|$ is at most exponentially small at rate θ_2.

Let $u^{(-n)}(\pi, \tau) = (q^{(-n)}(\pi, \tau), v^{(-n)}(\pi, \tau))$. According to the definition of the Zorich cocycle, we have

$$u_\beta^{(-n)} = u_\beta^{(-n-1)} \, ,$$

if β is *not* the loser of the arrow from $\pi^{(-n-1)}$ to $\pi^{(-n)}$ and

$$u_{\beta_l}^{(-n)} = u_{\beta_l}^{(-n-1)} + u_{\beta_w}^{(-n-1)} \, ,$$

if β_l (resp. β_w) is the loser (resp. the winner) of this arrow.

For $\alpha \in \mathcal{A}$, let $\Gamma_\alpha^{(-n)}$ be the broken line in $\mathbb{R}^2$ starting at the origin and obtained by adding successively the vectors $u_{\beta_i}^{(-n)}$, where $\beta_0, \beta_1, \ldots$ are defined as follows: if $T^{(0)}$ is any i.e.m. with combinatorial data $\pi^{(0)}$, and $T^{(-n)}$ is the i.e.m. whose n–times Rauzy–Veech induction is $T^{(0)}$, we have

$$[T^{(-n)}]^i (I_\alpha^{(0)}) \subseteq I_{\beta_i}^{(-n)} \, .$$

Here, i runs from 0 to the return time of $I_\alpha^{(0)}$ in $I^{(0)}$.

In other terms, $\beta_0, \beta_1, \dots$ is the itinerary of $I_\alpha^{(0)}$ with respect to the partition of $I^{(-n)}$ by the $I_\beta^{(-n)}$. When we go one step further to $T^{(-n-1)}$ on $I^{(-n-1)}$, the new itinerary is obtained by replacing β_l by $\beta_l\beta_w$ or $\beta_w\beta_l$ (depending whether the arrow from $\pi^{(-n-1)}$ to $\pi^{(-n)}$ has top or bottom type).

Consequently, the vertices of $\Gamma_\alpha^{(-n)}$ are also vertices of $\Gamma_\alpha^{(-n-1)}$. The following properties are now clear:

1. $\Gamma_\alpha^{(-n)}$ is the graph of a piecewise affine continuous map $V_\alpha^{(-n)}(\pi,\tau)$ on $[0, q_\alpha(\pi,\tau)]$ satisfying

$$V_\alpha^{(-n)}(\pi,\tau)(0) = 0\,,$$
$$V_\alpha^{(-n)}(\pi,\tau)(q_\alpha(\pi,\tau)) = v_\alpha(\pi,\tau)\,.$$

   (In particular $V_\alpha^{(0)}(\pi,\tau)$ is the affine map on $[0, q_\alpha(\pi,\tau)]$ with these boundary values).
2. The vertices of $\Gamma_\alpha^{(-n)}$ are also vertices of $\Gamma_\alpha^{(-n-1)}$.

From the behaviour of the coefficients $\Theta_1^{(-n)}$, $\Theta_2^{(-n)}$ and $\Theta^{(-n)}$ it also follows that

3. The sequence $V_\alpha^{(-n)}(\pi,\tau)$ converges uniformly exponentially fast (with respect to Zorich reparametrized time) at rate θ_2 to a continuous function $V_\alpha(\pi,\tau)$ on $[0, q_\alpha(\pi,\tau)]$ (with the same boundary values).
4. The function $V_\alpha(\pi,\tau)$ satisfies a Hölder condition of exponent θ, for any $\theta < \theta_2/\theta_1$.

We also define the following function $V_*(\pi,\tau)$: if α_b, α_t are the last letter of the bottom, top lines of π, we set:

$$V_*(\pi,\tau)(x) = \begin{cases} V_{\alpha_b}(\pi,\tau)(x) & \text{if } 0 \le x \le q_{\alpha_b}\,, \\ V_{\alpha_t}(\pi,\tau)(x - q_{\alpha_b}) + v_{\alpha_b} & \text{if } q_{\alpha_b} \le x \le q_{\alpha_b} + q_{\alpha_t}\,, \end{cases}$$

(with $q_{\alpha_b} = q_{\alpha_b}(\pi,\tau)$, etc.).

3.4.3 The functions $W_\alpha(\pi,\lambda,\tau)$. For π,τ as above, $\alpha \in \mathcal{A}$, $\lambda \in (\mathbb{R}^+)^{\mathcal{A}}$, we can perform with respect to the vector $w(\pi,\lambda,\tau) = v(\pi,\tau) - t(\pi,\lambda,\tau)q(\pi,\tau)$ of Section 3.4.1 the same construction that we did for $v(\pi,\tau)$. We denote by $w^{(-n)}(\pi,\lambda,\tau)$ the image of $w(\pi,\lambda,\tau)$ under the Zorich cocycle and we have

$$w^{(-n)}(\pi,\lambda,\tau) = \Theta_2^{(-n)} w(\pi^{(-n)}, \lambda^{(-n)}, \tau^{(-n)})\,.$$

We obtain functions $W_\alpha(\pi,\lambda,\tau)$, $W_*(\pi,\lambda,\tau)$ which are related to the previous ones by

$$W_\alpha(\pi,\lambda,\tau)(x) = V_\alpha(\pi,\tau)(x) - t(\pi,\lambda,\tau)x\,,$$
$$W_*(\pi,\lambda,\tau)(x) = V_*(\pi,\tau)(x) - t(\pi,\lambda,\tau)x\,.$$

3.4.4 Relation to Birkhoff sums. Let $\alpha \in \mathcal{A}$. Denote as above by $(\beta_0, \beta_1, \dots)$ the itinerary of $I_\alpha^{(0)}$ with relation to the partition $I_\beta^{(-n)}$ till its return to $I^{(0)}$.

Consider the Birkhoff sums

$$S_\alpha q^{(-n)}(i) = \sum_{j=0}^{i-1} q_{\beta_j}^{(-n)}(\pi, \tau)\,,$$

$$S_\alpha w^{(-n)}(i) = \sum_{j=0}^{i-1} w_{\beta_j}^{(-n)}(\pi, \lambda, \tau)\,.$$

We have then by definition of $\Gamma^{(-n)}$ (for $W(\pi, \lambda, \tau)$)

$$W_\alpha(S_\alpha q^{(-n)}(i)) = S_\alpha w^{(-n)}(i)\ .$$

If instead we look at the Birkhoff sums

$$S_\alpha q(i) = \sum_{j=0}^{i-1} q_{\beta_j}(\pi^{(-n)}, \tau^{(-n)})\,,$$

$$S_\alpha w(i) = \sum_{j=0}^{i-1} w_{\beta_j}(\pi^{(-n)}, \lambda^{(-n)}, \tau^{(-n)})\,,$$

we will have, in view of the relation between $q^{(-n)}, w^{(-n)}$ and q, w:

$$S_\alpha q(i) = (\Theta_1^{(-n)})^{-1} S_\alpha q^{(-n)}(i)\,,$$

$$S_\alpha w(i) = (\Theta_2^{(-n)})^{-1} S_\alpha w^{(-n)}(i)\,,$$

hence

$$S_\alpha w(i) = (\Theta_2^{(-n)})^{-1} W_\alpha(\Theta_1^{(-n)} S_\alpha q(i))\,.$$

In view of this formula one can think of W_α as the “limit shape” for the Birkhoff sum of w.

3.4.5 Functional equation. Here we relate the $W_\alpha(\pi, \lambda, \tau)$ to the $W_\alpha(\pi^{(-1)}, \lambda^{(-1)}, \tau^{(-1)})$. The relation is a consequence of the formulas

$$q^{(-1)}(\pi, \tau) = \Theta_1^{(-1)} q(\pi^{(-1)}, \tau^{(-1)})\,,$$

$$w^{(-1)}(\pi, \lambda, \tau) = \Theta_2^{(-1)} w(\pi^{(-1)}, \lambda^{(-1)}, \tau^{(-1)})\,.$$

Indeed, *if α is not the loser* of the arrow from $\pi^{(-1)}$ to $\pi^{(0)}$, we obtain

$$W_\alpha(\pi, \lambda, \tau)(x) = \Theta_2^{(-1)} W_\alpha(\pi^{(-1)}, \lambda^{(-1)}, \tau^{(-1)}) \left(\frac{x}{\Theta_1^{(-1)}} \right)\ .$$

If α is the loser of this arrow, we obtain

$$W_{\alpha_l}(\pi,\lambda,\tau)(x) = \Theta_2^{(-1)} W_*(\pi^{(-1)},\lambda^{(-1)},\tau^{(-1)}) \left(\frac{x}{\Theta_1^{(-1)}} \right) .$$

3.5 On the direction of w

Recall that in Section 3.3.1 we want to bound from above the Birkhoff sums of w at some point x^*. In Section 3.4.4 we have related the Birkhoff sums of $w(\pi^{(-n)},\lambda^{(-n)},\tau^{(-n)})$ to the limit shape $W_\alpha(\pi,\lambda,\tau)$. In Section 3.7 the point x^* will be defined using the maximum of $W_\alpha(\pi^{(n)},\lambda^{(n)},\tau^{(n)})$ (for $n > 0$ large). Therefore we need to compare these functions $W_\alpha(\pi^{(n)},\lambda^{(n)},\tau^{(n)})$ to their maximum values. In order to do this, the Proposition below is a crucial technical step.

3.5.1 The Rauzy operations R_t, R_b in $\mathcal{R}$ do not change the first letter of the bottom and top lines of elements of $\mathcal{R}$. So there is a letter $a \in \mathcal{A}$ which is the first letter in the top line of any element of $\mathcal{R}$. Consider the set Υ of (π,λ,τ) with $\pi \in \mathcal{R}$, $\lambda \in (\mathbb{R}^+)^{\mathcal{A}}$, $\tau \in \Theta_\pi$, which satisfy the following properties

(i) a is the last letter of the bottom line of π;

(ii) a is the loser of the next step of the Rauzy–Veech algorithm for (π,λ,τ): if α is the last letter of the top line of π, we have $\lambda_\alpha > \lambda_a$;

(iii) $w_a(\pi,\lambda,\tau)(w_a(\pi,\lambda,\tau) + w_\alpha(\pi,\lambda,\tau)) < 0$.

Here $w(\pi,\lambda,\tau)$ is the vector associated to the exponent θ_2 defined in 3.4.1. There were two possible choices for w but obviously property (iii) does not depend on this choice. Observe also that there are elements $\pi \in \mathcal{R}$ satisfying (i): since the Rauzy–Veech expansion of a standard i.e.m. with no connections produces an ∞–complete path (see e.g. [Y1]) the letter a must be the winner of at least one arrow in $\mathcal{D}$ and this can only occur when a is the last letter of the bottom line.

Proposition. *The set Υ has positive measure.*

Proof. The rest of this Section 3.5 is devoted to the proof of this assertion.

3.5.2 Recall that

$$w(\pi,\lambda,\tau) = v(\pi,\tau) - \frac{<\lambda, v>}{<\lambda, q>} q(\pi,\tau) .$$

In view of (ii), the vector λ is allowed to vary in a convex cone whose extremal vectors $\lambda^{(\beta)}$ are given by

- $\lambda_\gamma^{(\beta)} := \delta_{\gamma\beta}\,,\ \beta \neq a$
- $\lambda_\gamma^{(a)} := \delta_{\gamma a} + \delta_{\gamma\alpha}\,,$

where $\delta_{\gamma\beta}$, $\delta_{\gamma a}$ and $\delta_{\gamma\alpha}$ denote the Kronecker symbol. The corresponding values for w_a are

$$\bullet\, v_a - \frac{v_\beta}{q_\beta} q_a \,,\ \beta \neq a$$

$$\bullet\, v_a - \frac{v_\alpha + v_a}{q_\alpha + q_a} q_a \,.$$

We see that these values have the same sign if and only if $\frac{v_a}{q_a}$ is either larger than all other $\frac{v_\beta}{q_\beta}$ or smaller than these quantities. Furthermore, if a change of sign of w_a occurs, we want that $w_a + w_\alpha$ does not change sign at the same time, and this occurs if and only if $\frac{v_a}{q_a} = \frac{v_\alpha}{q_\alpha}$. We will prove below the following two results

Proposition 1. *Let $\pi \in \mathcal{R}$ such that a is the first top letter and last bottom letter of π. For all $\alpha \in \mathcal{A}$, $\alpha \neq a$ and almost all τ we have*

$$v_a(\pi,\tau) q_\alpha(\pi,\tau) - v_\alpha(\pi,\tau) q_a(\pi,\tau) \neq 0\,.$$

Proposition 2. *There exist $\pi \in \mathcal{R}$, with last bottom letter a, letters b, c and a positive measure set of τ on which*

$$\frac{v_c}{q_c} < \frac{v_a}{q_a} < \frac{v_b}{q_b}\,.$$

These two propositions do indeed imply that Υ has positive measure. Let a, b, c, π, τ be as in Proposition 2; almost surely the conclusion of Proposition 1 is also satisfied. We have $w_a(\pi,\lambda,\tau) < 0$ if and only if the linear form $l(\lambda) =< \lambda, v > -\frac{v_a}{q_a} < \lambda, q >$ is positive and $w_a(\pi,\lambda,\tau) + w_\alpha(\pi,\lambda,\tau) < 0$ if and only if the linear form $\tilde{l}(\lambda) =< \lambda, v > -\frac{v_a + v_\alpha}{q_a + q_\alpha} < \lambda, q >$ is positive (here α is the last letter in the top line of π). One has $l(\lambda^{(b)}) > 0$, $l(\lambda^{(c)}) < 0$. Moreover, l and $\tilde{l}$ are not proportional thus there exists a set of λ of positive measure where $l(\lambda)\tilde{l}(\lambda) < 0$. This concludes the proof of the Proposition. □

Obviously, *the statement obtained from the Proposition in 3.5.1 and the Propositions 1 and 2 in 3.5.2 by exchanging the role of the top and bottom lines are also true.*

3.5.3 Proof of Proposition 1. It is based on the *twisting property* of the Rauzy monoid proved by A. Avila and M. Viana [AV]. Let us recall the content of this property. For $\pi \in \mathcal{R}$, be the antisymmetric matrix $\Omega(\pi)$ has been defined by

$$\Omega_{\beta\gamma}(\pi) = \begin{cases} 1 & \text{if } \pi_t\beta < \pi_t\gamma \,,\ \pi_b\beta > \pi_b\gamma \,, \\ -1 & \text{if } \pi_t\beta > \pi_t\gamma \,,\ \pi_b\beta < \pi_b\gamma \,, \\ 0 & \text{otherwise.} \end{cases}$$

The subspaces $H(\pi) = \operatorname{Im}\Omega(\pi)$ have dimension $2g$ and are invariant under the Zorich cocycle, which acts symplectically on these subspaces. Let $\pi \in \mathcal{R}$, $F \subset H(\pi)$ a subspace of dimension k, $0 < k < 2g$, and $F_1^*, \ldots, F_l^* \subset H(\pi)$ be subspaces of codimension k. The twisting property asserts that there exists a loop σ of $\mathcal{D}$ at π such that the image of F under the matrix B_σ corresponding to σ under the Zorich cocycle is transverse to $F_1^*, \ldots, F_l^*$.

Consider the 2–dimensional subspace $F(\pi, \tau)$ generated by q and v. As it is associated to the positive Lyapunov exponents $\theta_1 > \theta_2$, it is contained in $H(\pi)$ (the Lyapunov exponents on $\mathbb{R}^{\mathcal{A}}/H(\pi)$ are equal to zero).

Let $\pi \in \mathcal{R}$ be such that a is the first top letter and the last bottom letter of π and let $\alpha \in \mathcal{A}$, $\alpha \neq a$. The relation $v_\alpha q_a - v_a q_\alpha = 0$ holds if and only if $F(\pi, \tau)$ is *not* transverse to the codimension 2 subspace

$$\{y \in \mathbb{R}^{\mathcal{A}}\,,\, y_a = y_\alpha = 0\,\}\,.$$

We claim that the intersection $F^*(\alpha)$ of this subspace with $H(\pi)$ is transverse, hence has codimension 2 in $H(\pi)$: indeed, let $\nu \in \mathbb{R}^{\mathcal{A}}$, $y = \Omega(\pi)\nu$; as a is the first top and the last bottom letter of π we have

$$y_a = \sum_{\beta \neq a} \nu_\beta\,,$$

On the other hand the coefficient of ν_a in y_α is -1. Therefore the linear forms (of the variable ν) y_a and y_α are not proportional and the claim follows.

Therefore, if the conclusion of Proposition 1 for π, α does not hold, there exists a set of positive measure $X \subset \mathbb{P}(\Theta_\pi)$ such that, for $\tau \in X$, the subspace $F(\pi, \tau)$ is not transverse to $F^*(\alpha)$.

The following Lemma will be proved below.

Lemma. *Let $\pi \in \mathcal{R}$, $X \subset \mathbb{P}(\Theta_\pi)$ a subset of positive measure. For any $\varepsilon > 0$, there exista a loop σ of $\mathcal{D}$ at π such that the measure of $\mathbb{P}(\Theta_\pi) \setminus ({}^tB_\sigma(X) \cap \mathbb{P}(\Theta_\pi))$ is $< \varepsilon$.*

From the twisting property and the compactness of the Grassmannians, there exist loops $\sigma_1, \ldots, \sigma_k$ of $\mathcal{D}$ at π such that, for any 2–dimensional subspace $F_0 \subset H(\pi)$, and any codimension 2 subspace $F_0^* \subset H(\pi)$, F_0^* is transverse to at least one of the $B_{\sigma_i} F_0$.

Let $\varepsilon > 0$, and let σ be as in the Lemma above. If $\varepsilon > 0$ is small enough, there exists a set of positive measure $Y \subset \mathbb{P}(\Theta_\pi)$ such that, for $\tau \in Y$, ${}^tB_{\sigma_i}^{-1}\tau$ belongs to ${}^tB_\sigma(X) \cap \mathbb{P}(\Theta_\pi)$ for all $1 \leq i \leq k$. Writing $\tau_i = {}^tB_\sigma^{-1}\,{}^tB_{\sigma_i}^{-1}\tau$, we have $\tau_i \in X$ for $1 \leq i \leq k$; this means that $F(\pi, \tau_i)$ is not transverse to $F^*(\alpha)$. As the F–bundle is invariant under the Rauzy–Veech dynamics, we have that $F(\pi, \tau_i) = B_\sigma B_{\sigma_i} F(\pi, \tau)$; setting $F_0 = F(\pi, \tau)$, $F_0^* = B_\sigma^{-1} F^*(\alpha)$, we see that, for $\tau \in Y$, $B_{\sigma_i} F_0$ is not transverse to F_0^* for all $1 \leq i \leq k$, a contradiction. □

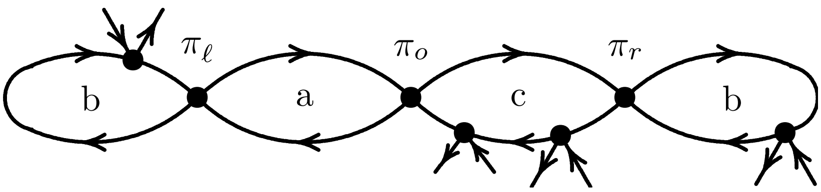
Proof of the Lemma. □

Remark. Proposition 1 is in general false if we replace a, α by any two distinct letters: consider in genus 2

$$\pi = \begin{pmatrix} A & B & C & D & E \\ D & E & C & B & A \end{pmatrix}.$$

Obviously we have $\{u_D = u_E\}$ as equation of $H(\pi)$, hence $q_D v_E - q_E v_D \equiv 0$.

3.5.4 Proof of Proposition 2. Let c be the first letter of the bottom line of all elements of $\mathcal{R}$: we have $c \neq a$; let b be any letter distinct from a and c. We will prove the inequalities of Proposition 2 up to exchanging b and c (which leaves invariant the statement of Proposition 2). Let $\pi_0 \in \mathcal{R}$ such that the last top and bottom letters are c, a respectively (if $\pi \in \mathcal{R}$ is such that a is the last letter of the bottom line such a π_0 is obtained by a suitable number of iterations of the Rauzy operation R_b). Consider in $\mathcal{D}$ the subdiagram obtained by erasing the arrows whose winner is not a, b or c and then keeping the connected component $\mathcal{D}'$ of π_0. It is easily seen to have the typical form shown in the figure (see [AV], [AGY])



(i.e. it is essentially the Rauzy diagram with $d = 3$, (see e.g. [Y1]) with some meaningless vertices added; only π_0, π_l and π_r have two arrows going out).

For paths contained in $\mathcal{D}'$, the a, b, c coordinates of vectors are changed under the Zorich cocycle exactly as in the Rauzy diagram with $d = 3$. Consider the vectors in the right halfplane:

$$\begin{aligned} u_a &= u_a(\pi_0, \tau) = (q_a(\pi_0, \tau), v_a(\pi_0, \tau)), \\ u_b &= u_b(\pi_0, \tau) = (q_b(\pi_0, \tau), v_b(\pi_0, \tau)), \\ u_c &= u_c(\pi_0, \tau) = (q_c(\pi_0, \tau), v_c(\pi_0, \tau)). \end{aligned}$$

By Proposition 1 (and its symmetric statement obtained by exchanging top and bottom), for almost all τ, no two among these 3 vectors are collinear (indeed, c has the same properties than a).

If there is a set of τ of positive measure such that u_a is between u_b and u_c in the right halfplane, the conclusion of Proposition 2 is satisfied; assume therefore that it is not the case.

Next assume that on a set of positive measure the vector $u_a + u_c$ is between u_a and u_b. Consider the path σ starting at π_0, going to π_l and making N–times the b–loop at π_l; the effect on the vectors is the following (we have for each arrow to add the winning vector to the losing one):

$$\begin{aligned} u_a &\longrightarrow u'_a = u_a + N u_b\,, \\ u_b &\longrightarrow u'_b = u_b\,, \\ u_c &\longrightarrow u'_c = u_a + u_c\,. \end{aligned}$$

If N is large enough then u'_a is between u'_b and u'_c hence the conclusion of Proposition 2 is again satisfied (at π_l).

Finally, in the remaining case, we would have that, for almost all τ, u_b is between u_a and $u_a + u_c$; the loop at π_0 obtained by going to π_r, making N times the b–loop at π_r and coming back to π_0 has for effect:

$$\begin{aligned} u_a &\longrightarrow u''_a = u_a + u_c\,, \\ u_b &\longrightarrow u''_b = u_c + (N+1) u_b\,, \\ u_c &\longrightarrow u''_c = u_c + N u_b\,. \end{aligned}$$

For large N, u''_c is between u''_a and u''_b, which contradicts the assumption. The proof of Proposition 2 is now complete. □

3.6 Consequences for limit shapes

3.6.1 Let (π, λ, τ) be a typical point for the Rauzy–Veech dynamics, let $\alpha \in \mathcal{A}$, and let $W_\alpha(\pi, \lambda, \tau)$ be the limit shape defined in Section 3.4.3.

Proposition *The extremal values of $W_\alpha(\pi, \lambda, \tau)$ (minimum and maximum) are not taken at the endpoints of the interval of definition $[0, q_\alpha(\pi, \tau)]$ of $W_\alpha(\pi, \lambda, \tau)$.*

Proof. As the set Υ of the proposition in 3.5.1 has positive measure and the invariant measure for Rauzy–Veech dynamics is conservative and ergodic, there exists (for almost all (π, λ, τ)) a positive integer N such that $(\pi^{(-N)}, \lambda^{(-N)}, \tau^{(-N)})$ belongs to Υ and the interval $I^{(0)}$ is contained in the first subinterval $I_a^{(-N+1)}$ of $I^{(-N+1)}$. We have then

$$\begin{aligned} W_\alpha(\pi, \lambda, \tau)(q_a^{(-N)}(\pi, \tau)) &= w_a^{(-N)}(\pi, \lambda, \tau)\,, \\ W_\alpha(\pi, \lambda, \tau)(q_a^{(-N+1)}(\pi, \tau)) &= w_a^{(-N+1)}(\pi, \lambda, \tau)\,, \end{aligned}$$

with

$$\begin{aligned} q_a^{(-N+1)}(\pi, \tau) &= q_a^{(-N)}(\pi, \tau) + q_\alpha^{(-N)}(\pi, \tau)\,, \\ w_a^{(-N+1)}(\pi, \lambda, \tau) &= w_a^{(-N)}(\pi, \lambda, \tau) + w_\alpha^{(-N)}(\pi, \lambda, \tau)\,, \end{aligned}$$

α being the winner of the arrow from $\pi^{(-N)}$ to $\pi^{(-N+1)}$. By the definition of Υ we have that

$$w_a^{(-N)}(\pi,\lambda,\tau)w_a^{(-N+1)}(\pi,\lambda,\tau) < 0$$

and therefore 0 is not an extremal value of $W_\alpha(\pi,\lambda,\tau)$. The other endpoint is treated in a similar manner, exchanging the top and the bottom lines. □

3.6.2 Smallest concave majorant. Let $F : [a,b] \to \mathbb{R}$ be continuous. The infimum of concave majorants of F on $[a,b]$ is the smallest concave majorant of F and will be denoted by $\hat{F}$; it is continuous and satisfies $\hat{F}(a) = F(a)$, $\hat{F}(b) = F(b)$; moreover, the maximum values of F and $\hat{F}$ are the same. We write $\hat{F}'_r$, $\hat{F}'_l$ for the right and left derivatives of $\hat{F}$.

Proposition *Let (π,λ,τ) be a typical point for Rauzy–Veech dynamics and let $\alpha \in \mathcal{A}$. We have*

$$\hat{W}'_{\alpha,r}(\pi,\lambda,\tau)(0) = +\infty\,,$$
$$\hat{W}'_{\alpha,l}(\pi,\lambda,\tau)(q_\alpha(\pi,\tau)) = -\infty\,,$$
$$\hat{W}'_{*,r}(\pi,\lambda,\tau)(q_{\alpha_b}) = \hat{W}'_{*,l}(\pi,\lambda,\tau)(q_{\alpha_b}) \neq 0\,.$$

Proof. The first two assertions are a very slight extension of the Proposition in 3.6.1: in the proof of this proposition we first replace the set Υ of Section 3.5.1 by the slightly smaller set Υ_δ obtained by replacing condition (iii) in 3.5.1 by

(iii)$_\delta$ $\qquad w_a(w_a + w_\alpha) < 0\,,$ and $|w_a| > \delta$ and $|w_a + w_\alpha| > \delta\,.$

If $\delta > 0$ is small enough, this has still positive measure. Now, the integer N in the proof of Proposition 3.6.1 can be taken arbitrarily large; as $q_a^{(-N)}$ and $w_a^{(-N)}$ go down exponentially fast (in Zorich time) at respective rates $\theta_1 > \theta_2$, this implies the first two assertions of the Proposition.

For the last assertion, it follows from the definition of V_* and the first two assertions that we have

$$\hat{V}_*(\pi,\tau)(q_{\alpha_b}) > V_*(\pi,\tau)(q_{\alpha_b})\,.$$

It follows that $\hat{V}_*$ is affine in a neighborhood of q_{α_b}, in particular $\hat{V}'_{*,r}(q_{\alpha_b}) = \hat{V}'_{*,l}(q_{\alpha_b})$.

Now, obviously we have

$$\hat{W}_*(\pi,\lambda,\tau)(x) = \hat{V}_*(\pi,\tau)(x) - \frac{<\lambda,v>}{<\lambda,q>}x\,,$$

(adding an affine function to F adds the same affine function to the smallest concave majorant). Therefore we have

$$\hat{W}'_*(\pi, \lambda, \tau)(q_{\alpha_b}) = 0$$

if and only if

$$\frac{< \lambda, v >}{< \lambda, q >} = \hat{V}'_*(\pi, \tau)(q_{\alpha_b})$$

which has λ–measure zero for any given (π, τ). □

3.6.3 Corollary. *The function $W_\alpha(\pi, \lambda, \tau)$ takes its maximum value at a unique point $x_\alpha^{\max}(\pi, \lambda, \tau)$ (for almost all (π, λ, τ)).*

Proof. Let (π, λ, τ) be a typical point and $\alpha \in \mathcal{A}$. By the functional equation of Section 3.4.5, $W_\alpha(\pi, \lambda, \tau)$ is a rescaled version of either $W_\alpha(\pi^{(-1)}, \lambda^{(-1)}, \tau^{(-1)})$ (if α is not the loser of the arrow from $\pi^{(-1)}$ to π) or $W_*(\pi^{(-1)}, \lambda^{(-1)}, \tau^{(-1)})$ (if α is the loser of this arrow).

In this last case, by the last assertion of Proposition 3.6.2, $W_*(\pi^{(-1)}, \lambda^{(-1)}, \tau^{(-1)})$ does not take its maximum value both in $[0, q_{\alpha_b}(\pi^{(-1)}, \tau^{(-1)})]$ and in $[q_{\alpha_b}(\pi^{(-1)}, \tau^{(-1)}), q_{\alpha_b}(\pi^{(-1)}, \tau^{(-1)}) + q_{\alpha_t}(\pi^{(-1)}, \tau^{(-1)})]$ (otherwise we would have $\hat{W}'_*(\pi^{(-1)}, \lambda^{(-1)}, \tau^{(-1)})(q_{\alpha_b}(\pi^{(-1)}, \tau^{(-1)})) = 0$).

In view of the definition of W_*, this means that the set $\mathcal{M}$ where $W_\alpha(\pi, \lambda, \tau)$ takes its maximum value is a rescaled version of the set where $W_{\alpha(1)}(\pi^{(-1)}, \lambda^{(-1)}, \tau^{(-1)})$ takes its maximum value, for some $\alpha(1) \in \mathcal{A}$. Iterating this procedure, we obtain that $\mathcal{M}$ is a rescaled version (by a factor $\Theta_1^{(-n)}$) of the set where $W_{\alpha(n)}(\pi^{(-n)}, \lambda^{(-n)}, \tau^{(-n)})$ takes its maximum value, for some letter $\alpha(n) \in \mathcal{A}$. As the q_α are bounded by 1 this proves that the diameter of $\mathcal{M}$ is smaller than $\Theta_1^{(-n)}$ for all $n \geq 0$, hence it is a point. The case of the minimum is similar. □

A similar result is true for minimum values. The function $W_*(\pi, \lambda, \tau)$ also takes its maximum value at a unique point $x_*^{\max}(\pi, \lambda, \tau)$. By the proposition in 3.6.1 we know that $x_*^{\max}(\pi, \lambda, \tau)$ is distinct from 0, q_{α_b}, $q_{\alpha_b} + q_{\alpha_t}$. Observe that we have

$$x_*^{\max}(\pi, \lambda, \tau) \in (0, q_{\alpha_b}) \Longleftrightarrow \hat{W}'_*(\pi, \lambda, \tau)(q_{\alpha_b}) < 0\,,$$
$$x_*^{\max}(\pi, \lambda, \tau) \in (q_{\alpha_b}, q_{\alpha_b} + q_{\alpha_t}) \Longleftrightarrow \hat{W}'_*(\pi, \lambda, \tau)(q_{\alpha_b}) > 0\,.$$

Assume for instance that $x_*^{\max}(\pi, \lambda, \tau) \in (0, q_{\alpha_b})$. As W_* and $\hat{W}_*$ coincide at $x_*^{\max}$, we have, for $x \in [q_{\alpha_b}, q_{\alpha_b} + q_{\alpha_t}]$

$$\begin{aligned} W_*(x) &\leq \hat{W}_*(x) \\ &\leq \hat{W}_*(q_{\alpha_b}) + \hat{W}'_*(q_{\alpha_b})(x - q_{\alpha_b}) \\ &\leq W_*(x_*^{\max}) + \hat{W}'_*(q_{\alpha_b})(x - q_{\alpha_b})\,. \end{aligned}$$

This will provide a satisfactory control of W_* if $|\hat{W}'_*(q_{\alpha_b})|$ is not too small and $(x - q_{\alpha_b})$ is not too small. When x is very close to q_{α_b}, we will rely on a direct control on $W_*(x_*^{\max}) - W_*(q_{\alpha_b})$, based on the Proposition in 3.5.1.

3.7 Proof of the Proposition in 3.3.1

3.7.1 Let (π, λ, τ) be a typical point for the Rauzy–Veech dynamics.

We observe first that, if $\tilde{w}$ is a vector in the subspace $E_2(\pi, \lambda)$, Zorich has proved [Z2] that the Birkhoff sums $S_n\tilde{w}$ satisfy, uniformly on $I^{(0)}$, an estimate

$$\|S_n(\tilde{w})\|_{C^0} \leq C(\varepsilon)|n|^{\omega+\varepsilon},$$

for all $\varepsilon > 0$; here ω is either 0 if $g = 2$ or θ_3/θ_1 if $g \geq 3$. In any case, we have $\omega+\varepsilon < \theta_2/\theta_1 - \varepsilon$ for small ε, hence the order is smaller than the one in Proposition 3.3.1.

It follows that it is sufficient to prove the estimate of Proposition 3.3.1 when w is "the" vector $w(\pi, \lambda, \tau)$ considered above (there are actually two vectors to consider, opposite to each other).

3.7.2 Recall the relation between Birkhoff sums and limit shapes from Section 3.4.4:

$$S_\alpha w(i) = \Theta_2^{(n)} W_\alpha((\Theta_1^{(n)})^{-1} S_\alpha q(i)),$$

where

- $S_\alpha q(i) = \sum_{j=0}^{i-1} q_{\beta_j}(\pi, \tau)$,
- $S_\alpha w(i) = \sum_{j=0}^{i-1} w_{\beta_j}(\pi, \lambda, \tau)$,
- $\beta_0, \beta_1, \ldots$ is the itinerary of $I_\alpha^{(n)}$ with relation to the partition $I_\beta^{(0)}$ of $I^{(0)}$,
- $W_\alpha = W_\alpha(\pi^{(n)}, \lambda^{(n)}, \tau^{(n)})$ is the limit shape at $(\pi^{(n)}, \lambda^{(n)}, \tau^{(n)})$,
- the real number $\Theta_1^{(n)} = \Theta_1^{(n)}(\pi, \lambda, \tau) > 0$ is defined by the relation $q^{(n)}(\pi, \tau) = \Theta_1^{(n)} q(\pi^{(n)}, \tau^{(n)})$ where $q^{(n)}(\pi, \tau)$ is the image of $q(\pi, \tau)$ under the Zorich cocycle,
- the real number $\Theta_2^{(n)} = \Theta_2^{(n)}(\pi, \lambda, \tau)$ is similarly defined by $w^{(n)}(\pi, \lambda, \tau) = \Theta_2^{(n)} w(\pi^{(n)}, \lambda^{(n)}, \tau^{(n)})$,
- i varies from 0 to the return time of $I_\alpha^{(n)}$ in $I^{(n)}$ under $T^{(0)}$.

We assume that the choices of signs for $w(\pi, \lambda, \tau)$ and $w(\pi^{(n)}, \lambda^{(n)}, \tau^{(n)})$ are such that

$$\Theta_2^{(n)} > 0.$$

By Corollary 3.6.3, for almost all (π, λ, τ), all $\alpha \in \mathcal{A}$, all $n \geq 0$, $W_\alpha(\pi^{(n)}, \lambda^{(n)}, \tau^{(n)})$ has a unique maximum at some $x_\alpha^{\max} = x_\alpha^{\max}(\pi^{(n)}, \lambda^{(n)}, \tau^{(n)})$. Let i be the integer such that

$$S_\alpha q(i) < \Theta_1^{(n)} x_\alpha^{\max} < S_\alpha q(i+1), \tag{4}$$

where the inequalities are strict, by Proposition 3.6.1.

Let $I_\alpha^{\max}(n)$ be the image of $I_\alpha^{(n)}$ by $(T^{(0)})^i$.

Consider what happens when going from n to $n+1$. If α is *not the loser* of the arrow from $\pi^{(n)}$ to $\pi^{(n+1)}$, $W_\alpha(\pi^{(n+1)}, \lambda^{(n+1)}, \tau^{(n+1)})$ is a rescaled version of $W_\alpha(\pi^{(n)}, \lambda^{(n)}, \tau^{(n)})$, hence the respective maxima correspond. Therefore the values of i are the same, and $I_\alpha^{\max}(n+1)$ is equal to (if α is not the winner) or contained in (if α is the winner) $I_\alpha^{\max}(n)$ (because $I_\alpha^{(n+1)}$ is equal to, resp. contained in, $I_\alpha^{(n)}$).

If α *is the loser* of the arrow from $\pi^{(n)}$ to $\pi^{(n+1)}$, $W_\alpha(\pi^{(n+1)}, \lambda^{(n+1)}, \tau^{(n+1)})$ is a rescaled version of $W_*(\pi^{(n)}, \lambda^{(n)}, \tau^{(n)})$. Write as usual α_b (resp. α_t) for the last letters in the bottom (resp. top) lines of $\pi^{(n)}$. The maximum $x_*^{\max}$ is either $x_{\alpha_b}^{\max}$ or $q_{\alpha_b} + x_{\alpha_t}^{\max}$; in the first case, the values of i for $I_\alpha^{\max}(n+1)$ and $I_{\alpha_b}^{\max}(n)$ are again the same, and $I_\alpha^{(n+1)}$ is a subinterval of $I_{\alpha_b}^{(n)}$, hence $I_\alpha^{\max}(n+1) \subset I_{\alpha_b}^{\max}(n)$; in the second case, the values of i for $I_\alpha^{\max}(n+1)$ and $I_{\alpha_t}^{\max}(n)$ differ by the return time Q of $I_{\alpha_b}^{(n)}$ in $I^{(n)}$, and the image of $I_\alpha^{(n+1)}$ under $(T^{(0)})^Q$ is contained in $I_{\alpha_t}^{(n)}$, hence $I_\alpha^{\max}(n+1)$ is contained in $I_{\alpha_t}^{\max}(n)$.

Thus, we have the following

Lemma. *For each n, the intervals $I_\alpha^{\max}(n)$ are disjoint. They satisfy*

$$I_\alpha^{\max}(n+1) \subset I_{\eta_n(\alpha)}^{\max}(n)$$

where $\eta_n(\alpha) = \alpha$ except possibly when α is the loser of the arrow from $\pi^{(n)}$ to $\pi^{(n+1)}$; in this case $\eta_n(\alpha)$ is either α or the winner of the same arrow.

Proof. The last assertion has been proved above, the first one is clear because the orbits of the $I_\alpha^{(n)}$ are disjoint till their return time. □

We can now specify the point x^* in Proposition 3.3.1. Indeed, take any sequence $(\alpha_n)_{n\geq 0} \subset \mathcal{A}$ such that

$$\eta_n(\alpha_{n+1}) = \alpha_n \,.$$

Remark. It is reasonable to expect that for almost all (π, λ, τ) such a sequence is unique.

The point x^* is defined to be

$$x^* = \cap_{n\geq 0} \overline{I_{\alpha_n}^{\max}(n)} \,.$$

3.7.3 The Birkhoff sums of w at x^* and the functions W_α are related as follows.

Denote by $Q^+(n) \geq 0$ (respectively $Q^-(n) \leq 0$) the first entrance time in the future (resp. in the past) of x^* in $I^{(n)}$ under $T^{(0)}$. The sequence $Q^+(n)$ is non decreasing and the sequence $Q^-(n)$ is non increasing.

Moreover, for almost all (π, λ, τ), one has $(\pi^{(n)}, \lambda^{(n)}, \tau^{(n)}) \in \Upsilon_\delta$ for infinitely many $n \geq 0$, where Υ_δ is the set defined in 3.6.2. It follows that there are arbitrarily large values of n such that the maximum $x^{\max}_{\alpha_n}(\pi^{(n)}, \lambda^{(n)}, \tau^{(n)})$ of $W_{\alpha_n}(\pi^{(n)}, \lambda^{(n)}, \tau^{(n)})$ is not exponentially small w.r.t. Zorich time $Z(n)$. This implies that the integer i in formula (4) above goes to $+\infty$ and thus

$$\lim_{n \to +\infty} Q^-(n) = -\infty \, ,$$

and similarly one has

$$\lim_{n \to +\infty} Q^+(n) = +\infty \, .$$

Given some integer j, we want to estimate the Birkhoff sum $S_j w(x^*)$.

Assume for instance that j is positive (the other case is symmetric) and let n be such that

$$Q^+(n) < j \leq Q^+(n+1) \, .$$

For $m \geq 0$, let $i_m \geq 0$ be the integer such that

$$I^{\max}_{\alpha_m}(m) = T^{i_m}(I^{(m)}_{\alpha_m}) \, .$$

Claim. *α_{n+1} is the loser of the arrow from $\pi^{(n)}$ to $\pi^{(n+1)}$.*

Proof. Assume that this is not the case. Then the discussion before the lemma in Section 3.7.2 shows that $i_n = i_{n+1}$; on the other hand, the return times of $I^{(n)}_{\alpha_{n+1}}$ in $I^{(n)}$ and $I^{(n+1)}_{\alpha_{n+1}}$ in $I^{(n+1)}$ are the same. Then we would have $Q^+(n) = Q^+(n+1)$, a contradiction. □

It follows from the claim that $W_{\alpha_{n+1}}(\pi^{(n+1)}, \lambda^{(n+1)}, \tau^{(n+1)})$ is a rescaled version of $W_* = W_*(\pi^{(n)}, \lambda^{(n)}, \tau^{(n)})$.

We have then

$$\begin{aligned} S_j w(x^*) &= \sum_{k=i_{n+1}}^{i_{n+1}+j-1} w_{\beta_k}(\pi, \lambda, \tau) \\ &= \Theta_2^{(n)} \left(W_*([\Theta_1^{(n)}]^{-1} S_{\alpha_{n+1}} q(i_{n+1} + j)) - W_*([\Theta_1^{(n)}]^{-1} S_{\alpha_{n+1}} q(i_{n+1})) \right) \, . \end{aligned}$$

Claim. *We have $i_{n+1} = i_n$, $\pi_b^{(n)}(\alpha_{n+1}) = d$ and*

$$[\Theta_1^{(n)}]^{-1} S_{\alpha_{n+1}} q(i_{n+1}) \in [0, q_{\alpha_b}(\pi^{(n)}, \lambda^{(n)}, \tau^{(n)})]$$

and

$$[\Theta_1^{(n)}]^{-1} S_{\alpha_{n+1}} q(i_{n+1}+j) \in [q_{\alpha_b}(\pi^{(n)},\lambda^{(n)},\tau^{(n)}), q_{\alpha_b}(\pi^{(n)},\lambda^{(n)},\tau^{(n)})+q_{\alpha_t}(\pi^{(n)},\lambda^{(n)},\tau^{(n)})]\,. \blacksquare$$

Proof. We refer again to the discussion before the lemma in Section 3.7.2. We claim that in this discussion we must have that $x_*^{\max}$ is $x_{\alpha_b}^{\max}$. (Otherwise this discussion shows that $Q^+(n+1) = Q^+(n)$). We have seen in Section 3.7.2 that then we have $i_n = i_{n+1}$, $\pi_b^{(n)}(\alpha_{n+1}) = d$ and thus

$$[\Theta_1^{(n)}]^{-1} S_{\alpha_{n+1}} q(i_{n+1}) \in [0, q_{\alpha_b}(\pi^{(n)},\lambda^{(n)},\tau^{(n)})]\,.$$

Moreover, we have

$$[\Theta_1^{(n)}]^{-1} S_{\alpha_{n+1}} q(i_{n+1} + Q^+(n)) = q_{\alpha_b}(\pi^{(n)},\lambda^{(n)},\tau^{(n)})\,,$$

and

$$[\Theta_1^{(n)}]^{-1} S_{\alpha_{n+1}} q(i_{n+1} + Q^+(n+1)) = q_{\alpha_b}(\pi^{(n)},\lambda^{(n)},\tau^{(n)}) + q_{\alpha_t}(\pi^{(n)},\lambda^{(n)},\tau^{(n)})\,,$$

Hence

$$[\Theta_1^{(n)}]^{-1} S_{\alpha_{n+1}} q(i_{n+1}+j) \in [q_{\alpha_b}(\pi^{(n)},\lambda^{(n)},\tau^{(n)}), q_{\alpha_b}(\pi^{(n)},\lambda^{(n)},\tau^{(n)})+q_{\alpha_t}(\pi^{(n)},\lambda^{(n)},\tau^{(n)})]\,. \blacksquare$$

Let

$$\begin{aligned} y^\dagger &= (\Theta_1^{(n)})^{-1} S_{\alpha_{n+1}} q(i_{n+1}+j)\,, \\ y^* &= (\Theta_1^{(n)})^{-1} S_{\alpha_{n+1}} q(i_{n+1})\,. \end{aligned}$$

From the construction of W_* we have

$$|\Theta_2^{(n)}(W_*(y^*) - W_*(x_*^{\max}))| \leq C\,,$$

where the majorant C depends on (π,λ,τ) but *not* on n. We therefore are left with the estimation of

$$\Theta_2^{(n)}(W_*(y^\dagger) - W_*(x_*^{\max}))\,,$$

when $x_*^{\max} \in [0, q_{\alpha_b}]$, $y^\dagger \in [q_{\alpha_b}, q_{\alpha_b} + q_{\alpha_t}]$.

3.7.4 For $n \geq 0$, write $W_*^{\max}(n)$ for the maximum value of $W_*(\pi^{(n)},\lambda^{(n)},\tau^{(n)})$ in its domain $[0, q_{\alpha_b} + q_{\alpha_t}]$. If the maximum value is taken in $[0, q_{\alpha_b}]$, let $\tilde{W}_*^{\max}(n)$ be the maximum value of W_* in $[q_{\alpha_b}, q_{\alpha_b} + q_{\alpha_t}]$; if the maximum value of W_* is taken in $[q_{\alpha_b}, q_{\alpha_b} + q_{\alpha_t}]$, let $\tilde{W}_*^{\max}(n)$ be the maximum value in $[0, q_{\alpha_b}]$.

To complete the proof of Proposition 3.3.1, it is therefore sufficient to prove the following estimate:

Proposition. *For almost all* (π, λ, τ) *one has*

$$\lim_{n \to +\infty} \frac{1}{Z(n)} \log(W_*^{\max}(n) - \tilde{W}_*^{\max}(n)) = 0\,,$$

where $Z(n)$ *is the Zorich time defined in Section 1.4.*

Proof. We apply Birkhoff ergodic theorem to the Rauzy–Veech dynamics (in Zorich time) and to the characteristic function of the set Υ_δ. We see that for any n there exists $n' < n$ such that $I^{(n)}$ is contained in the first interval $I_a^{(n'+1)}$, $(\pi^{(n')}, \lambda^{(n')}, \tau^{(n')})$ belongs to Υ_δ, and the ratio $\frac{Z(n)-Z(n')}{Z(n)}$ converges to 0 as $n \to +\infty$.

By definition of Υ_δ and the scaling rules, there exists a point $x_1 \in [q_{\alpha_b}, q_{\alpha_b} + q_{\alpha_t}]$ such that

$$W_*(x_1) - W_*(q_{\alpha_b}) \geq \delta \frac{\mathrm{Min}\,[\Theta_2^{(n')}, \Theta_2^{(n'+1)}]}{\Theta_2^{(n)}}\,.$$

Exchanging the top and bottom lines, we find similarly that there exists a point $x_0 \in [0, q_{\alpha_b}]$ such that

$$W_*(x_0) - W_*(q_{\alpha_b}) \geq \delta \frac{\mathrm{Min}\,[\Theta_2^{(n')}, \Theta_2^{(n'+1)}]}{\Theta_2^{(n)}}\,.$$

On the other hand, take $n'' < n'$ such that $\frac{Z(n)-Z(n'')}{Z(n)}$ still goes to zero but

$$\|W(\pi^{(n'')}, \lambda^{(n'')}, \tau^{(n'')})\| \frac{\Theta_2^{(n'')}}{\mathrm{Min}\,[\Theta_2^{(n')}, \Theta_2^{(n'+1)}]}$$

is small; in view of the choice of normalization for W, this is possible because of the following

Lemma. *For almost all* (π, τ) *we have*

$$\lim_{n \to +\infty} \frac{1}{Z(n)} \log \mathrm{Inf}_\alpha\, q_\alpha(\pi^{(n)}, \tau^{(n)}) = 0\,.$$

Proof. This follows easily from the boundary behaviour of the Zorich invariant measure, see [Y1] for instance. □

Putting together the properties of n' and n'', we see that for

$$|y - q_{\alpha_b}| \le r(n) := \mathrm{Min}_\alpha\, q_\alpha(\pi^{(n'')}, \tau^{(n'')}) \frac{\Theta_1^{(n'')}}{\Theta_1^{(n)}} \ ,$$

we have

$$W_*(y) \le W_*^{\max}(n) - \frac{1}{2}\delta \frac{\mathrm{Min}\,[\Theta_2^{(n')}, \Theta_2^{(n'+1)}]}{\Theta_2^{(n)}} \, .$$

Observe that by the claim and the choice of n'' we have

$$\lim_{n\to+\infty} \frac{1}{Z(n)} \log r(n) = 0 \ .$$

To estimate W_* outside the neighborhood of q_{α_b} we go back to the smallest concave majorant $\hat{W}_*$ of Section 3.6.2. By the proposition in this Section the derivative at q_{α_b} of $\hat{W}_*$ almost surely exists and is non zero.

Observe that the maximum value of W_* is taken in $[0, q_{\alpha_b}]$ (respectively $[q_{\alpha_b}, q_{\alpha_b} + q_{\alpha_t}]$) if and only if $\hat{W}_*'(q_{\alpha_b}) < 0$ (resp. $\hat{W}_*'(q_{\alpha_b}) > 0$).

In the first case, we have, for $y \ge q_{\alpha_b} + r(n)$

$$W_*(y) \le W_*^{\max}(n) + \tilde{W}_*'(q_{\alpha_b}) r(n) \ .$$

We claim that

Claim *Almost surely in* (π, λ, τ) *we have*

$$\limsup_{n\to+\infty} \frac{1}{n} \log |\hat{W}_*'(\pi^{(n)}, \lambda^{(n)}, \tau^{(n)})(q_{\alpha_b})| \ge 0 \, .$$

Proof. Recall that (Section 3.6.2)

$$\hat{W}_*'(\pi, \lambda, \tau)(q_{\alpha_b}) = \hat{V}_*'(\pi, \tau)(q_{\alpha_b}) - \frac{<\lambda, v>}{<\lambda, q>} \, .$$

Therefore one has $|\hat{W}_*'(q_{\alpha_b})| < \varepsilon$ if and only if

$$\left| \frac{<\lambda, v>}{<\lambda, q>} - \hat{V}_*'(\pi, \tau)(q_{\alpha_b}) \right| < \varepsilon \, .$$

For fixed (π, τ), the set of λ such that $|\hat{W}_*'(\pi, \lambda, \tau)(q_{\alpha_b})| < \varepsilon$ has therefore a Lebesgue measure which is at most $C\varepsilon$ (because q and v are normalized to have l^2 norm 1, and q is positive). Going to the Zorich invariant measure (with the control of [Y1] for instance) and using a Borel–Cantelli argument gives the claim. □

Combining the estimate for $|y - q_{\alpha_b}| < r(n)$ and the one for $|y - q_{\alpha_b}| > r(n)$ now gives the Proposition. $\square$

3.7.5 End of the proof of Proposition 3.3.1 We have just seen that the quantity at the end of Section 3.7.3

$$\Theta_2^{(n)}(W_{\alpha_n}(y^\dagger) - W_{\alpha_n}(x_{\alpha_n}^{\max}))$$

(with $y^\dagger \in [q_{\alpha_b}, q_{\alpha_b} + q_{\alpha_t}]$ if $x_{\alpha_n}^{\max} \in [0, q_{\alpha_b}]$ and $y^\dagger \in [0, q_{\alpha_b}]$ if $x_{\alpha_n}^{\max} \in [q_{\alpha_b}, q_{\alpha_b} + q_{\alpha_t}]$) grows exponentially fast at rate θ_2 (in Zorich time $Z(n)$). This quantity was seen in Section 3.7.3 to control $S_j w(x^*)$ for $Q^+(n-1) < j \leq Q^+(n)$ (in the case $x_{\alpha_n}^{\max} \in [0, q_{\alpha_b}]$).

But as we have

$$\lim_{n\to+\infty} \frac{1}{Z(n)} \log r(n) = 0$$

we will have by the scaling rules

$$\lim_{n\to+\infty} \frac{1}{Z(n)} \log Q^+(n-1) = \theta_1 \ .$$

The proof of Proposition 3.3.1 is now complete. $\square$

Remark. The dimension $r - 1$ of $\mathrm{Aff}^{(1)}(\underline{\gamma}, w)$ is obtained as follows. With the notations of Section 2.2, we have $r_c = 0$ and $r = r_d$ is the number of sequences α_n such that $\eta_n(\alpha_{n+1}) = \alpha_n$. Indeed, observe first that for n large and $n' >> n$ the image L_n of the composition $\eta_n \circ \ldots \circ \eta_{n'}$ is independent of n' and has r elements; moreover, η_n is 1-to-1 from L_{n+1} onto L_n. Take then T^* in the interior of $\mathrm{Aff}^{(1)}(\underline{\gamma}, w)$. For $\alpha \in L_n$, $I_\alpha^{\max}(n)$ contains a wandering interval such that the complement has small Lebesgue measure (for large n). Taking then $n' >> n$ and decomposing $(0,1)$ into the union of the orbits of the $I_\beta^{\max}(n')$, one has that the measure of each orbit is no more than the measure of the largest interval in the orbit , which is contained in some $I_\alpha^{\max}(n)$, $\alpha \in L_n$; hence one concludes that the complement of the orbits of the r wandering intervals has 0 Lebesgue measure.

From $r_c = 0$ it follows that *any* affine i.e.m. $T \in \mathrm{Aff}(\underline{\gamma}, w)$ has a wandering interval (consider the segment from T^* to T as in Section 2).

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